

Triangle-Free Triple Systems

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This paper is dedicated to the memory of Gábor Simonyi, who unexpectedly passed away in April 2025, just as the paper was being submitted. As his friends, colleagues and collaborators, we miss him dearly.

Abstract

There are four non-isomorphic configurations of triples that can form a triangle in a 3-uniform hypergraph. Forbidding different combinations of these four configurations, fifteen extremal problems can be defined, several of which already appeared in the literature. Here we systematically study all of these problems solving the new cases exactly or asymptotically. In many cases we also characterize the extremal constructions.

Mathematics Subject Classifications: 05C35, 05B07

1 Introduction

1.1 Notations

A *hypergraph* $\mathcal{F}$ is a pair $(V(\mathcal{F}), E(\mathcal{F}))$ where the set of *edges*, $E(\mathcal{F})$, is a family of subsets of the set of *vertices* $V(\mathcal{F})$ (see Berge [1], Lovász [22]). To simplify notations we frequently identify $\mathcal{F}$ with $E(\mathcal{F})$ (and omit parentheses, etc.) if it does not cause ambiguity. If all edges of $\mathcal{F}$ have the same size, say r , then we speak of an r -graph, or r -uniform hypergraph. With this terminology 2-graphs are simple graphs without loops and 3-uniform hypergraphs are also called *triple systems*. We usually identify $V = V(\mathcal{F})$ with the set of the first n positive integers, $[n] := \{1, 2, 3, \dots, n\}$. Then $\binom{V}{r}$ stands for the complete r -graph with vertex set V , it is $\{F : |F| = r, F \subseteq V\}$.

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We write $\partial\mathcal{F}$ for $\{F \setminus \{x\} : F \in \mathcal{F}, x \in F\}$, it is called the *shadow* of $\mathcal{F}$. If $\mathcal{F}$ is a triple system then its shadow is a simple graph. The *degree* of $S \subseteq V(\mathcal{F})$ is the number of edges containing S , in notation

$$\deg_{\mathcal{F}}(S) := |\{F : F \in \mathcal{F}, S \subseteq F\}|.$$

Given a triple system $\mathcal{F}$ we write $\mathcal{F}[x]$ for the *link* of x in $\mathcal{F}$, i.e., the graph on $V(\mathcal{F}) \setminus \{x\}$ whose edges are those pairs that form with x a triple in $\mathcal{F}$. If F is an edge of a 3-uniform hypergraph $\mathcal{F}$ such that $x, y \in F$, $x \neq y$, and there is no other $F' \in \mathcal{F}$ containing both x and y , then we say that the pair xy is an *own pair* of F . With our previous notation, $\deg_{\mathcal{F}}(xy) = 1$.

1.2 Cycles in hypergraphs

In hypergraph theory a cycle of length ℓ is usually defined as a sequence of *distinct* vertices $x_1, x_2, \dots, x_\ell$ together with a sequence of *distinct* edges $E_1, \dots, E_\ell$ such that E_i contains x_i and x_{i+1} for $1 \leq i < \ell$ and E_ℓ contains x_ℓ and x_1 . We refer to a cycle of ℓ edges as an ℓ -cycle. For ordinary graphs, the above definition coincides with the definition of a cycle C_ℓ in graphs. We call a 3-cycle a *triangle*.

One of the oldest problems in extremal graph theory is to investigate the maximum number of edges of graphs without given short cycles, e. g., the Turán-Mantel theorem says that a triangle-free graph on n vertices has at most $\lfloor \frac{1}{4}n^2 \rfloor$ edges and the only extremal graph is the complete bipartite graph with parts of sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$, respectively. For further graph results see the book of Bollobás [4].

Hypergraph problems are more complicated, but there are some results, e. g., that of Lazebnik and Verstraëte [21] saying that if $\mathcal{F}$ is 3-uniform, has n vertices, girth five (i.e., no cycles of length less than five) and a maximum number of edges, then $|E(\mathcal{F})| = \frac{1}{6}n^{3/2} + o(n^{3/2})$.

In this paper we deal with 3-uniform hypergraphs containing no triangles of certain types.

1.3 Triangles in triple systems

One can easily realize that there are altogether four non-isomorphic configurations of three triples that create a triangle. We will call these configurations **A**, **B**, **C**, and **D** (see Figure 1 below). The main goal of this paper is to study the maximum number of triples one can have on n vertices while excluding certain combinations of the four triangle creating configurations.

Configuration **A** is the one needing six vertices, that is, it is isomorphic to the hypergraph with $V = \{1, \dots, 6\}$, $E = \{124, 135, 236\}$. It is known as the loose 3-cycle.

Configuration **B** is on five vertices, where two of the three triples involved intersect in two vertices and each of them meets the third triple in a different vertex. Thus **B** is isomorphic to the hypergraph with $V = \{1, \dots, 5\}$, $E = \{125, 134, 234\}$.

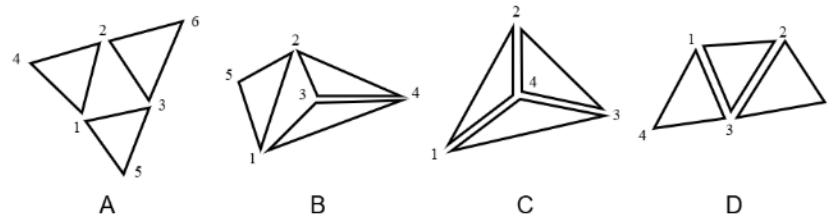


Figure 1: The four configurations of three triples forming a triangle

Configuration **C** is the one on four vertices, that is, it is isomorphic to the hypergraph with $V = \{1, \dots, 4\}$, $E = \{124, 134, 234\}$. It can be viewed as a 3-edge subhypergraph of the tight 4-cycle.

Configuration **D** is on five vertices again, with one triple sharing with each of the other two a different pair of vertices. It is isomorphic to the hypergraph with $V = \{1, \dots, 5\}$, $E = \{123, 134, 235\}$. It is known as the tight 3-path.

Our starting point is a result (first proved by Győri [17]) giving the maximum number of edges in a triangle-free 3-uniform hypergraph, i.e., in case all of **A**, **B**, **C** and **D** are forbidden subconfigurations. Győri's result, however, does not include a characterization of the extremal constructions. In Section 2 we reprove his result and also describe the extremal families. Then we consider the fourteen other cases when some proper subset of $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is forbidden. Five of these subsets define problems of long history, most of them are solved, but not all. Then we go on to investigate the other nine. For all but one of those nine combinations, we obtain an exact solution of the extremal problem (though sometimes only for large enough n), often along with a characterization of the extremal constructions. In the one exceptional case we give an asymptotic solution.

To state our results let us introduce the notation $\text{ex}(n, X_1 \dots X_k)$ for $\{X_1, \dots, X_k\} \subseteq \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$. It denotes the maximum number of triples in a 3-uniform hypergraph on n vertices which does not contain any of the configurations $X_1, \dots, X_k$. Table 1 below summarizes the results, including earlier ones and those in the present paper.

2 The starting case: no triangle at all

How large can a triple system $\mathcal{F}$ on n vertices be, if it contains no triangle at all? Here is a construction originally suggested by András Gyárfás (personal communication) as a conjectured optimum. Let $n \geq 3$ be given, and let k be a positive integer such that $2k < n$. Partition the set V of n vertices into k pairs $A_1, \dots, A_k$ and a set B of size $n - 2k$. Let $\mathcal{F}_{n,k}$ consist of all $k(n - 2k)$ triples of the form $A_i \cup \{b\}$ where $1 \leq i \leq k$ and $b \in B$. Then $\mathcal{F}_{n,k}$ contains no triangles. To maximize the size $k(n - 2k)$ for a given n , we choose k to be as close as possible to $\frac{n}{4}$ (this choice is unique unless $n \equiv 2 \pmod{4}$, in which case the two choices $k = \frac{n \pm 2}{4}$ are equally good). With this optimal choice of k we have $|\mathcal{F}_{n,k}| = \lfloor \frac{1}{8}n^2 \rfloor$.

AB $\binom{n-1}{2}$ for $n \geq 5$ and char. ext. Theorem 3			
A $\binom{n-1}{2}$ for $n \geq n_0$ and char. ext. Frankl&Füredi [12]	AC $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ for even n $\frac{1}{4}(n-1)^2 + 1$ for odd $n \geq 5$ and char. ext. Theorem 13	ABC $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ and char. ext. Theorem 11	
B $\lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor$ for $n \geq n_0$ Frankl&Füredi [10] char. ext. Keevash&Mubayi [20]	AD $\lfloor \frac{1}{4}(n-1)^2 \rfloor + 2$ for even $n \geq 4$ $\frac{1}{4}(n-1)^2$ for odd n and char. ext. Theorem 10	ABD $\lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 8$ and char. ext. Theorem 19	ABCD $\lfloor \frac{1}{8}n^2 \rfloor$ Győri [17] char. ext. Theorem 1
C $\geq \frac{2}{7}\binom{n}{3}(1 - o(1))$ Frankl&Füredi [11] $\leq .287\binom{n}{3}$ for $n \geq n_0$ Falgas-Ravry&Vaughan [9]	BC $\lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor$ and char. ext. Bollobás [3]	ACD $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ and char. ext. Corollary 9	
D $\frac{n(n-1)}{3}, n \equiv 1, 4$ $\frac{n(n-1)}{3} - 4, n \equiv 7, 10$ $\frac{n(n-2)}{3}, n \equiv 0, 2, 3, 6, 8, 9$ $\frac{n(n-2)}{3} - 1, n \equiv 5, 11$ $n \geq n_0$ (residues mod 12) Theorem 4	BD $\frac{n(n-1)}{3}, n \equiv 1, 4$ $\frac{n(n-1)}{3} - 4, n \equiv 7, 10$ $\frac{n(n-2)}{3}, n \equiv 0, 2, 3, 8$ $\frac{n(n-2)}{3} - 1, n \equiv 5, 6, 9, 11$ $n \geq n_0$ (residues mod 12) Theorem 4	BCD $\frac{1}{4}n^2(1 - o(1))$ Theorem 22	
CD $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ and char. ext. Theorem 8			

Table 1: A summary of results. For each combination of excluded configurations, we indicate the value of the extremum (the largest size of a triple system on n vertices obeying those exclusions) to the extent that it is known. We also point out the existence of a characterization of the extremal constructions when available. For earlier results we cite the papers, for new ones we refer to the relevant theorems in the present paper.

The optimality of this construction among all triangle-free triple systems was proved by Győri [17]. Here we reproduce the proof and extend the analysis to obtain also a characterization of the extremal constructions.

Theorem 1.

$$\text{ex}(n, \mathbf{ABCD}) = \lfloor \frac{1}{8}n^2 \rfloor$$

and equality is attained by a triple system $\mathcal{F}$ if and only if it is of the form $\mathcal{F}_{n,k}$ with k chosen optimally as indicated above.

Before turning to the proof of Theorem 1, we state and prove an observation that will be used both here and subsequently.

Observation 2. *The triple system $\mathcal{F}$ avoids **C** and **D** if and only if each triple $F \in \mathcal{F}$ contains at least two own pairs.*

Proof of Observation 2. Recall that xy is an own pair of $F \in \mathcal{F}$ if F is the only triple in $\mathcal{F}$ that contains both x and y .

If $F \in \mathcal{F}$ has fewer than two own pairs, then at least two of its pairs are shared with other triples in $\mathcal{F}$. Say $F = xyz$, the pair xz is shared with $F' \in \mathcal{F} \setminus \{F\}$ and the pair yz is shared with $F'' \in \mathcal{F} \setminus \{F\}$. Then $F' \neq F''$ (otherwise $F' = F'' = F$) and F, F', F'' form a triangle with F supplying the pair xy . If $F' \setminus \{x, z\} = F'' \setminus \{y, z\}$ this triangle is a **C**, otherwise it is a **D**. This proves one direction of the observation. The other direction follows from the fact that both **C** and **D** have a triple with at most one own pair. $\square$

Proof of Theorem 1. Let $\mathcal{F}$ be a triple system not containing any of **A**, **B**, **C**, **D**. By Observation 2, each triple in $\mathcal{F}$ has at least two own pairs. Consider a graph G on $V(\mathcal{F})$ whose edge set contains exactly two own pairs of each $F \in \mathcal{F}$ (if F has three own pairs, we arbitrarily choose two of them to be edges of G). Thus $|E(G)| = 2|\mathcal{F}|$. We claim that G is triangle-free. Indeed, if a triangle were formed by own pairs of three different triples, then we would have a triangle in $\mathcal{F}$ in the hypergraph sense, that is, one of our forbidden configurations would appear. Thus at least two edges of a triangle in G should be own pairs of the same triple F . However, in that case the third edge of the triangle is also contained in F , thus it cannot be the own pair of another triple F' . But it also cannot be an edge of G as a third own pair of F since every triple in $\mathcal{F}$ is represented by exactly two pairs in G . Therefore a triangle in G cannot occur. This implies by Turán's theorem that $|E(G)| \leq \frac{1}{4}n^2$. Since $|E(G)| = 2|\mathcal{F}|$, we got $|\mathcal{F}| \leq \lfloor \frac{1}{8}n^2 \rfloor$.

It remains to show that if $|\mathcal{F}| = \lfloor \frac{1}{8}n^2 \rfloor$ then $\mathcal{F}$ must be of the form $\mathcal{F}_{n,k}$. Note that $|E(G)| = 2\lfloor \frac{1}{8}n^2 \rfloor$, and this number equals $\lfloor \frac{1}{4}n^2 \rfloor$ unless $n \equiv 2 \pmod{4}$, in which case it equals $\lfloor \frac{1}{4}n^2 \rfloor - 1$. By known characterizations (see [7]) of the extremal and next-to-extremal cases for Turán's theorem, G must be bipartite with parts A and B satisfying one of the following:

- (a) $n \not\equiv 2 \pmod{4}$, $|A|$ and $|B|$ differ by at most 1 and G is complete bipartite.
- (b) $n \equiv 2 \pmod{4}$, $|A| = |B|$ and G is complete bipartite minus one edge.

(c) $n \equiv 2 \pmod{4}$, $|A|$ and $|B|$ differ by 2 and G is complete bipartite.

Each triple in $\mathcal{F}$ contributes two own pairs to G , hence contains two vertices in A and one in B or vice versa. We claim that all triples in $\mathcal{F}$ must be of the same kind, i.e., the same part contains two vertices from each triple. Assume, for the sake of contradiction, that there are triples $F_A, F_B \in \mathcal{F}$ such that $F_A \cap A = \{x, x'\}$ and $F_B \cap B = \{y, y'\}$. Out of the four pairs in $\{x, x'\} \times \{y, y'\}$, at most one is a non-edge of G . Hence without loss of generality $x'y, yx, xy'$ form a path in G . Say these edges of G represent the triples F_1, F_2, F_3 in $\mathcal{F}$, respectively. Clearly, F_1, F_2 and F_3 cannot all be the same triple, so without loss of generality $F_1 \neq F_2$, and neither F_1 nor F_2 contains $\{x, x'\}$. But then F_A, F_1, F_2 are three distinct triples in $\mathcal{F}$ which form a forbidden triangle.

Thus, we assume without loss of generality that every triple in $\mathcal{F}$ contains two vertices in A and one in B . For each vertex $b \in B$, its link in $\mathcal{F}$ consists of disjoint pairs in A (otherwise, we would have an edge of G that belongs to two triples in $\mathcal{F}$). This gives the upper bound

$$|\mathcal{F}| \leq \lfloor \frac{|A|}{2} \rfloor \cdot |B|.$$

If $|A|$ is odd, this bound is strictly smaller than $\lfloor \frac{1}{8}n^2 \rfloor$. Thus $|A|$ must be even, ruling out case (b) above. Moreover, the bound must hold as an equality, so the link of each $b \in B$ is a partition of A into k pairs $A_1, \dots, A_k$, where $k = \frac{|A|}{2}$. Furthermore, this partition must be the same for all $b \in B$, otherwise there would be a forbidden triangle. This shows that $\mathcal{F}$ is of the form $\mathcal{F}_{n,k}$, and the conditions on the difference between $|A|$ and $|B|$ in cases (a), (c) above imply the optimal choice of k . $\square$

3 Earlier results

The four cases where we exclude just one type of triangle creating configurations **A**, **B**, **C**, or **D**, were already investigated in some different context. In particular, it was proven in [12] (as Theorem 3.3) that

$$\text{ex}(n, \mathbf{A}) = \binom{n-1}{2} \text{ for } n \geq n_0.$$

This result also determined that all extremal constructions are *full stars*, 3-uniform hypergraphs containing all edges through a fixed vertex. Observing that this construction also does not contain configuration **B**, we have that the same holds for **AB**-free hypergraphs, too. Here (in Section 4) we prove that $n_0 = 5$ for $\text{ex}(n, \mathbf{AB})$.

An early attempt to generalize Turán's theorem to hypergraphs (for triple systems) was proposed by Erdős and Katona (see [19]), who conjectured that the unique largest triple system on n vertices such that no three distinct triples A, B, C satisfy $A \Delta B \subset C$ is the complete 3-partite hypergraph with part sizes as equal as possible. Here Δ is the symmetric difference, $A \Delta B = (A \setminus B) \cup (B \setminus A)$. One can see that (for $A \neq B$) $A \Delta B \subset C$ can only occur when A and B have two common vertices and the three triples form either

a configuration **B** or **C**. The conjecture was verified by Bollobás [3], thus showing

$$\text{ex}(n, \mathbf{BC}) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor \quad \text{for all } n,$$

with the (nearly) balanced complete 3-partite hypergraph as the only extremal family. Frankl and Füredi (in [10] as Theorem 4) sharpened Bollobás's theorem proving that for $n \geq 3000$ it is sufficient to exclude only **B**. Subsequently, a new proof was given by Keevash and Mubayi [20] who also significantly lowered the constraint for n :

$$\text{ex}(n, \mathbf{B}) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor \quad \text{for } n \geq 33,$$

with equality only for the (nearly) balanced complete 3-partite hypergraph. They also proved that the extremal family is rather stable, namely for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{H}$ is a **B**-free triple system on n vertices with at least $(1 - \delta)\frac{1}{27}n^3$ edges, then there exists a partition of the vertex set of $\mathcal{H}$ as $V(\mathcal{H}) = V_1 \cup V_2 \cup V_3$ so that all but at most εn^3 edges of $\mathcal{H}$ have one vertex in each V_i . Incidentally, the same (nearly) balanced complete 3-partite hypergraph was shown by Gyárfás [16] to be extremal among K_4 -free triple systems for $n \geq 6$.

The value of $\text{ex}(n, \mathbf{C})$ is not known, it is a famous problem proposed by Erdős and Sós [8] that is thought to be difficult. Frankl and Füredi [11] gave a recursive construction based on the triple system $\mathcal{F}$ on 6 vertices with $\deg_{\mathcal{F}}(xy) = 2$ for all $x \neq y$, yielding the best known lower bound $\text{ex}(n, \mathbf{C}) \geq \frac{2}{7} \binom{n}{3} (1 - o(1))$. An upper bound with $\frac{1}{3}$ instead of $\frac{2}{7}$ was proved by de Caen [5] and Sidorenko [23]. A sequence of small improvements of the coefficient $\frac{1}{3}$ followed. The current record, obtained by Falgas-Ravry and Vaughan [9] using semi-definite programming, is slightly less than 0.287 (note that $\frac{2}{7} \approx 0.286$). Thus

$$\frac{2}{7} \binom{n}{3} (1 - o(1)) \leq \text{ex}(n, \mathbf{C}) \leq 0.287 \binom{n}{3} (1 + o(1)).$$

Configuration **D** is the most restrictive in terms of the largest size of a triple system excluding it: the upper bound $\text{ex}(n, \mathbf{D}) \leq \frac{n(n-1)}{3}$ follows from Theorem 3.8 of Frankl and Füredi [12]. Determining the exact value of $\text{ex}(n, \mathbf{D})$ takes more work, which we carry out below in Section 5.

4 Excluding **A** and **B**

As mentioned in the previous section, it follows from [12] that $\text{ex}(n, \mathbf{AB}) = \binom{n-1}{2}$ for $n \geq n_0$. Here we strengthen this statement, replacing **A** and **B** by the augmented configurations **A**⁺ and **B**⁺ defined below, and also specifying $n_0 = 5$.

Define configuration **A**⁺ as a six vertex triple system of four edges with $V = \{1, \dots, 6\}$, $E = \{123, 124, 135, 236\}$. This is the same as **A** with the middle triangle 123 as an additional triple. Similarly, configuration **B**⁺ on five vertices is the same as **B** with one more central triple 123, i.e., $V = \{1, \dots, 5\}$, $E = \{123, 125, 134, 234\}$. We obviously

have $\text{ex}(n, \mathbf{AB}) \leq \text{ex}(n, \mathbf{A}^+\mathbf{B}^+)$. Since the full star contains neither configuration $\mathbf{A}$ nor configuration $\mathbf{B}$, the following theorem implies the similar statement for $\text{ex}(n, \mathbf{AB})$ in place of $\text{ex}(n, \mathbf{A}^+\mathbf{B}^+)$.

Theorem 3.

$$\text{ex}(n, \mathbf{A}^+\mathbf{B}^+) = \binom{n-1}{2} \text{ for } n \geq 5$$

and the only extremal family is the full star.

Proof. Clearly, $\text{ex}(n, \mathbf{A}^+\mathbf{B}^+) \geq \binom{n-1}{2}$ as the full star is $\mathbf{A}^+\mathbf{B}^+$ -free. Now let $\mathcal{F} \subseteq \binom{[n]}{3}$ be an $\mathbf{A}^+\mathbf{B}^+$ -free family. Define $\mathcal{F}_1 \subseteq \mathcal{F}$ as the family of triples with an own pair, $\mathcal{F}_1 := \{xyz \in \mathcal{F} : \min\{\deg_{\mathcal{F}}(xy), \deg_{\mathcal{F}}(yz), \deg_{\mathcal{F}}(zx)\} = 1\}$, and let $\mathcal{F}_2 := \mathcal{F} \setminus \mathcal{F}_1$. Our main observation is that for every $F = xyz \in \mathcal{F}_2$ we have $\deg_{\mathcal{F}}(xy) = \deg_{\mathcal{F}}(yz) = \deg_{\mathcal{F}}(zx) = 2$. Indeed, by definition, all these three pairs have degrees at least 2, so one can find triples $xya, yzb, zxc \in \mathcal{F}$ such that $\{x, y, z\} \cap \{a, b, c\} = \emptyset$. These three triples together with xyz form a configuration $\mathbf{A}^+$ or $\mathbf{B}^+$ unless $a = b = c$, in which case they form a complete $\binom{[4]}{3}$. We obtain that a is unique so $\deg_{\mathcal{F}}(xy) = 2$, and similarly for b and c .

Define a nonnegative function $w(e, F) : \binom{[n]}{2} \times \binom{[n]}{3} \rightarrow \mathbb{R}$ as follows. It is always 0 except when $e \subset F$, $F \in \mathcal{F}$, when it is $\frac{1}{\deg_{\mathcal{F}}(e)}$. Note that here $\frac{1}{\deg_{\mathcal{F}}(e)} \geq \frac{1}{n-2}$. For every pair $e \in \partial\mathcal{F}$ we have

$$w(e) := \sum_{F \in \mathcal{F}} w(e, F) = \sum_{F: e \subset F \in \mathcal{F}} \frac{1}{\deg_{\mathcal{F}}(e)} = 1.$$

For every $F \in \mathcal{F}_1$, $F = xyz$ we have

$$w(F) := \sum_{e \subset F, |e|=2} w(e, F) = \frac{1}{\deg_{\mathcal{F}}(xy)} + \frac{1}{\deg_{\mathcal{F}}(yz)} + \frac{1}{\deg_{\mathcal{F}}(zx)} \geq 1 + \frac{2}{n-2} = \frac{n}{n-2}.$$

Assume first that $\mathcal{F} = \mathcal{F}_1$. Then

$$\binom{n}{2} \geq |\partial\mathcal{F}| = \sum_{e \in \partial\mathcal{F}} w(e) = \sum_e \sum_{F \in \mathcal{F}} w(e, F) = \sum_{F \in \mathcal{F}} w(F) \geq \frac{n}{n-2} |\mathcal{F}|. \quad (1)$$

This yields the required inequality $|\mathcal{F}| \leq \binom{n-1}{2}$. Suppose equality holds. Then each $F \in \mathcal{F}$ has a unique own pair. In this case, enumerating the triples in $\mathcal{F}$ as $F_1, \dots, F_m$ with $m = \binom{n-1}{2}$, and denoting their own pairs as $e_1, \dots, e_m$ respectively, we consider the set-pair system $([n] \setminus F_i, e_i)_{i=1, \dots, m}$. This system satisfies Bollobás's condition: $([n] \setminus F_i) \cap e_j = \emptyset$ if and only if $i = j$. Bollobás [2] proved that such a system satisfies $m \leq \binom{|[n] \setminus F_i| + |e_i|}{|e_i|} = \binom{n-1}{2}$ and equality holds only if the pairs are all possible partitions of a fixed set of size $n-1$ into sets of sizes $n-3$ and 2 , respectively. We conclude that equality can hold only if $\mathcal{F}$ is a full star.

Next, assume that $\mathcal{F}_2 \neq \emptyset$. For each $F \in \mathcal{F}_2$ we have $w(F) = \frac{3}{2}$. If $n \geq 7$ then $\frac{3}{2} > \frac{n}{n-2}$, so (1) holds as a strict inequality. It remains to check that for $n = 5, 6$ we

also have $|\mathcal{F}| < \binom{n-1}{2}$. Because $\mathcal{F}_2 \neq \emptyset$, the family $\mathcal{F}$ must contain a complete $\binom{[4]}{3}$ as explained above. To avoid a $\mathbf{B}^+$, there can be no triple $F \in \mathcal{F}$ with $|F \cap [4]| = 2$. When $n = 5$ this gives $|\mathcal{F}| = \binom{4}{3} < \binom{4}{2}$. When $n = 6$, triples not in $\binom{[4]}{3}$ must contain both 5 and 6, hence $|\mathcal{F}| \leq \binom{4}{3} + \binom{4}{1} < \binom{5}{2}$. $\square$

5 Excluding D (or B and D)

As mentioned, the upper bound $\text{ex}(n, \mathbf{D}) \leq \frac{n(n-1)}{3}$ follows from [12]. Here we conduct a careful analysis showing that this bound is either tight or nearly tight, depending on the residue of n modulo 12. The same analysis also determines $\text{ex}(n, \mathbf{BD})$. The following theorem summarizes our findings.

Theorem 4. *There exists n_0 such that for all $n \geq n_0$ we have:*

$$\text{ex}(n, \mathbf{D}) = \text{ex}(n, \mathbf{BD}) = \begin{cases} \frac{n(n-1)}{3} & \text{if } n \equiv 1, 4 \pmod{12} \\ \frac{n(n-1)}{3} - 4 & \text{if } n \equiv 7, 10 \pmod{12} \\ \frac{n(n-2)}{3} & \text{if } n \equiv 0, 2, 3, 8 \pmod{12} \\ \frac{n(n-2)}{3} - 1 & \text{if } n \equiv 5, 11 \pmod{12} \end{cases}$$

$$\text{ex}(n, \mathbf{D}) = \frac{n(n-2)}{3} \text{ and } \text{ex}(n, \mathbf{BD}) = \frac{n(n-2)}{3} - 1 \text{ if } n \equiv 6, 9 \pmod{12}.$$

Without the requirement $n \geq n_0$, the indicated quantities are always valid as upper bounds, and as exact values for $n \equiv 0, 1, 3, 4 \pmod{12}$.

5.1 Preparations for the proof

In preparation for the proof, we develop or recall some tools that will be used in it. Let $\mathcal{F}$ be a triple system on a vertex set V , $|V| = n$. We call two triples in $\mathcal{F}$ *contiguous* if they share two vertices. The contiguity relation defines a graph on $\mathcal{F}$, which allows us to partition $\mathcal{F}$ uniquely into its connected components with respect to contiguity:

$$\mathcal{F} = \bigcup_{i=1}^r \mathcal{F}_i$$

Let us denote by $\partial\mathcal{F}_i$ the shadow of $\mathcal{F}_i$, i.e., the set of those pairs contained in some triple in $\mathcal{F}_i$. Viewing the $\partial\mathcal{F}_i$ as graphs, they are edge-disjoint.

Two special types of $\mathcal{F}_i$ serve as building blocks of $\mathbf{D}$ -free triple systems. One, that we call a *cluster*, consists of 3 or 4 triples on 4 vertices; its shadow is a K_4 graph. The other, that we call a *k-crown*, consists of some number $k \geq 1$ of triples, all sharing a fixed pair of vertices x and y ; its shadow is a book graph B_k on $k+2$ vertices, consisting of k triangles sharing an edge. In the case $k = 1$ we get a single triple (triangle). The following simple fact is recorded here for later reference.

Observation 5. *A triple system $\mathcal{F}$ is $\mathbf{D}$ -free if and only if each of its connected components $\mathcal{F}_i$ is either a cluster or a k -crown for some $k \geq 1$.*

Proof. Suppose $\mathcal{F}$ is $\mathbf{D}$ -free, and let $\mathcal{F}_i$ be one of its connected components. If $|\mathcal{F}_i| = 1$ then $\mathcal{F}_i$ is a 1-crown. If $|\mathcal{F}_i| \geq 2$, then let F, F' be two contiguous triples in $\mathcal{F}_i$ which share the pair xy . If $\mathcal{F}_i = \{F, F'\}$ then it is a 2-crown. Otherwise, there are two ways to add triples to $\{F, F'\}$ that are contiguous to an existing triple without creating a $\mathbf{D}$ configuration. Either each additional triple shares xy , in which case we get a k -crown for some k , or each additional triple is contained in $F \cup F'$ and we obtain a cluster. Conversely, if every $\mathcal{F}_i$ is of one of these two forms, then no $\mathbf{D}$ configuration arises. $\square$

Thus, constructing a $\mathbf{D}$ -free triple system $\mathcal{F}$ on V amounts to packing edge-disjoint copies of K_4 and some B_k 's in the complete graph K_n . As we are interested in the largest possible $\mathcal{F}$, we may assume that each copy of K_4 contributes 4 triples to $\mathcal{F}$. The ratio $\frac{|\mathcal{F}_i|}{|E(\partial\mathcal{F}_i)|}$ is $\frac{2}{3}$ in each cluster and less than $\frac{1}{2}$ in each crown. Hence we would like to pack in K_n as many copies of K_4 as we can, and use the leftover edges, if any, to add copies of some B_k 's if possible. The maximum number of copies of a fixed graph H that can be packed in K_n was determined for large enough n by Caro and Yuster [6]. Their main tool, which will also be ours, was the following decomposition theorem of Gustavsson, which extends Wilson's theorem (see Theorem 23 below) from complete graphs to sufficiently dense ones.

Theorem 6 (Gustavsson [15], Glock et al. [14]). *For every fixed graph H there exist $n_0 = n_0(H)$ and $\varepsilon = \varepsilon(H) > 0$ such that the following holds for all $n \geq n_0$ and every graph G on n vertices with all vertex degrees greater than $(1 - \varepsilon)n$: There exists a decomposition of the edge set $E(G)$ into copies of H if and only if the number of edges of H divides that of G , and the greatest common divisor of the vertex degrees in H divides that of the vertex degrees in G .*

We will apply this theorem to $H = K_4$. In this case, the divisibility conditions become: $6|E(G)|$ and $3|\deg_G(v)$ for all $v \in V$. These conditions are satisfied by $G = K_n$ if and only if $n \equiv 1, 4 \pmod{12}$. For all (not just large enough) $n \equiv 1, 4 \pmod{12}$, a full packing of copies of K_4 in K_n is known to exist (Hanani [18]). When such a full packing does not exist, we will use a decomposition into K_4 's of a large subgraph G of K_n , specified by describing its complement $\overline{G}$. In terms of $\overline{G}$ the divisibility conditions become:

$$|E(\overline{G})| \equiv \binom{n}{2} \pmod{6} \text{ and } \deg_{\overline{G}}(v) \equiv n - 1 \pmod{3} \text{ for all } v \in V.$$

The graphs $\overline{G}$ that we will specify have constant maximum degree, which ensures that all vertex degrees in G will be greater than $(1 - \varepsilon)n$ for large enough n . The K_4 -decomposition of G provided by Theorem 6, plus some edge-disjoint B_k 's in $\overline{G}$ when available, will naturally define a large $\mathbf{D}$ -free triple system $\mathcal{F}$. In most cases this $\mathcal{F}$ will also be $\mathbf{B}$ -free; when this is not the case we will modify $\overline{G}$ to get a slightly smaller triple system which is $\mathbf{BD}$ -free.

Another useful way to analyze triple systems $\mathcal{F}$ is by describing their links. Recall that the link $\mathcal{F}[x]$ of a vertex x is the graph on $V \setminus \{x\}$ whose edges are those pairs of vertices that form with x a triple in $\mathcal{F}$. Here is another simple characterization of **D**-free triple systems.

Observation 7. *A triple system $\mathcal{F}$ is **D**-free if and only if each of its links $\mathcal{F}[x]$ is a vertex-disjoint union of triangles and stars (including the cases of an isolated vertex or edge). For such $\mathcal{F}$, denoting by $\text{st}(\mathcal{F}[x])$ the number of stars in $\mathcal{F}[x]$, we have*

$$|\mathcal{F}| = \frac{1}{3} \sum_{x \in V} (n - 1 - \text{st}(\mathcal{F}[x])).$$

Proof. A vertex x is the common vertex of three triples forming a **D** configuration if and only if $\mathcal{F}[x]$ contains a path of three edges. Thus, $\mathcal{F}$ is **D**-free exactly when no $\mathcal{F}[x]$ contains such a path. Note that any non-star tree and any cycle of length 4 or more contains a path of three edges, and so does a triangle with an adjacent edge. It follows that $\mathcal{F}$ is **D**-free if and only if every connected component of every link $\mathcal{F}[x]$ is a triangle or a star. To check the formula for $|\mathcal{F}|$, observe that $3|\mathcal{F}|$ equals the total, over $x \in V$, of the number of edges in $\mathcal{F}[x]$. For a fixed x , since $\mathcal{F}[x]$ is a vertex-disjoint union of triangles and stars, the number of edges is $n - 1$ minus the number of stars. $\square$

Noting that when $n \not\equiv 1 \pmod{3}$, i.e., $n - 1$ is not divisible by 3, there must be at least one star in each $\mathcal{F}[x]$, we immediately get the following upper bound:

$$\text{ex}(n, \mathbf{D}) \leq \begin{cases} \frac{n(n-1)}{3} & \text{if } n \equiv 1 \pmod{3} \\ \frac{n(n-2)}{3} & \text{if } n \not\equiv 1 \pmod{3} \end{cases}$$

In our detailed analysis, depending on the residue of n modulo 12, we will construct **D** (and even **BD**)-free triple systems that match this upper bound exactly or up to a small constant difference, and prove their optimality.

5.2 Proof of Theorem 4

Case 1 $n \equiv 1, 4 \pmod{12}$

We saw that $\frac{n(n-1)}{3}$ is an upper bound on the size of a **D**-free triple system. A matching construction is obtained by placing 4 triples in each copy of K_4 in a full packing in K_n , which exists by [18]. This gives a triple system $\mathcal{F}$ of size $|\mathcal{F}| = 4 \cdot \frac{1}{6} \binom{n}{2} = \frac{n(n-1)}{3}$, which is **B**-free as well.

Case 2 $n \equiv 0, 3 \pmod{12}$

We saw that $\frac{n(n-2)}{3}$ is an upper bound on the size of a **D**-free triple system in this case. For a matching construction, we use a packing of $\frac{n(n-3)}{12}$ copies of K_4 and $\frac{n}{3}$ triangles in K_n . Such a packing can be obtained by starting with a full packing of copies of K_4 in

K_{n+1} , which exists by [18], and deleting one vertex. Placing 4 triples in each copy of K_4 and one in each triangle gives a **BD**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{n(n-3)}{3} + \frac{n}{3} = \frac{n(n-2)}{3}$.

Case 3 $n \equiv 2, 8 \pmod{12}$

Again, $\frac{n(n-2)}{3}$ is an upper bound. For a matching construction for large enough n , we use Theorem 6 as follows. We take $\overline{G}$ to be a perfect matching in K_n , noting that it satisfies the required conditions: $|E(\overline{G})| = \frac{n}{2} \equiv \binom{n}{2} \pmod{6}$ and $\deg_{\overline{G}}(v) = 1 \equiv n-1 \pmod{3}$. Placing 4 triples in each copy of K_4 in a K_4 -decomposition of G we get a **BD**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{2}{3} \left(\binom{n}{2} - \frac{n}{2} \right) = \frac{n(n-2)}{3}$.

Case 4 $n \equiv 5, 11 \pmod{12}$

We start with the construction for large enough n , using Theorem 6. We take $\overline{G}$ to consist of a matching of $\frac{n-5}{2}$ edges and a star on the remaining 5 vertices. The required conditions are satisfied: $|E(\overline{G})| = \frac{n-5}{2} + 4 \equiv \binom{n}{2} \pmod{6}$ and $\deg_{\overline{G}}(v) \in \{1, 4\} \equiv n-1 \pmod{3}$ for all $v \in V$. Placing 4 triples in each copy of K_4 in a decomposition of G , we get a **BD**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{2}{3} \left(\binom{n}{2} - \frac{n+3}{2} \right) = \frac{n(n-2)}{3} - 1$.

To see that this is optimal, assume that there is a **D**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{n(n-2)}{3}$. By Observation 7, we must have $\text{st}(\mathcal{F}[x]) = 1$ for every vertex x . Consider the mapping defined on V by $x \mapsto y$, where $y \neq x$ is the center of the star in the link $\mathcal{F}[x]$. This is well defined, because there is a unique star and it has a unique center (the star cannot be a single edge because it has $1 \pmod{3}$ vertices). Note also that $x \mapsto y$ implies $y \mapsto x$, so this mapping partitions V into pairs, which is impossible because n is odd.

Case 5a $n \equiv 6, 9 \pmod{12}$, excluding **D**

We saw that $\frac{n(n-2)}{3}$ is an upper bound in this case. For a matching construction for large enough n , we use Theorem 6 as follows. We take $\overline{G}$ to consist of $\frac{n-6}{3}$ vertex-disjoint triangles and a B_4 on the remaining vertices. The required conditions are satisfied: $|E(\overline{G})| = (n-6) + 9 \equiv \binom{n}{2} \pmod{6}$ and $\deg_{\overline{G}}(v) \in \{2, 5\} \equiv n-1 \pmod{3}$ for all $v \in V$. Placing 4 triples in each copy of K_4 in a decomposition of G , one in each triangle in $\overline{G}$, and 4 in the book B_4 , we get a **D**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{2}{3} \left(\binom{n}{2} - n - 3 \right) + \frac{n-6}{3} + 4 = \frac{n(n-2)}{3}$.

Case 5b $n \equiv 6, 9 \pmod{12}$, excluding **BD**

Note that the construction in Case 5a is not **B**-free, because two triples xyz, xyw from the crown on B_4 form a **B** configuration with a triple containing $\{z, w\}$. We modify the construction, letting $\overline{G}$ consist of $\frac{n-9}{3}$ vertex-disjoint triangles and 4 more triangles on the remaining vertices, all sharing one vertex. The modification does not change the number of edges in $\overline{G}$, and the only new degree is 8, hence the required conditions are satisfied. Placing 4 triples in each copy of K_4 in a decomposition of G and one in each triangle in $\overline{G}$, we get a **BD**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{2}{3} \left(\binom{n}{2} - n - 3 \right) + \frac{n-9}{3} + 4 = \frac{n(n-2)}{3} - 1$.

To see that this is optimal, assume that there is a **BD**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{n(n-2)}{3}$. Since the upper bound $|\mathcal{F}| \leq \frac{n(n-2)}{3}$ for $n \not\equiv 1 \pmod{3}$ was obtained from Observation 7 by the fact that $\text{st}(\mathcal{F}[x]) \geq 1$ for every x , equality implies that $\text{st}(\mathcal{F}[x]) = 1$

in each link $\mathcal{F}[x]$. We claim that none of the $\mathcal{F}_i$ (the connected components of $\mathcal{F}$ with respect to contiguity, as defined at the beginning of Subsection 5.1) can be a k -crown with $k \geq 2$. Suppose that $\mathcal{F}_i$ is such a k -crown, consisting of the triples xyz_i , $i = 1, \dots, k$. Then the link $\mathcal{F}[z_1]$ contains xy as an isolated edge, and also $z_2, \dots, z_k$ as isolated vertices (the latter is due to **B**-freeness). This contradicts the uniqueness of the star in each link. By Observation 5, it follows that each $\mathcal{F}_i$ is either a cluster or a 1-crown (i.e., a triple contiguous to no other triple). This implies that each link is a vertex-disjoint union of $\frac{n-3}{3}$ triangles and an isolated edge. But this is impossible: if every vertex appears in $\frac{n-3}{3}$ clusters, then the number of clusters is $\frac{1}{4}n \cdot \frac{n-3}{3}$, which is not an integer.

Case 6 $n \equiv 7, 10 \pmod{12}$

We start with the construction for large enough n , using Theorem 6. We take $\overline{G}$ to be the triangular prism graph (two vertex-disjoint triangles with a perfect matching of their vertices) plus $n - 6$ isolated vertices. The required conditions are satisfied: $|E(\overline{G})| = 9 \equiv \binom{n}{2} \pmod{6}$ and $\deg_{\overline{G}}(v) \in \{0, 3\} \equiv n - 1 \pmod{3}$ for all $v \in V$. Placing 4 triples in each copy of K_4 in a decomposition of G and one in each of the two triangles in $\overline{G}$, we get a **BD**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{2}{3}(\binom{n}{2} - 9) + 2 = \frac{n(n-1)}{3} - 4$.

To see that this is optimal, assume that there is a **D**-free triple system $\mathcal{F}$ of size $|\mathcal{F}| = \frac{n(n-1)}{3} - 3$. Consider the decomposition of $\mathcal{F}$ into its connected components $\mathcal{F}_i$ with respect to contiguity. Let the graph G be the edge-disjoint union of the K_4 's that are the shadows of the cluster $\mathcal{F}_i$'s. Let $\overline{G}$ be the complement of G . The divisibility conditions are:

$$|E(\overline{G})| \equiv 3 \pmod{6} \text{ and } \deg_{\overline{G}}(v) \equiv 0 \pmod{3} \text{ for all } v \in V.$$

The shadows of the crown $\mathcal{F}_i$'s are edge-disjoint subgraphs of $\overline{G}$, and the number of triples that each of them carries is less than half its number of edges. Thus we have

$$\frac{n(n-1)}{3} - 3 = |\mathcal{F}| \leq \frac{2}{3}|E(G)| + \frac{1}{2}|E(\overline{G})| = \frac{2}{3}\binom{n}{2} - \frac{1}{6}|E(\overline{G})|$$

from which we deduce that $|E(\overline{G})| \leq 18$. Together with the fact that $|E(\overline{G})| \equiv 3 \pmod{6}$, this leaves only three possibilities for $|E(\overline{G})|$, that we can rule out one-by-one:

- $|E(\overline{G})| = 3$
By $\deg_{\overline{G}}(v) \equiv 0 \pmod{3}$, all non-zero degrees must be at least 3, which is impossible with 3 edges.
- $|E(\overline{G})| = 9$
All non-zero degrees must be exactly 3, because there are not enough edges to allow a degree of 6 or more. Ignoring isolated vertices, $\overline{G}$ is a 3-regular graph on 6 vertices. There are only two such graphs: $K_{3,3}$ and the above-mentioned triangular prism. Thus $\overline{G}$ contains at most two triangles. Hence the total number of triples in $\mathcal{F}$ is at most the number in our construction, namely $\frac{n(n-1)}{3} - 4$.
- $|E(\overline{G})| = 15$
Let t be the number of triples in $\mathcal{F}$ coming from $\overline{G}$. From $\frac{n(n-1)}{3} - 3 = |\mathcal{F}| \leq$

$\frac{2}{3}(\binom{n}{2} - 15) + t$ we get that $t \geq 7$. The only way to obtain 7 triples from crowns whose edge-disjoint shadows have at most 15 edges in total is to use one 7-crown. But then $\overline{G}$ is a book graph B_7 (plus isolated vertices), having vertex degrees 2 and 8 which are not $0 \pmod{3}$.

This completes the proof of Theorem 4. $\square$

We note that after the arXiv version of this paper appeared, Yu, Yu and Ji [25] have shown that the values of $\text{ex}(n, \mathbf{D})$ and $\text{ex}(n, \mathbf{BD})$ given in Theorem 4 are valid for all $n \geq 92$. Instead of relying on Gustavsson's theorem, their constructions use explicit designs.

6 Excluding C and D (and A, optionally)

First, note that Observation 2 from Section 2 immediately gives $\text{ex}(n, \mathbf{CD}) \leq \frac{1}{2}\binom{n}{2} = \frac{1}{4}n^2(1-o(1))$ and a construction of matching order of magnitude is not hard to find. In this case, however, we will give the exact value of $\text{ex}(n, \mathbf{CD})$ together with a characterization of all the extremal constructions. As it turns out it will be just one little step further to give a similar characterization of all extremal $\mathbf{ACD}$ -free constructions.

6.1 A construction based on a tournament

First we describe a construction we will call the *tournament construction*. Consider a maximum size matching $\{e_1, \dots, e_m\}$ in the graph K_n . Fix a tournament T on a vertex set consisting of the edges of this matching. (Thus our tournament is on $\lfloor \frac{n}{2} \rfloor$ vertices.) For $1 \leq i \neq j \leq m$, write $e_i \rightarrow_T e_j$ to denote the existence of an arc of T from e_i to e_j . We construct a triple system $\mathcal{F}_T$ on $[n]$ as follows. Whenever $e_i \rightarrow_T e_j$ and $e_j = \{x, y\}$, we place the triples $e_i \cup \{x\}$ and $e_i \cup \{y\}$ in $\mathcal{F}_T$. In case n is odd, we also place in $\mathcal{F}_T$ all triples of the form $e_i \cup \{z\}$, where $1 \leq i \leq m$ and z is the vertex left out of the matching $\{e_1, \dots, e_m\}$. The number of triples in $\mathcal{F}_T$ is $2\binom{\frac{n}{2}}{2}$ if n is even and $2\binom{\frac{n-1}{2}}{2} + \frac{n-1}{2}$ if n is odd. (Both of the latter can be written as $\lfloor \frac{1}{4}(n-1)^2 \rfloor$.) It is easy to check that each triple in $\mathcal{F}_T$ (for any possible T) contains two own pairs, thus $\mathbf{C}$ and $\mathbf{D}$ do not appear in $\mathcal{F}_T$ by Observation 2. This gives $\text{ex}(n, \mathbf{CD}) \geq \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

Theorem 8.

$$\text{ex}(n, \mathbf{CD}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor$$

and equality is attained by a triple system $\mathcal{F}$ if and only if it is of the form $\mathcal{F}_T$ for some tournament T .

Before the proof we obtain the following immediate consequence.

Corollary 9.

$$\text{ex}(n, \mathbf{ACD}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor$$

and equality is attained by a triple system $\mathcal{F}$ if and only if it is of the form $\mathcal{F}_T$ for the transitive tournament T .

Proof of Corollary 9. Using Theorem 8 we get $\text{ex}(n, \mathbf{ACD}) \leq \text{ex}(n, \mathbf{CD}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor$. To prove equality and the “if” part of the characterization, we show that when T is transitive the family $\mathcal{F}_T$ also avoids **A**. Suppose, for the sake of contradiction, that three triples containing pairs e_i, e_j, e_k from the matching form an **A** configuration in $\mathcal{F}_T$. Note that i, j, k must be distinct because two triples in **A** intersect in exactly one vertex. The six vertices of the configuration must be those of $e_i \cup e_j \cup e_k$. The transitivity of T implies that it is not cyclic on $\{e_i, e_j, e_k\}$, so without loss of generality $e_i \not\rightarrow_T e_j$ and $e_i \not\rightarrow_T e_k$. But then e_i does not form a triple in $\mathcal{F}_T$ with any vertex of $e_j \cup e_k$, a contradiction.

To prove the “only if” part of the characterization, suppose that $\mathcal{F}$ is **ACD**-free and $|\mathcal{F}| = \lfloor \frac{1}{4}(n-1)^2 \rfloor$. By Theorem 8, $\mathcal{F}$ must be of the form $\mathcal{F}_T$ for some tournament T . Suppose that T is not transitive. Then it contains a cyclically oriented triangle, say $e_i \rightarrow_T e_j \rightarrow_T e_k \rightarrow_T e_i$. Picking $u \in e_i, v \in e_j, w \in e_k$ we see that $e_i \cup \{v\}, e_j \cup \{w\}, e_k \cup \{u\}$ are triples in $\mathcal{F}_T$ that form a forbidden **A** configuration. Hence T must be transitive. $\square$

6.2 Proof of Theorem 8

Let $\mathcal{F}$ be a triple system not containing **C** and **D** and having the maximum number of triples. Then $|\mathcal{F}| \geq |\mathcal{F}_T| = \lfloor \frac{1}{4}(n-1)^2 \rfloor$. By Observation 2, each $F \in \mathcal{F}$ has at least two own pairs. We form a graph G on $V(\mathcal{F})$ by taking as edges exactly two own pairs from each $F \in \mathcal{F}$. Let $\overline{G}$ be the complement of G . Note that every $F \in \mathcal{F}$ has two of its pairs in G and the third in $\overline{G}$.

We claim that the edges of $\overline{G}$ form a matching of size $\lfloor \frac{n}{2} \rfloor$. First, observe that $|E(G)| = 2|\mathcal{F}| \geq 2\lfloor \frac{1}{4}(n-1)^2 \rfloor = \lfloor \frac{1}{2}(n-1)^2 \rfloor$, where the last equality holds because every perfect square is 0 or 1 mod 4. Hence $|E(\overline{G})| \leq \binom{n}{2} - \lfloor \frac{1}{2}(n-1)^2 \rfloor = \lfloor \frac{n}{2} \rfloor$. Now we distinguish two cases. If every vertex $x \in V(\mathcal{F})$ is incident to an edge of $\overline{G}$, then $\overline{G}$ must be a perfect matching (and n is even). In the other case, consider a vertex x all of whose pairs are edges of G . These $n-1$ pairs are own pairs of triples containing x , so n is odd and these triples are of the form $xy_1z_1, \dots, xy_{(n-1)/2}z_{(n-1)/2}$ where $y_1z_1, \dots, y_{(n-1)/2}z_{(n-1)/2}$ form a perfect matching in $V(\mathcal{F}) \setminus \{x\}$ and each of them is an edge of $\overline{G}$ (as the third pair of its triple). In this case, too, we get that $\overline{G}$ is a matching of size $\lfloor \frac{n}{2} \rfloor$. In particular, the inequalities above must hold as equalities, proving the first part of the theorem.

To establish the tournament structure, we take the edges of $\overline{G}$ as the maximum size matching (the vertices of the tournament). We already know that each triple in $\mathcal{F}$ contains an edge of $\overline{G}$. Given any two distinct edges of $\overline{G}$, the four crossing pairs between them must be edges of G , hence own pairs of triples. This prevents the existence of triples going both ways between two edges of $\overline{G}$, and implies the tournament structure for $\mathcal{F}$. $\square$

7 Excluding **A** and **D**

For the combination **AD**, by our previous analysis for **ACD**, we can write

$$\text{ex}(n, \mathbf{AD}) \geq \text{ex}(n, \mathbf{ACD}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor.$$

We recall that $\text{ex}(n, \mathbf{ACD})$ is attained uniquely by the construction $\mathcal{F}_T$ based on a transitive tournament T .

It turns out that the answer for $\mathbf{AD}$ is similar, but a bit more subtle. For any even $n \geq 4$ we can add two triples to $\mathcal{F}_T$ as indicated next, and still avoid $\mathbf{A}$ and $\mathbf{D}$. Say the pairs 12 and 34 are the elements of the tournament T with the lowest and second lowest outdegree, respectively. On these four vertices, $\mathcal{F}_T$ contains the two triples 134, 234, and we can add the other two triples 123, 124 without forming an $\mathbf{A}$ or $\mathbf{D}$. We denote this augmented construction by $\mathcal{F}_T^+$. For odd n we cannot do better than $\frac{1}{4}(n-1)^2$, but we can modify $\mathcal{F}_T$ as indicated next, and get another extremal construction (assuming $n \geq 5$). Say 0, 12, 34 are the singleton and the two lowest outdegree elements of T , respectively. On these five vertices, $\mathcal{F}_T$ contains the four triples 012, 034, 134, 234. We can replace 012 and 034 by 123 and 124 without forming an $\mathbf{A}$ or $\mathbf{D}$. We denote this modified construction by $\mathcal{F}_T^\pm$.

Theorem 10.

$$\text{ex}(n, \mathbf{AD}) = \begin{cases} \lfloor \frac{1}{4}(n-1)^2 \rfloor + 2 & \text{if } n \geq 4 \text{ is even} \\ \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd} \end{cases}$$

and equality is attained by a triple system $\mathcal{F}$ if and only if it is of the form $\mathcal{F}_T^+$ (for even $n \geq 4$) or one of the two forms $\mathcal{F}_T$ or $\mathcal{F}_T^\pm$ (for odd $n \geq 5$) for a transitive tournament T .

Proof. We argue by induction on n . The base cases $n = 3, 4$ are trivial, so we assume $n \geq 5$. Let $\mathcal{F}$ be an $\mathbf{AD}$ -free triple system on the vertex set V , with $|V| = n$ and

$$|\mathcal{F}| \geq f(n) := \begin{cases} \lfloor \frac{1}{4}(n-1)^2 \rfloor + 2 & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd} \end{cases}$$

We have to show that $|\mathcal{F}| = f(n)$ and $\mathcal{F}$ is of one of the specified forms.

Assume first that $\mathcal{F}$ is $\mathbf{C}$ -free as well. Then by Corollary 9 we have $|\mathcal{F}| \leq \lfloor \frac{1}{4}(n-1)^2 \rfloor$, so this is possible only when n is odd, in which case we get $|\mathcal{F}| = f(n)$ and $\mathcal{F}$ is of the form $\mathcal{F}_T$ (again by Corollary 9).

From now on, we assume that $\mathcal{F}$ does contain a $\mathbf{C}$ configuration, say on vertices 1, 2, 3, 4. Note that any triple in $\mathcal{F}$ that is not contained in $\{1, 2, 3, 4\}$ intersects it in at most one vertex – otherwise we get a $\mathbf{D}$ configuration. When $n = 5$ this immediately implies that $\mathcal{F}$ consists only of the 4 triples in $\{1, 2, 3, 4\}$, which form an $\mathcal{F}_T^\pm$. When $n = 6$ this implies that $\mathcal{F}$ consists of those 4 triples and another 4 triples of the form $i56$, $i = 1, 2, 3, 4$, yielding an $\mathcal{F}_T^+$. Henceforth we assume $n \geq 7$, which will allow us to apply the induction hypothesis to the family of those triples in $\mathcal{F}$ contained in $V' := V \setminus \{1, 2, 3, 4\}$.

For $i = 1, 2, 3, 4$, let G_i be the graph whose edges are those pairs in V' that form with the vertex i a triple in $\mathcal{F}$; in other words, G_i is the graph induced on V' by the link $\mathcal{F}[i]$. By Observation 7, each connected component of G_i is either a star (including the cases of an isolated vertex or edge) or a triangle. By the absence of $\mathbf{A}$, an edge of G_i and an edge of G_j , $j \neq i$, cannot share exactly one vertex. Let G be the multigraph on V' formed by the G_i , $i = 1, 2, 3, 4$. By the two observations just made, each connected component of

G is either a pair of vertices with 2, 3 or 4 parallel edges between them (say there are m such pairs $P_1, \dots, P_m$), or a star or triangle whose edges belong to a single G_i .

As the number of edges is at most 4 in each P_j , and at most the number of vertices in every other component of G , we can write

$$|E(G)| \leq 4m + (n - 4 - 2m) = n + 2m - 4.$$

On the other hand, using the induction hypothesis and the equality $f(n) - f(n - 4) = 2n - 6$, we get

$$f(n) \leq |\mathcal{F}| \leq 4 + |E(G)| + f(n - 4) = |E(G)| + f(n) - 2n + 10,$$

implying that $|E(G)| \geq 2n - 10$. Comparing these two bounds on $|E(G)|$, we immediately get $2m \geq n - 6$, so $\cup_{j=1}^m P_j$ covers all but at most two vertices of V' . Actually, it cannot leave out two vertices, because then our upper bound on $|E(G)|$ is not tight (those two vertices carry at most one edge). It follows that $P_1, \dots, P_m$ is a maximum matching in V' , i.e., $m = \lfloor \frac{n-4}{2} \rfloor$. Our bounds on $|E(G)|$ become

$$2n - 10 \leq |E(G)| \leq 4 \lfloor \frac{n-4}{2} \rfloor.$$

For odd n the bounds coincide, implying that each P_j has 4 parallel edges. For even n there is a slack of 2, so there may be one exceptional P_j with 2 parallel edges, or at most two exceptional P_j with 3 parallel edges each. Note that in any case, for any two distinct P_j, P_k , there is a graph G_i having an edge in both P_j and P_k .

We claim that any triple $F \in \mathcal{F}$, $F \subseteq V'$ must contain one of the pairs P_j , $j = 1, \dots, m$. If not, then F is of the form xyz where $x \in P_j$, $y \in P_k$, $k \neq j$, and z is either in a third P_ℓ or (when n is odd) is the unpaired vertex of V' . Let x' and y' be the other vertices of P_j and P_k , respectively, and let $i \in \{1, 2, 3, 4\}$ be such that $E(G_i)$ contains both xx' and yy' . Then xyz, ixx', iyy' give an **A** configuration, which is forbidden.

We claim further that for any two distinct pairs P_j and P_k , there cannot coexist in $\mathcal{F}$ a triple made of P_j and a vertex of P_k and a triple made of P_k and a vertex of P_j . Indeed, two such triples and a triple of the form $P_j \cup \{i\}$, $i \in \{1, 2, 3, 4\}$, would yield a **D** configuration.

To summarize, the possible triples $F \in \mathcal{F}$, $F \subseteq V'$ are either contained in some $P_j \cup P_k$, $j \neq k$ (and there are at most two such triples for each $j \neq k$), or consist of some P_j and the unpaired vertex of V' (when n is odd). This gives the following upper bound on $|\mathcal{F}|$:

$$|\mathcal{F}| \leq \begin{cases} 4 + 4\frac{n-4}{2} + 2\binom{\frac{n-4}{2}}{2} & \text{if } n \text{ is even} \\ 4 + 4\frac{n-5}{2} + 2\binom{\frac{n-5}{2}}{2} + \frac{n-5}{2} & \text{if } n \text{ is odd} \end{cases}$$

A calculation shows that for either parity, this bound equals $f(n)$. We conclude that $|\mathcal{F}| = f(n)$ and all the triples counted in the upper bound are in fact in $\mathcal{F}$. This $\mathcal{F}$ is easily seen to be of the form $\mathcal{F}_T^+$ in the even case and $\mathcal{F}_T^\pm$ in the odd case, for a tournament T in which P_j beats P_k if the two triples in $P_j \cup P_k$ contain P_j , and every P_j beats 12 and 34. The tournament must be transitive in order to avoid an **A** configuration. $\square$

8 Excluding A, B and C

8.1 A construction based on the Turán graph

Let $\mathcal{F}_n^*$ be the following triple system on a set V of n vertices. Fix a vertex $x \in V$, and consider a complete bipartite graph on $V \setminus \{x\}$ with parts of size $\lfloor \frac{n-1}{2} \rfloor$ and $\lceil \frac{n-1}{2} \rceil$, respectively. Let the triple F be in $\mathcal{F}_n^*$ if and only if F consists of x and an edge of the complete bipartite graph. It is easy to check that none of A, B and C is contained in $\mathcal{F}_n^*$, and its size is $\lfloor \frac{1}{4}(n-1)^2 \rfloor$.

Theorem 11.

$$\text{ex}(n, \text{ABC}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor$$

and equality is attained by a triple system $\mathcal{F}$ if and only if it is isomorphic to $\mathcal{F}_n^*$.

8.2 The structure of nearly extremal triangle-free graphs

We will need the following lemma about triangle-free graphs on n vertices having at least $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ edges. Given a graph $G = (V, E)$, two vertices $x, y \in V$ are called *remote* if the distance between them in G is greater than 2.

Lemma 12. *Let $G = (V, E)$ be a triangle-free graph with $|V| = n$ and $|E| \geq \lfloor \frac{1}{4}(n-1)^2 \rfloor$. Then either*

1. *G is bipartite, and if $|E| > \lfloor \frac{1}{4}(n-1)^2 \rfloor$ then no two vertices of the same part are remote, or*
2. *G is non-bipartite, and there are at most two pairs of remote vertices (none if $|E| > \lfloor \frac{1}{4}(n-1)^2 \rfloor$).*

Proof. Our proof extends a classical one by Andrásfai, Erdős and Gallai (see [7]), who showed that a triangle-free graph with at least $\lfloor \frac{1}{4}(n-1)^2 \rfloor + 2$ edges must be bipartite.

Suppose first that G is bipartite with parts A and B of cardinality a and b , respectively, where $a + b = n$, $a \leq b$. If some two vertices of the same part are remote, then at least a pairs in $A \times B$ are non-edges in G . This implies that $|E| \leq a(b-1) \leq \lfloor \frac{1}{4}(n-1)^2 \rfloor$, proving the assertion in the first part of the lemma.

Next, suppose that G is non-bipartite, and let C_ℓ be a shortest odd cycle in G , of length $\ell \geq 5$. To avoid a shorter odd cycle, C_ℓ must be induced, and every vertex in $V \setminus V(C_\ell)$ can have at most two neighbors in $V(C_\ell)$. Using these facts, and Turán's theorem for the graph induced on $V \setminus V(C_\ell)$, we get the upper bound

$$|E| \leq \ell + 2(n - \ell) + \lfloor \frac{1}{4}(n - \ell)^2 \rfloor.$$

This bound is a decreasing function of ℓ , and for $\ell = 7$ it is strictly less than $\lfloor \frac{1}{4}(n-1)^2 \rfloor$. Hence we must have $\ell = 5$, for which the bound becomes $|E| \leq \lfloor \frac{1}{4}(n-1)^2 \rfloor + 1$.

If $|E| = \lfloor \frac{1}{4}(n-1)^2 \rfloor + 1$ then G has the following structure: a 5-cycle C_5 , a complete bipartite graph $K_{\lfloor \frac{n-5}{2} \rfloor, \lceil \frac{n-5}{2} \rceil}$ on $V \setminus V(C_5)$, and every vertex of the latter is joined by edges to two non-adjacent vertices of the former. This graph is easily seen to have diameter 2, i.e., no pair of remote vertices. If $|E| = \lfloor \frac{1}{4}(n-1)^2 \rfloor$ then one of the two terms $2(n-5)$ and $\lfloor \frac{1}{4}(n-5)^2 \rfloor$ in the upper bound on $|E|$ was off by 1. If it was the former, then there is a unique vertex x in $V \setminus V(C_5)$ that has a single neighbor y in $V(C_5)$. Denoting the two vertices of C_5 at distance 2 from y by z and w , we see that there are at most two pairs of remote vertices, $\{x, z\}$ and $\{x, w\}$. If it was the latter, then the triangle-free graph induced on $V \setminus V(C_5)$ has $\lfloor \frac{1}{4}(n-5)^2 \rfloor - 1$ edges, one less than the Turán bound. This induced graph can only be of one of the following forms: the Turán graph minus an edge, or (when $n-5$ is even) a $K_{\frac{n-5}{2}-1, \frac{n-5}{2}+1}$, or (when $n-5=5$) another copy of C_5 . In each of these cases, one can check that there are at most two pairs of remote vertices. $\square$

8.3 Proof of Theorem 11

Let $\mathcal{F}$ be an **ABC**-free triple system on n vertices. Then each triple in $\mathcal{F}$ has at least one own pair. Choosing exactly one own pair from each triple, we get a graph $G = (V, E)$ with $|E| = |\mathcal{F}|$. The absence of **ABC** in $\mathcal{F}$ implies that G is triangle-free.

We claim that for each triple in $\mathcal{F}$, one of its pairs is an edge of G and the other two pairs are remote in G . First, observe that a triple in $\mathcal{F}$ cannot have two pairs that are edges of G , because one of them would have to be an own pair of a different triple, which is impossible. Thus, if $xyz \in \mathcal{F}$ and xy is its chosen own pair, then none of xz , yz is an edge of G . It remains to show that none of these pairs is joined by a path of length 2 in G . Suppose, for the sake of contradiction, that xu and uz are edges of G , for some $u \notin \{x, y, z\}$. By the observation above, xuz is not a triple in $\mathcal{F}$, so xu and uz must be own pairs of $xuv \in \mathcal{F}$ and $uzw \in \mathcal{F}$, respectively, for some $v, w \notin \{x, u, z\}$. Also $v \neq y$, because xy is an own pair of xyz . Now, the triples xyz , xuv , uzw form an **A** (if $w \notin \{y, v\}$) or a **B** (if $w \in \{y, v\}$) configuration, both of which are forbidden. This contradiction shows that x and z are remote, and a symmetric argument works for y and z .

We assume that $|E| = |\mathcal{F}| \geq \lfloor \frac{1}{4}(n-1)^2 \rfloor$, and apply Lemma 12 to the graph G . As each triple in $\mathcal{F}$ contributes two pairs of remote vertices, case 2 of the lemma cannot occur. Thus we are in case 1, and G is bipartite with bipartition $V = A \cup B$. Each triple in $\mathcal{F}$ must contain an edge of G , hence two vertices in one part and one in the other. By the above, the two vertices in the same part are remote. If $|E| > \lfloor \frac{1}{4}(n-1)^2 \rfloor$ we get a contradiction to the statement of case 1 of the lemma. This proves that $\text{ex}(n, \mathbf{ABC}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

It remains to show that if $|\mathcal{F}| = \lfloor \frac{1}{4}(n-1)^2 \rfloor$ then $\mathcal{F}$ is of the form $\mathcal{F}_n^*$. For this, it suffices to show that all triples in $\mathcal{F}$ share some fixed vertex x . Indeed, by the absence of **C**, the link of such x in $\mathcal{F}$ has to be a triangle-free graph on $V \setminus \{x\}$ with $\lfloor \frac{1}{4}(n-1)^2 \rfloor$ edges, hence it must be the Turán graph.

Let $\overline{G}$ be the bipartite complement of G , i.e., $E(\overline{G})$ is the set of pairs in $A \times B$ which are not edges of G . As $|A| + |B| = n$, we have $|E(\overline{G})| = |A| \cdot |B| - \lfloor \frac{1}{4}(n-1)^2 \rfloor \leq \min\{|A|, |B|\}$. On the other hand, if there is a triple in $\mathcal{F}$ with two vertices in A then they are remote

in G , hence every vertex in B is incident to an edge of $\overline{G}$. Similarly, if there is a triple in $\mathcal{F}$ with two vertices in B then every vertex in A is incident to an edge of $\overline{G}$.

Assume first that there are triples of both kinds. We conclude that $|A| = |B|$ and $\overline{G}$ is a perfect matching between A and B . Now, if $|A| = |B| \geq 3$ then no two vertices of the same part are remote and we get a contradiction. The only possibility is $|A| = |B| = 2$, in which case $n = 4$, $|\mathcal{F}| = 2$ and the two triples in $\mathcal{F}$ must share two vertices.

Thus, we may assume that all triples in $\mathcal{F}$ are of the same kind, having without loss of generality two vertices in A and one in B . Let $x \in A$ and $y \in B$ form an edge of $\overline{G}$. Since every vertex in B is incident to an edge of $\overline{G}$, and there are at most $|B|$ such edges, xy is the only edge of $\overline{G}$ incident to y . Hence, no two vertices in $A \setminus \{x\}$ are remote. Therefore, every triple in $\mathcal{F}$ must contain x , completing the proof. $\square$

9 Excluding A and C

9.1 AC-free constructions

We have already encountered two different equally large constructions of **AC**-free triple systems: $\mathcal{F}_T$, the one based on the transitive tournament (which also avoids **D**), and $\mathcal{F}_n^*$, the one based on the Turán graph (which also avoids **B**), both of size $\lfloor \frac{1}{4}(n-1)^2 \rfloor$. It turns out that these are optimal **AC**-free constructions for even n , but we can do better by 1 for odd $n \geq 5$.

Given an **AC**-free family $\mathcal{F}$ on the n -element vertex set V one can define another **AC**-free family $\mathcal{F}^{+2}$ on the vertex set $V \cup \{1, 2\}$ as follows. $\mathcal{F}^{+2} := \mathcal{F} \cup \{12v : v \in V\}$. We call this new family a 2-*extension* of $\mathcal{F}$. One can define an **AC**-free family $\mathcal{F}^{+4}$ on the vertex set $V \cup \{1, 2, 3, 4\}$ as

$$\mathcal{F}^{+4} := \mathcal{F} \cup \{12v : v \in V\} \cup \{34v : v \in V\} \cup \{123, 234\}.$$

We call this a 4-*extension* of $\mathcal{F}$. Note that if $V \neq \emptyset$ then $(\mathcal{F}^{+2})^{+2}$ is not the same as $\mathcal{F}^{+4}$, although they only differ by replacing 124 by 234. For even $n \geq 2$ let $\mathbf{F}_{(n)}$ denote the class of n -vertex triple systems one can obtain by an arbitrary sequence of 2-extensions and 4-extensions starting with a system of the form $\mathcal{F}_{n-2k}^*$ for some $0 \leq k \leq \frac{n-2}{2}$ (the case $k = 0$ gives $\mathcal{F}_n^*$ without extensions, the case $k = \frac{n-2}{2}$ applies a sequence of extensions starting from the empty triple system on 2 vertices and gives $\mathcal{F}_T$ when it uses only 2-extensions). All families $\mathcal{F} \in \mathbf{F}_{(n)}$, n even, have the same size $|\mathcal{F}| = \lfloor \frac{1}{4}(n-1)^2 \rfloor$.

For $n = 5$ we arrange the vertices 1, 2, 3, 4, 5 cyclically, and let $\mathcal{F}_5^5$ be the *tight 5-cycle* consisting of the five triples of consecutive vertices 123, 234, 345, 451, 512. Note that it contains no **A** or **C**, and its size is $5 = \frac{1}{4}(5-1)^2 + 1$. For odd $n \geq 5$ the members of the class $\mathbf{F}_{(n)}$ are those n -vertex triple systems obtained from a tight 5-cycle by an arbitrary sequence of 2- and 4-extensions. By induction we have $|\mathcal{F}| = \frac{1}{4}(n-1)^2 + 1$ for all odd $n \geq 5$ when $\mathcal{F} \in \mathbf{F}_{(n)}$.

Theorem 13. *Define*

$$f(n) := \begin{cases} \lfloor \frac{1}{4}(n-1)^2 \rfloor & \text{if } n \text{ is even or } n = 1, 3 \\ \frac{1}{4}(n-1)^2 + 1 & \text{if } n \text{ is odd, } n \neq 1, 3 \end{cases}$$

Then $\text{ex}(n, \mathbf{AC}) = f(n)$. Moreover, for $n \geq 4$ the only extremal $\mathbf{AC}$ -free families are the members of $\mathbf{F}_{(n)}$.

We can express the number of non-isomorphic extremal families for a given n in terms of the Fibonacci sequence defined by $f_1 = f_2 = 1$ and $f_k = f_{k-1} + f_{k-2}$ for $k > 2$. For even n we claim that $|\mathbf{F}_{(n)}| = 2f_{n/2} - 1$. Indeed, this holds for $n = 2, 4$ because in these cases there is a single extremal family. For even $n > 4$ we note that $\mathcal{F} \in \mathbf{F}_{(n)}$ is either a 2-extension of a member of $\mathbf{F}_{(n-2)}$, or a 4-extension of a member of $\mathbf{F}_{(n-4)}$, or is $\mathcal{F}_n^*$. Hence by induction $|\mathbf{F}_{(n)}| = 2f_{(n-2)/2} - 1 + 2f_{(n-4)/2} - 1 + 1 = 2f_{n/2} - 1$ as claimed. For odd $n \geq 5$ a similar induction shows that $|\mathbf{F}_{(n)}| = f_{(n-3)/2}$. Indeed, the extremal family is unique for $n = 5, 7$, and for odd $n > 7$ we have $|\mathbf{F}_{(n)}| = |\mathbf{F}_{(n-2)}| + |\mathbf{F}_{(n-4)}| = f_{(n-5)/2} + f_{(n-7)/2} = f_{(n-3)/2}$. The multitude of extremal constructions may explain why the proof of Theorem 13 turned out to be more complicated than other proofs in this paper. Before turning to the proof itself, we further investigate the structure of possible maximal families and we also need a graph theoretic lemma (presented in Subsection 9.5).

9.2 Reducing $\mathbf{AC}$ -free families

Suppose that the $\mathbf{AC}$ -free triple system $\mathcal{F}$ on n vertices has a pair $u, v \in V$ such that $\deg_{\mathcal{F}}(uv) = n - 2$ and let $\mathcal{H}$ be the family of triples of $\mathcal{F}$ contained in $V \setminus \{u, v\}$. Then $\mathcal{F} = \mathcal{H}^{+2}$. This observation leads to the following useful lemma.

Lemma 14. *Let $\mathcal{F}$ be an $\mathbf{AC}$ -free triple system on the n -vertex set V . Suppose that there are disjoint pairs $e_1, \dots, e_k \subset V$ ($1 \leq k < \frac{n}{2}$), $K := \cup e_i$, having the following properties:*

(K1) *There is no triple $F \in \mathcal{F}$ such that $|F \cap K| = 1$.*

(K2) *If $F \in \mathcal{F}$ meets both e_i and e_j then $F \subset e_i \cup e_j$ (for all $1 \leq i \neq j \leq k$).*

Let $\mathcal{H}$ denote the family of triples of $\mathcal{F}$ contained in $V \setminus K$. Then

$$|\mathcal{F}| - \lfloor \frac{1}{4}(n-1)^2 \rfloor \leq |\mathcal{H}| - \lfloor \frac{1}{4}(n-2k-1)^2 \rfloor, \quad (2)$$

and here equality holds if and only if there is a sequence of 2- and 4-extensions of $\mathcal{H}$ resulting in $\mathcal{F}$.

If $\mathcal{F}$ possesses such a set of pairs, then we call it *reducible*.

Proof of Lemma 14. The family $\mathcal{F} \setminus \mathcal{H}$ contains two kinds of triples, those meeting $V \setminus K$ and containing an e_i and those contained in some $e_i \cup e_j$. Since every four vertices can carry at most two triples we get

$$|\mathcal{F}| - |\mathcal{H}| \leq k(n-2k) + 2 \binom{k}{2} = \lfloor \frac{1}{4}(n-1)^2 \rfloor - \lfloor \frac{1}{4}(n-2k-1)^2 \rfloor.$$

We are done with inequality (2).

In the case of equality we have $e_i \cup \{v\} \in \mathcal{F}$ for all $1 \leq i \leq k$ and $v \in V \setminus K$, and each $e_i \cup e_j$ contains exactly two triples from $\mathcal{F}$. Define an oriented graph T_K with vertex set $\{e_1, \dots, e_k\}$ as follows. Add an arrow from e_i to e_j if there is a triple F containing e_i and meeting e_j . Note that T_K has at least one edge between e_i and e_j , and it also might have two edges in opposite directions. However, T_K does not contain oriented triangles because $\mathcal{F}$ is **A**-free. This implies that the pairs $\{e_i, e_j\}$ with double arrows are pairwise disjoint, so they form a matching, say L . We obtain that T_K consists of a transitive tournament (say, there is an arrow from e_i to e_j for $1 \leq j < i \leq k$) together with a few reversed arrows $e_i \rightarrow e_{i+1}$ for $\{e_i, e_{i+1}\} \in L$. Every e_i outside L corresponds to a 2-extension, and the pairs from L define 4-extensions. $\square$

9.3 The structure of AC-free families: partitioning the triples and the pairs

Let $\mathcal{F}$ be an **AC**-free triple system on $[n]$. Recall that a tight 5-cycle is a triple system consisting of 5 triples on 5 vertices that is isomorphic to $\mathcal{F}_5^5$ (defined in Subsection 9.1). Tight 5-cycles that are contained in the triple system $\mathcal{F}$ will play a special role in our analysis. Suppose that $\mathcal{P} \subseteq \mathcal{F}$ is a tight 5-cycle. Then $V(\mathcal{P})$ contains no more triples from $\mathcal{F} \setminus \mathcal{P}$. Also, the case of $|F \cap V(\mathcal{P})| = 2$ for some $F \in \mathcal{F}$ is impossible because it would create an **A** configuration. Indeed, suppose that $V(\mathcal{P}) = \{1, 2, 3, 4, 5\}$ in this cyclic order and $F \in \mathcal{F}$ contains 6 and two vertices of $V(\mathcal{P})$. By symmetry we may assume that $F = 126$ or $F = 136$, and in both cases F forms an **A** configuration with 234, 451 from $\mathcal{P}$. We conclude that triples in $\mathcal{F} \setminus \mathcal{P}$ can meet $V(\mathcal{P})$ in at most one vertex. So the 10 pairs of $\mathcal{P}$ are contained only in the triples from $\mathcal{P}$. Note that a pair of cyclically adjacent vertices is contained in two triples of $\mathcal{P}$, while a pair of non-adjacent vertices is contained in a single triple of $\mathcal{P}$. Let $\mathcal{R} \subseteq \mathcal{F}$ be the family of triples contained in tight 5-cycles and let G_R be the graph on $[n]$ whose edges are the pairs contained in members of $\mathcal{R}$. We further split the graph G_R into two graphs: G_{R1} consists of the pairs that are non-adjacent in their tight 5-cycle (and hence contained in a single triple), G_{R2} consists of the pairs that are adjacent in their tight 5-cycle (and hence contained in two triples). We have $|\mathcal{R}| = |E(G_{R1})| = |E(G_{R2})|$ and $|E(G_R)| = 2|\mathcal{R}|$.

Every triple $F \in \mathcal{F}$ contains at least one own pair. We partition $\mathcal{F} \setminus \mathcal{R}$ into $\mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_1$ consists of the triples with exactly one own pair, and $\mathcal{F}_2$ of those with at least two own pairs. Let G_1 be the graph on $[n]$ whose edges are the own pairs of the triples in $\mathcal{F}_1$. Picking exactly two own pairs of every triple in $\mathcal{F}_2$, let G_2 be the graph on $[n]$ formed by them. Thus $|E(G_1)| = |\mathcal{F}_1|$ and $|E(G_2)| = 2|\mathcal{F}_2|$. As all edges in G_1 and G_2 are own pairs of some $F \in \mathcal{F}$ we have $E(G_1) \cap E(G_2) = \emptyset$.

Define the graph G_3 as

$$E(G_3) = \{xy : x, y \text{ are joined by a path of length 2 in } G_1\}.$$

We will require the following fact about the edges of G_3 .

Lemma 15. *If $xy \in E(G_3)$ then no triple in $\mathcal{F}$ contains both x and y .*

Proof. Let v be a vertex such that $xv, vy \in E(G_1)$, and let F_1 and F_2 be the triples in $\mathcal{F}_1$ having xv and vy , respectively, as their unique own pair. Assume for the sake of

contradiction that the triple F_3 in $\mathcal{F}$ contains x and y . Clearly, xvy cannot be a triple in $\mathcal{F}$ because it contains both xv and vy . Therefore F_1, F_2, F_3 are all distinct and each of them contains a different pair from $\{x, v, y\}$ and a vertex outside it. Say $F_1 = xvw$, $F_2 = vyz$, $F_3 = xyu$. To avoid forming an **A** or a **C**, two of the vertices w, z, u must be the same and the third must be different.

We consider first the case when $w = z$. Since xw is not an own pair of F_1 , there is another triple in $\mathcal{F}$ containing it, say $F_4 = xws$. If $s \notin \{y, u\}$ then F_2, F_3, F_4 form a forbidden **A**. If $s = y$ then F_1, F_2, F_4 form a forbidden **C**. Thus $s = u$ is the only possibility. Repeating the same arguments for yw which is not an own pair of F_2 , we find the triple $F_5 = ywu$ in $\mathcal{F}$. But now F_3, F_4, F_5 form a **C**, giving a contradiction.

Thus, we are left with the case when $w \neq z$ and u is equal to one of them, without loss of generality $u = w$. Since vw is not an own pair of F_1 , there is another triple in $\mathcal{F}$ containing it, say $F_4 = vws$. If $s \notin \{y, z\}$ then F_2, F_3, F_4 form a forbidden **A**. If $s = y$ then F_1, F_3, F_4 form a forbidden **C**. Thus $s = z$ is the only possibility. Since yz is not an own pair of F_2 , there is another triple in $\mathcal{F}$ containing it, say $F_5 = yzt$. If $t \notin \{x, w\}$ then F_3, F_4, F_5 form a forbidden **A**. If $t = w$ then F_2, F_4, F_5 form a forbidden **C**. Thus $t = x$ is the only possibility. Summarizing, we have the following five triples in $\mathcal{F}$:

$$F_1 = vwx, F_3 = wxy, F_5 = xyz, F_2 = yzv, F_4 = zvw.$$

They form a tight 5-cycle, contradicting the fact that F_1 and F_2 are in $\mathcal{F}_1$, hence not part of any such cycle. $\square$

Our Lemma 15 implies that $E(G_3)$ is disjoint from $E(G_R) \cup E(G_1) \cup E(G_2)$, so the following four edge sets are pairwise disjoint:

$$E(G_R), E(G_1), E(G_2), E(G_3).$$

By adding a fifth one

$$E(G_4) := \{xy: x \neq y, xy \text{ is in none of the above}\},$$

we obtain a partition of $E(K_n)$ into five graphs. Figure 2 shows, for every possible triple in $\mathcal{F}$, to which of the above graphs its pairs belong. Again, if needed, we split G_4 into three graphs, G_{40} , G_{41} and G_{42} to indicate if an edge is not covered, covered once or at least twice by members of $\mathcal{F}$, respectively.

The discussion above yields the following observations, to be used later:

$$\deg_{\mathcal{F}}(xy) > 2 \Rightarrow xy \in E(G_{42}) \tag{3}$$

$$\deg_{\mathcal{F}}(xy) = 2 \Rightarrow xy \in E(G_{R2}) \cup E(G_{42}) \tag{4}$$

$$\deg_{\mathcal{F}}(xy) = 0 \Leftrightarrow xy \in E(G_3) \cup E(G_{40}) \tag{5}$$

9.4 Proof of Theorem 13: three cases without induction

Let $\mathcal{F}$ be an **AC**-free triple system on $[n]$ and suppose that $|\mathcal{F}| = \text{ex}(n, \text{AC})$. Constructions showing $|\mathcal{F}| \geq f(n)$ were given above. It is trivial to check their optimality for $n \leq 4$, so from now on we may suppose that $n \geq 5$. We will prove that $|\mathcal{F}| \leq f(n)$ and $\mathcal{F} \in \mathbf{F}_{(n)}$. We break the argument into six cases.

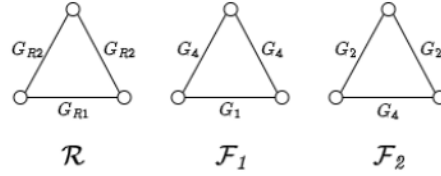


Figure 2: Assignment of the pairs of each triple in $\mathcal{F}$ to graphs

9.4.1 Case 1: There is an isolated vertex in G_4

Here we assume that x is an isolated vertex in G_4 . In this case all edges xy belong to either G_2 or G_R . Indeed, $xy \notin E(G_4)$. Also $xy \in E(G_1)$ implies that there is an $xzy \in \mathcal{F}_1$, so there is a path $x-z-y$ in G_4 . But this is impossible since there is no $xz \in E(G_4)$, so $xy \notin E(G_1)$. Finally, $xy \in E(G_3)$ would require the existence of a vertex z such that $xz, zy \in E(G_1)$, but we have just shown that xz is not in $E(G_1)$ for any z .

Consider the graph $\mathcal{F}[x]$, the link of x in $\mathcal{F}$. If the pair xy is contained in d triples (i.e., $\deg_{\mathcal{F}}(xy) = d$) then the degree of the vertex y in the graph $\mathcal{F}[x]$ is exactly d . Since $xy \notin E(G_4)$ all the degrees in $\mathcal{F}[x]$ are at most 2 by observation (3). Moreover, if a component C of $\mathcal{F}[x]$ has a vertex of degree 2 then by observation (4), $V(C) \cup \{x\}$ is the vertex set of a tight 5-cycle, and C is a path of length 3. Thus, any component C of $\mathcal{F}[x]$ that is not an isolated vertex is of one of two forms:

I A single edge e .

We let $k \geq 0$ be the number of such components, and denote them by $e_1, \dots, e_k$. We denote by K the set of vertices of these components, $|K| = 2k$.

II A path P of length 3.

We let $\ell \geq 0$ be the number of such components, and denote them by $P_1, \dots, P_\ell$. We denote by L the set of vertices of these components, $|L| = 4\ell$.

Note that $xy \in E(G_2)$ for any $y \in K$, and $xy \in E(G_R)$ for any $y \in L$. We claim that there are no isolated vertices in $\mathcal{F}[x]$. Indeed, suppose that z is an isolated vertex. Then $\deg_{\mathcal{F}}(xz) = 0$ and hence by observation (5), $xz \in E(G_3) \cup E(G_{40})$. This contradicts the observation above that all edges incident to x belong to either G_2 or G_R .

Thus, $\mathcal{F}[x]$ gives the following decomposition of $[n] \setminus \{x\}$:

$$[n] \setminus \{x\} = K \cup L, \text{ where } |K| = 2k, |L| = 4\ell.$$

Note that n must be odd in this case since $n = |K| + |L| + 1$. Next, we bound the number of triples contained in $K \cup L$:

- Each P_i contains two triples (those triples in the tight 5-cycles that do not include x). Other than those, no triple in $K \cup L$ contains two vertices of the same P_i .
- To avoid an **A** configuration, no triple in $K \cup L$ meets three components of $\mathcal{F}[x]$, and no triple meets P_i and P_j when $i \neq j$.

- Thus, any triple in $K \cup L$ that is not contained in a P_i , consists of both vertices of some e_i , and one vertex of another component.
- To avoid a **C** configuration, for every $i \neq j$ there are at most 2 triples contained in $e_i \cup e_j$.

In total, we have

$$|\mathcal{F}| \leq k + 5\ell + 2\binom{k}{2} + k \cdot 4\ell = (k + 2\ell)^2 + 5\ell - 4\ell^2 \leq \left(\frac{n-1}{2}\right)^2 + 1 = f(n), \quad (6)$$

which completes the proof of the upper bound in Case 1.

If equality holds in (6) then $\ell = 1$, $k = \frac{n-5}{2}$. If $k = 0$ then $n = 5$ and $\mathcal{F}$ is a tight 5-cycle, i.e., $\mathcal{F} \in \mathbf{F}_{(5)}$. Suppose now that $k \geq 1$. According to the above observations K and $\mathcal{F}$ satisfy the two constraints of Lemma 14, so $\mathcal{F}$ is reducible. Inequality (2) and $|\mathcal{F}| = f(n)$ yield

$$|\mathcal{F}| - \lfloor \frac{1}{4}(n-1)^2 \rfloor = |\mathcal{H}| - \lfloor \frac{1}{4}(n-2k-1)^2 \rfloor = |\mathcal{H}| - 4 = 1$$

where $\mathcal{H}$ is a tight 5-cycle. Then Lemma 14 gives that $\mathcal{F}$ is obtained from $\mathcal{H}$ by a sequence of 2- and 4-extensions, i.e., $\mathcal{F} \in \mathbf{F}_{(n)}$. $\square$

9.4.2 Case 2: There is a pendant G_4 edge which is in G_{40}

Let x be a degree one vertex of G_4 , and let y be its unique neighbor in G_4 . In this case we suppose that $\deg_{\mathcal{F}}(xy) = 0$, the pair xy does not occur in any triple of $\mathcal{F}$. Again, as in Case 1, there is no G_1 edge incident to x , so there is no such G_3 edge either.

Consider the structure of the link $\mathcal{F}[x]$, similar to what we did in Case 1. The vertex y is isolated in $\mathcal{F}[x]$ and $[n] \setminus \{x\} = K \cup L \cup \{y\}$. In particular n is even here, $n = 2k + 4\ell + 2$. The only possible triples in $\mathcal{F}$ that contain y and do not give rise to an **A** configuration are of the form $e_i \cup \{y\}$. There are at most k of those. Thus, like in (6), we get

$$|\mathcal{F}| \leq k + 5\ell + 2\binom{k}{2} + k \cdot 4\ell + k = \lfloor \left(\frac{n-1}{2}\right)^2 \rfloor + 3\ell - 4\ell^2 \leq f(n).$$

If equality holds here then $\ell = 0$, $k = \frac{n-2}{2}$, and Lemma 14 can be applied to obtain $\mathcal{F} \in \mathbf{F}_{(n)}$. $\square$

9.4.3 Case 3: There is a pendant G_4 edge which is in G_{41}

Let x be a degree one vertex of G_4 , and let y be its unique neighbor in G_4 . In this case we suppose that $\deg_{\mathcal{F}}(xy) = 1$, the pair xy occurs in a triple $xyz \in \mathcal{F}$ with $\deg_{\mathcal{F}}(xz) = \deg_{\mathcal{F}}(yz) = 1$. In the process creating the graph G_2 we selected xz and yz and put xy to G_{41} . Redefine these graphs on these three pairs, put yz into G_{41} and the other pairs to G_2 . Then x becomes isolated in the new G_4 . That case has been solved as Case 1. $\square$

9.5 Pairs of vertices joined by a path of length 2

Given a graph G , we define a new graph $T = T(G)$ on the same vertex set as follows: two vertices $x \neq y$ form an edge in T if they are joined by a path of length 2 in G , i.e., there exists a vertex z such that both xz and zy are edges of G (note that xy itself may or may not be an edge of G).

Lemma 16. *Let G be a graph on n vertices, and let $T = T(G)$ be the graph defined above. Then*

$$|E(T)| \geq |E(G)| - \lfloor \frac{n}{2} \rfloor. \quad (7)$$

This inequality holds with equality if and only if one of the following is the case:

- (L1) *n is even, and G is the disjoint union of balanced complete bipartite graphs $K(A_i, B_i)$ with $|A_i| = |B_i| = d_i$, $i = 1, \dots, r$ where $d_i \geq 1$, $r \geq 1$, $\sum_{i=1}^r d_i = \frac{n}{2}$.*
- (L2) *n is odd, and G is as above except that one of the components is of the form $K(A_i, B_i)$ with $|A_i| = d_i$ and $|B_i| = d_i + 1$ or is an isolated vertex.*

Moreover, if n is even and $|E(T)| = |E(G)| - \frac{n}{2} + 1$, then either

- (L3) *one component of G is a $K_{d,d}$ with one edge left out, $d \geq 2$ (denote this as $K_{d,d}^-$), and the other components are balanced complete bipartite graphs, or*
- (L4) *there are positive integers $d_1, d_2, \dots, d_r$, $r \geq 2$ such that $\sum_{i=1}^r d_i = \frac{n}{2} + 1$ and the components of G are complete bipartite graphs K_{d_1, d_1-1} , K_{d_2, d_2-1} , and K_{d_i, d_i} for $i > 2$.*

The inequality (7) was proved in dual form by Füredi [13]. Below we also prove the characterization of equality and at the same time re-prove the inequality.

Note that in each of the cases (L1)–(L4) the graph $T(G)$ is a disjoint union of complete graphs. Also, in cases (L1) and (L3) the graph G has no isolated vertex.

Proof of Lemma 16. The graphs described in (L1) and (L2) satisfy inequality (7) with equality. We show that all other graphs satisfy (7) with strict inequality.

For a graph H and a vertex v , we denote by $N_H(v)$ the set of neighbors of v in H . We observe that for every vertex x ,

$$N_T(x) = \bigcup_{y \in N_G(x)} (N_G(y) \setminus \{x\}).$$

In particular, if x has a neighbor y with $\deg_G(x) < \deg_G(y)$ then $\deg_T(x) \geq \deg_G(x)$. This implies that there always exists a vertex x with $\deg_T(x) \geq \deg_G(x)$, unless every connected component C_i of G is d_i -regular for some $d_i \geq 1$, and moreover, in C_i the d_i neighbors of x all have the same $d_i - 1$ neighbors in addition to x , which means that $C_i \cong K_{d_i, d_i}$. In other words, if G is not of the form indicated in (L1), then there exists a vertex x with $\deg_T(x) \geq \deg_G(x)$.

We now proceed by induction on n . The base case $n = 1$ is trivial. For the induction step, we assume that G and $T = T(G)$ are graphs on $n \geq 2$ vertices, and the lemma is true for $n - 1$.

Assuming that G is not of the form indicated in (L1), let x be a vertex with $\deg_T(x) \geq \deg_G(x)$. Applying the induction hypothesis to $G \setminus \{x\}$ we get

$$|E(T(G \setminus \{x\}))| \geq |E(G \setminus \{x\})| - \lfloor \frac{n-1}{2} \rfloor. \quad (8)$$

Therefore

$$\begin{aligned} |E(T)| &\geq |E(T(G \setminus \{x\}))| + \deg_T(x) \\ &\geq |E(G \setminus \{x\})| - \lfloor \frac{n-1}{2} \rfloor + \deg_G(x) = |E(G)| - \lfloor \frac{n-1}{2} \rfloor. \end{aligned} \quad (9)$$

When n is odd, equality in (7) is possible, but it requires that all the above inequalities hold with equality. In particular, we must have equality in (8), hence by induction $G \setminus \{x\}$ has to be of the form (L1). Consider $N_G(x)$. If it meets two or more components of $G \setminus \{x\}$, or two sides of the same component, then $T(G)$ has at least one edge with vertices in $G \setminus \{x\}$ that is not in $E(T(G \setminus \{x\}))$. This makes the first inequality in (9) strict. We conclude that there is a component $K(A_i, B_i)$ of $G \setminus \{x\}$ with $|A_i| = |B_i| = d_i$ such that $N_G(x)$ is contained in one of its parts, say in A_i . If $\emptyset \subsetneq N_G(x) \subsetneq A_i$ then $\deg_T(x) = d_i > \deg_G(x)$, making the second inequality in (9) strict. This leaves only two possibilities: either $N_G(x) = A_i$, in which case we can add x to B_i and get a K_{d_i, d_i+1} , or $N_G(x) = \emptyset$ making x an isolated vertex in G . In either case, G fits the description in part (L2) of the lemma.

If n is even then $\lfloor \frac{n-1}{2} \rfloor < \lfloor \frac{n}{2} \rfloor$, hence (9) implies that inequality (7) holds strictly. This completes the proof of (7) together with (L1) and (L2).

Consider now the case $|E(T)| = |E(G)| - \frac{n}{2} + 1$. Again equality must hold in (8) and (9) so $\deg_T(x) = \deg_G(x)$, $G \setminus \{x\}$ has to be of the form (L2). If $N_G(x) = \emptyset$ then G is of the form (L4). If $N_G(x) \neq \emptyset$ then it meets only one component C . This component could not be an isolated vertex, because that would give $\deg_G(x) = 1$, $\deg_T(x) = 0$. So $C = K(A_i, B_i)$ and $N_G(x)$ is a subset of one side of C while $N_T(x)$ is the entire other side. Given that $\deg_T(x) = \deg_G(x)$, there are two possibilities. If $|A_i| = |B_i|$ then $N_G(x)$ is a full side of C , say B_i , so we get $K(A_i \cup \{x\}, B_i)$, yielding case (L4). If $|A_i| = |B_i| - 1$ then $N_G(x) = B_i \setminus \{y\}$ for some $y \in B_i$, so we get $K(A_i \cup \{x\}, B_i)$ minus xy , yielding case (L3). $\square$

9.5.1 Remark

We do not use but can prove an even stronger version of Lemma 16.

Claim 17. *Let G be a graph, and let $T = T(G)$ be the graph defined above. Suppose that $\{e_1, \dots, e_\ell\}$ is a maximal matching of G , i.e., $A := V(G) \setminus (\cup e_i)$ is an independent set of vertices. Then*

$$|E(T)| \geq |E(G)| - \ell. \quad (10)$$

This inequality holds with equality if and only if each component of G is either a balanced complete bipartite graph, or a $K_{d,d}^-$ (with $d \geq 2$) contributing $d - 1$ edges to the maximal matching, or a $K_{d,d+1}$ ($d \geq 1$), or an isolated vertex.

Sketch of proof. For two distinct edges e_i, e_j in the maximal matching, let $E(G[e_i, e_j])$ denote the set of edges of G with one end in e_i and the other in e_j , and let $E(T[e_i, e_j])$ denote the analogous set for T . Similarly, for an edge e_i and a vertex $a \in A$, let $E(G[e_i, a])$ and $E(T[e_i, a])$ denote the corresponding sets of edges with one end in e_i and the other equal to a .

By inspection, we have $|E(T[e_i, e_j])| \geq |E(G[e_i, e_j])|$ and $|E(T[e_i, a])| \geq |E(G[e_i, a])|$ for any two edges e_i, e_j and vertex $a \in A$. As this accounts for all edges of G apart from $e_1, \dots, e_\ell$, it implies (10). In case of equality we must have equalities above and moreover, no e_i and no two vertices in A can form an edge in T . We obtain that $E(G[e_i, e_j])$ should be either empty or two disjoint edges, and similarly $E(G[e_i, a])$ is either empty or a single edge. The rest of the proof is just to check how these edges can fit together so as to maintain the above conditions for equality. $\square$

9.6 The core inequality of the proof, and more reductions in case of a pendant G_4 edge

We will prove by induction on n that $|\mathcal{F}| \leq f(n)$ and $\mathcal{F} \in \mathbf{F}_{(n)}$.

Look at the partition $[n] = C_1 \cup \dots \cup C_r$ where the C_i are the connected components of G_4 . By Case 1 we may suppose that $p_i := |C_i| \geq 2$ for every i . Note that every edge of G_1 is contained in some C_i and the same is true for every edge of G_3 . If C is one of the C_i , we write G_4^C, G_1^C, G_3^C for the corresponding graph restricted to C . We have

$$\begin{aligned} 2|\mathcal{F}| &= 2|\mathcal{R}| + 2|\mathcal{F}_1| + 2|\mathcal{F}_2| \\ &= |E(G_R)| + 2|E(G_1)| + |E(G_2)| \\ &= |E(G_R)| + |E(G_1)| + |E(G_2)| + |E(G_3)| + |E(G_4)| - \frac{n}{2} \\ &\quad + (n - |E(G_4)|) + \left(|E(G_1)| - |E(G_3)| - \frac{n}{2}\right) \\ &= \binom{n}{2} - \frac{n}{2} + \sum_i \left((p_i - |E(G_4^{C_i})|) + \left(|E(G_1^{C_i})| - |E(G_3^{C_i})| - \frac{p_i}{2}\right)\right). \end{aligned} \tag{11}$$

In the last line for every component C_i we have $p_i - |E(G_4^{C_i})| \leq 1$ and Lemma 16 applied to $G_3^{C_i} = T(G_1^{C_i})$ states $|E(G_1^{C_i})| - |E(G_3^{C_i})| - \lfloor \frac{p_i}{2} \rfloor \leq 0$. So (11) yields a pretty good upper bound for $|\mathcal{F}|$ and to improve it further we will investigate the components where $G_4^{C_i}$ is a tree, and also the cases when $|E(G_1^{C_i})| - |E(G_3^{C_i})| - \frac{p_i}{2} \geq -1$.

Call a component C of size p *near optimal* if

$$|E(G_3^C)| \leq |E(G_1^C)| - \frac{p-2}{2}. \tag{12}$$

For *not* near optimal components the quantity

$$\sigma(C_i) := (p_i - |E(G_4^{C_i})|) + \left(|E(G_1^{C_i})| - |E(G_3^{C_i})| - \frac{p_i}{2}\right)$$

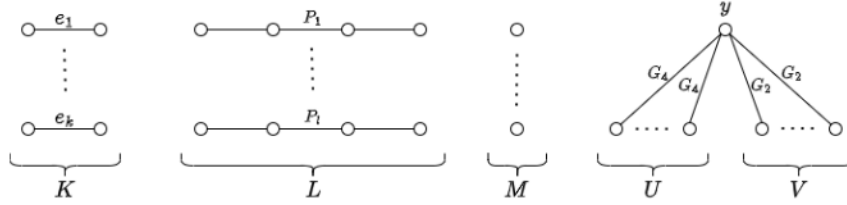


Figure 3: The structure of $\mathcal{F}[x]$ in Case 4

is at most $-\frac{1}{2}$. Since $|\mathcal{F}| \geq f(n)$ we must have at least one near optimal component. In such a component, Lemma 16 and (12) imply that G_3^C consists of vertex disjoint complete graphs (including isolated vertices, of course).

9.6.1 Case 4: There is a pendant G_4 edge in a near optimal component

In this subsection we suppose that C is a near optimal component and $x \in C$ is a degree one vertex of G_4 . Let y be its unique neighbor in G_4 . Since we have already discussed the case $\deg_{\mathcal{F}}(xy) = 0$ (Case 2) and $\deg_{\mathcal{F}}(xy) = 1$ (Case 3) we may suppose that $\deg_{\mathcal{F}}(xy) = q \geq 2$.

Consider the structure of the link $\mathcal{F}[x]$, similar to what we did in Case 1. There are two further forms of components in $\mathcal{F}[x]$ in addition to forms I and II which appear in K and L , respectively.

III A star S_q centered at y and having q leaves, $q \geq 2$.

There is exactly one such component in $\mathcal{F}[x]$. The leaves of S_q can be partitioned into two (possibly empty) parts:

$$\begin{aligned} U &:= \{u: xyu \in \mathcal{F}_1, xu \in E(G_1), yu \in E(G_4)\}, \\ V &:= \{v: xyv \in \mathcal{F}_2, xv, yv \in E(G_2)\}. \end{aligned}$$

We write $|U| = s$, $|V| = t$, $s + t = q$.

IV An isolated vertex z .

We let $m \geq 0$ be the number of isolated vertices, and denote by M the set of isolated vertices. We remark that isolated vertices in $\mathcal{F}[x]$ are now permissible because x has neighbors in G_1 (provided that $s > 0$), which allows the isolated vertices to be neighbors of x in G_3 .

We summarize: $\mathcal{F}[x]$ looks as in Figure 3, and $[n] \setminus \{x\} = K \cup L \cup M \cup \{y\} \cup U \cup V$ with $K = e_1 \cup \dots \cup e_k$ and $L = P_1 \cup \dots \cup P_\ell$.

Our next arguments exploit and combine structural information about $\mathcal{F}[x]$ and about G_1^C, G_3^C, G_4^C based on Lemma 16. Since the G_1 and G_3 neighbors of x are in the same component, we already know that

(C1) $\{x, y\}, M, U \subseteq C$.

Since G_3^C consists of cliques and the G_3 neighborhood of x is exactly M we get

(C2) $M \cup \{x\}$ is a (maximal) clique in G_3 .

Similarly, since U is the G_1 neighborhood of x , by definition

(C3) U is a clique in G_3 .

It follows from Lemma 15 that no two vertices in $M \cup \{x\}$ belong together to a triple in $\mathcal{F}$ and similarly U meets every triple in $\mathcal{F}$ in at most one vertex.

(C4) We may assume that $K = \emptyset$.

Proof. Suppose that $k \geq 1$. First we show that the constraints (K1) and (K2) of Lemma 14 are valid for $\mathcal{F}$. Suppose, on the contrary to (K1), that there is a $vwz \in \mathcal{F}$, $K \cap vwz = \{v\}$, $v \in e_i$. We have $x \notin vwz$, because we know $\mathcal{F}[x]$. If $wz \cap L \neq \emptyset$ then we get an **A** configuration. If one of w, z is y , say $w = y$, we can choose a $z' \in U \cup V$, $z' \neq z$ (since $|U \cup V| \geq 2$) and find the **A** configuration $e_i \cup \{x\}, vyz, xyz'$. Thus we may assume that $w, z \in M \cup U \cup V$, and by (C2) at least one of them, say z , is in $U \cup V$. But now $e_i \cup \{x\}, vwz, xyz$ form an **A**. To check (K2) is obvious. Applying (2), we get

$$\varepsilon(n) := f(n) - \lfloor \frac{1}{4}(n-1)^2 \rfloor \leq |\mathcal{F}| - \lfloor \frac{1}{4}(n-1)^2 \rfloor \leq |\mathcal{H}| - \lfloor \frac{1}{4}(n-2k-1)^2 \rfloor,$$

implying $|\mathcal{H}| \geq f(n-2k) - \varepsilon(n-2k) + \varepsilon(n)$. Thus $|\mathcal{H}| \geq f(n-2k)$. By induction, this must hold with equality and $\mathcal{H} \in \mathbf{F}_{(n-2k)}$. Also (2) holds with equality, so $\mathcal{F}$ is obtained from $\mathcal{H}$ by a sequence of 2- and 4-extensions, i.e., $\mathcal{F} \in \mathbf{F}_{(n)}$. $\square$

(C5) $L = \emptyset$.

Proof. Suppose that $\ell \geq 1$. By a case check similar to the proof of (C4), one can verify that any triple $vwz \in \mathcal{F}$ that meets P_1 in a single vertex creates an **A** configuration. Thus, the only triples in $\mathcal{F}$ that meet P_1 are those in its tight 5-cycle. Let $\mathcal{H}$ be the rest of the triples. We obtain

$$f(n) \leq |\mathcal{F}| = |\mathcal{H}| + 5 \leq f(n-4) + 5.$$

This is a contradiction, noting that under our conditions $n \geq 8$. $\square$

(C6) We may assume that $M \neq \emptyset$, therefore $U \neq \emptyset$.

Indeed, $M = \emptyset$ implies $[n] = \{x, y\} \cup U \cup V$ with $\deg_{\mathcal{F}}(xy) = n-2$. As we have seen before Lemma 14, this implies that $\mathcal{F}$ is a 2-extension of $\mathcal{F} \setminus \{x, y\}$, and we are done by induction.

(C7) y is an isolated vertex in G_1^C .

Indeed, suppose that $yz \in E(G_1^C)$. Clearly $z \in M$. Every $z \in M$ has a G_1 neighbor in U , say $zw \in E(G_1)$. Then, by definition, the two G_1 edges yz and zw place yw in G_3 . However, yw is a G_4 pair.

Since y is isolated in G_1^C , according to the structure of G_1^C described in Lemma 16, only cases (L2) or (L4) are possible, so every other component of G_1^C is a (balanced or almost balanced) complete bipartite graph. Consider the G_1 component $\tilde{C}$ to which x belongs. We know from $\mathcal{F}[x]$ that $N_{G_1^C}(x) = U$ and $N_{G_3^C}(x) = M$. Thus, $\tilde{C}$ is a complete bipartite graph of G_1 -edges with parts $\{x\} \cup M$ and U .

Let us conclude this case with the following statement.

Claim 18. *Suppose that $|C| = p$ is even and C is a near optimal component with a pendant G_4 edge. Then equality holds in (12), i.e., $|E(G_1^C)| - |E(G_3^C)| - \frac{p}{2} = -1$.*

Proof. By (C7) the graph G_1^C has an isolated vertex (namely y). Then (L1) is not possible in Lemma 16. $\square$

9.6.2 Case 5: The G_4 edges form a tree in a near optimal component

It is, in fact a subcase and a continuation of Case 4. We maintain the assumptions and conclusions of Case 4, and add the assumption that G_4^C is a tree. Again let x be a degree one vertex of G_4^C , and let y be its unique neighbor in G_4 .

(C8) The G_4 neighborhood of y is $\{x\} \cup M \cup U$.

Proof. We already know that the vertices in $\{x\} \cup U$ are in the G_4 neighborhood of y , and those in V are not. As $K = L = \emptyset$, it remains to show that $M \subseteq N_{G_4}(y)$.

The arguments leading to (C7) can be carried out for any choice of a leaf of G_4^C . We may assume that all such leaves are incident to G_{42} edges, otherwise we are done. Thus, for every leaf of G_4^C its unique neighbor in G_4^C is an isolated G_1^C vertex. If there is only one isolated vertex in G_1^C (namely y), then it must be that vertex, regardless of the choice of the leaf of G_4^C . This implies that G_4^C is a star, and it is centered at y . In particular, $M \subseteq N_{G_4}(y)$.

Consider the case when G_1^C has at least two isolated vertices, y and y' . The near optimality of C and Lemma 16 imply that only the case (L4) is possible, there are no more isolated vertices. By (C7) each leaf of the tree G_4^C is adjacent to an isolated vertex of G_1^C . So the tree G_4^C consists of two stars with centers y and y' and a path connecting these centers. In particular, all but one of the G_4^C neighbors of y are leaves. According to (L4) the non-isolated components of G_1^C are balanced complete bipartite graphs, in particular such is $\tilde{C} = K(\{x\} \cup M, U)$, so $|U| = |\{x\} \cup M| \geq 2$. All vertices in U are G_4^C neighbors of y , so at least one of them, say u , is a leaf in G_4^C adjacent to y . Now consider any vertex $z \in M$. As $zu \in E(G_1^C)$, z and u must have a common G_4^C neighbor, which can only be y . This shows that $z \in N_{G_4}(y)$, completing the proof. $\square$

(C9) For every pair ab with $a \in \{x\} \cup M$, $b \in U$, the unique triple in $\mathcal{F}$ containing it is aby .

Indeed, suppose abz is that unique triple. Then $az, bz \in E(G_4^C)$, so z is the middle vertex of a path of length 2 in G_4^C . Since G_4^C is a tree, this path $a-z-b$ is the unique path connecting a to b in the tree. But ay and by are tree edges by (C8) so $z = y$. $\square$

(C10) $|U| \geq 2$.

Indeed, by Lemma 16 and the near optimality of C we know that $K(\{x\} \cup M, U)$ is balanced or almost balanced. We have $(|M| + 1) - |U| \leq 1$, so $|U| \leq 1$ gives $|M| \leq 1$. Writing $M = \{z\}$, $U = \{u\}$, we have $zuy \in \mathcal{F}$ by (C9). As zu is the only own pair of this triple, there must be another triple $zyv \in \mathcal{F}$. The only remaining options are $v = x$ or $v \in V$. Both are impossible: the former because $\deg_{\mathcal{F}}(xz) = 0$, the latter because the pair yv , $v \in V$, is an own pair of xyv . $\square$

Next, we give an upper bound on the number of triples in $\mathcal{F}$.

- Triples containing no vertex of V :

Since $\{x\} \cup M$ and U are cliques in G_3 , any triple in $\mathcal{F}$ can contain at most one vertex from each of them (Lemma 15). Thus triples avoiding V must be of the form aby with $a \in \{x\} \cup M$, $b \in U$. By (C9) all these are indeed triples in $\mathcal{F}$, so we have $(m + 1)s$ of them.

- Triples containing exactly one vertex of V :

There are t such triples containing xy . There are no other such triples. Indeed, any other such triple would have to be of the form zuv , $z \in M, u \in U, v \in V$. But then, taking $w \in U \setminus \{u\}$, we get an **A** configuration: zuv, zwy, xyv .

- Triples containing exactly two vertices of V :

There are no such triples. Indeed, suppose zvw is such a triple, with $v, w \in V$. Then $z \neq x$, because vw is not an edge of $\mathcal{F}[x]$. If $z = y$ we get a **C** configuration: yvw, xyv, xyw . If $z \in M$ then taking $u \in U$ we get an **A** configuration: zvw, zuv, xyv . Finally, suppose $z \in U$. Since zy is not an own pair of xyz , there is another triple $zya \in \mathcal{F}$ containing it. The vertex a cannot be in V (by the previous paragraph), so we get an **A** configuration: zvw, zya, xyv .

- Triples contained in V :

By the induction hypothesis, there are at most $f(t)$ such triples.

Altogether, we get that the number of triples in $\mathcal{F}$ is at most $(m + 1)s + t + f(t)$. Using $(m + 1) + s = n - t - 1$, $n \geq t + 5$, and the notation $\delta(t) := f(t) - \frac{1}{4}(t - 1)^2$ with the convention $\delta(0) = -\frac{1}{4}$, we obtain

$$\begin{aligned} |\mathcal{F}| - \frac{1}{4}(n - 1)^2 &\leq -\frac{1}{4}(n - 1)^2 + \frac{1}{4}(n - t - 1)^2 + t + \frac{1}{4}(t - 1)^2 + \delta(t) \\ &= \frac{1}{4}(2t^2 + 4t + 1 - 2nt) + \delta(t) \leq \frac{1}{4}(1 - 6t) + \delta(t). \end{aligned}$$

Note that $\delta(t) = -\frac{1}{4}$ for even t , and for odd t we have $\delta(1) = \delta(3) = 0$ and $\delta(t) = 1$ when $t \geq 5$. We know that $|\mathcal{F}| - \frac{1}{4}(n-1)^2 \geq \delta(n) \geq -\frac{1}{4}$, whereas the upper bound $\frac{1}{4}(1-6t) + \delta(t)$ is at most $-\frac{5}{4}$ for $t \geq 1$. So we must have $t = 0$ and n even. In this case, as we have seen, $\mathcal{F}$ consists of all triples aby with $a \in \{x\} \cup M$, $b \in U$. Thus $\mathcal{F}$ is the construction based on the Turán graph with the vertex y common to all triples, i.e., $\mathcal{F} = \mathcal{F}_n^*$, completing the proof of $\mathcal{F} \in \mathbf{F}_{(n)}$ in Case 5. $\square$

9.7 Proof of Theorem 13: the end

Recall that we partitioned $[n]$ into the connected components $C_1, \dots, C_r$ of G_4 of sizes $p_1, \dots, p_r$, $p_i \geq 2$ for all i . We may also suppose that each pendant G_4 edge is a G_{42} edge.

9.7.1 Case 6: The opposite of Case 5

This is the last case, and we will show that it leads to a contradiction. We suppose that for each component C of size p either G_4^C is not a tree, i.e., $p - |E(G_4^C)| \leq 0$, call this Case 6/1, or it is not near optimal, i.e., $|E(G_1^C)| - \frac{p-2}{2} < |E(G_3^C)|$ (Case 6/2). Note that in both cases

$$\lceil \frac{p}{2} \rceil - |E(G_4^C)| + |E(G_1^C)| - |E(G_3^C)| \leq 0. \quad (13)$$

Indeed, Lemma 16 states $|E(G_1^C)| - \lfloor \frac{p}{2} \rfloor \leq |E(G_3^C)|$. Adding this to $p - |E(G_4^C)| \leq 0$ we obtain (13) in Case 6/1. In Case 6/2 we add $|E(G_1^C)| - \frac{p-3}{2} \leq |E(G_3^C)|$ to the obvious $p - 1 - |E(G_4^C)| \leq 0$ and get $\frac{p+1}{2} - |E(G_4^C)| + |E(G_1^C)| - |E(G_3^C)| \leq 0$.

Hence in the sum in the last line of (11) each term $\sigma(C_i)$ is non-positive. Even more, in Case 6/1 we have $\sigma(C_i) \leq -\frac{1}{2}$ when p_i is odd, and $\sigma(C_i)$ is negative in Case 6/2 as well. Therefore in Case 6 we get $|\mathcal{F}| \leq \frac{1}{4}(n^2 - 2n) \leq f(n)$. Here equality must hold, so to avoid negative terms each p_i should be even and Case 6/2 is also excluded.

So from now on, we may suppose 6/1 holds with equality in (13) for each i . So each p_i is even, $p_i = |E(G_4^{C_i})|$, and equality holds in Lemma 16, i.e., (L1) holds. Thus C_i contains a perfect matching of G_1 edges. Then Claim 18 implies that $G_4^{C_i}$ has no pendant edge, each degree is at least 2. So $G_4^{C_i}$ should be 2-regular, connected, i.e., each $G_4^{C_i}$ is an even cycle, $p_i \geq 4$.

We are going to show that each p_i is exactly 4. Let the vertex set of C be $[p]$ with G_4^C edges $12, 23, \dots, (p-1)p$, and $p1$. The G_1^C edges are diagonals of length 2 in G_4^C . They also contain a matching of size $\frac{p}{2}$, and this is only possible if there are crossing edges like 13 and 24 with $123, 234 \in \mathcal{F}_1$.

Suppose that there exists a triple containing 14, say $14z \in \mathcal{F}$. Since 13 and 24 are covered exactly once by the triples of $\mathcal{F}$ we have $z \notin \{1, 2, 3, 4\}$. The pair 12 is not a G_{41} edge (because 123 does not have two own pairs) so it is a G_{42} edge, there exists a triple $12w \in \mathcal{F}$, $w \neq 3$, containing it. The triples $12w, 14z, 234$ form an **A** configuration unless $w \in \{4, z\}$. To avoid **C** we have $w \neq 4$, so $w = z$ and $12z \in \mathcal{F}$. One can have the same argument starting with the pair 34 in place of 12 and obtain $34z \in \mathcal{F}$. Then we got a tight 5-cycle with vertex set $\{1, 2, 3, 4, z\}$, a contradiction, because 12, etc., are G_4 edges.

We conclude that the pair 14 is uncovered. If it is a G_3 edge then there are two G_1 pairs $w1$ and $w4$, both diagonals of length 2. But this is impossible since p is even and the arc 1234 is of odd length. So 14 is a G_{40} edge. Thus 1234 form a 4-cycle in G_4 , $p = 4$. This must be true for each component, so G_4 is a vertex disjoint union of 4-cycles such that each contains two triples and a G_{40} pair. So G_1 is a perfect matching and there is no G_3 edge at all.

Consider any 4-cycle C in G_4 , say with $123, 234 \in \mathcal{F}$, $14 \in E(G_{40})$ and $12, 23, 34 \in E(G_{42})$. We have a triple $12z \in \mathcal{F}$ with $z \neq 3$, and (to avoid **C**) $z \neq 4$. Every pair $4z$ with $z \notin C$ is contained in a triple, so we have some triple $4zw \in \mathcal{F}$. The triples $12z, 234, 4zw$ form an **A** configuration unless $w \in \{1, 2, 3\}$. The case $w = 1$ is not allowed because $14 \in E(G_{40})$, and the case $w = 2$ is not allowed because $24 \in E(G_1)$. So $w = 3$, $34z \in \mathcal{F}$. The triples $12z, 34z$ have no G_1 pair, hence their pairs $1z, 2z, 3z, 4z$ are G_2 pairs. The pair 23 cannot appear in any triple in $\mathcal{F}$ except 123, 234. Indeed, $23z$ is excluded because $2z, 3z$ are own pairs of other triples, and $23w$ for $w \notin C \cup \{z\}$ creates the **A** configuration $12z, 34z, 23w$. We conclude that the set of triples in $\mathcal{F}$ that meet C in exactly two vertices is of the form $\{12z, 34z : z \in Z_C\}$ where Z_C is a non-empty set disjoint from C .

Next, we discuss tight 5-cycles. We claim that they must be vertex disjoint. Indeed, any two of them can meet in at most one vertex. If two of them on vertex sets R_1, R_2 share a vertex x , then in order to avoid an **A** configuration all 16 pairs ab , $a \in R_1 \setminus \{x\}$, $b \in R_2 \setminus \{x\}$ must be uncovered. However, the only uncovered pairs are G_{40} edges and they are pairwise disjoint.

Now, for any two distinct 4-cycles C_i, C_j in G_4 , we write $C_i \rightarrow C_j$ if $Z_{C_i} \cap C_j \neq \emptyset$. As shown above, in this case all pairs in $C_i \times (Z_{C_i} \cap C_j)$ are G_2 edges. For any $i \neq j$, exactly one of $C_i \rightarrow C_j$ and $C_j \rightarrow C_i$ holds. Indeed, if both held then pairs in $(Z_{C_j} \cap C_i) \times (Z_{C_i} \cap C_j)$ would be G_2 edges contained in two triples in $\mathcal{F}$. If neither held then all pairs in $C_i \times C_j$ would have to be G_R edges (nothing else is allowed), which is impossible because adjacent G_R edges must come from the same tight 5-cycle (as those are disjoint).

We have thus defined a tournament T on $\{C_1, \dots, C_r\}$. As $Z_{C_i} \neq \emptyset$ for every i , every C_i has at least one out-neighbor in T , so T cannot be transitive. But then it contains a directed triangle, which gives rise to an **A** configuration. This contradiction completes the proof. $\square$

10 Excluding A, B and D

Recall that $\text{ex}(n, \text{ABCD}) = \lfloor \frac{1}{8}n^2 \rfloor$ for all $n \geq 1$ and in Theorem 1 in Section 2 we have described $\mathcal{F}_{n,k}$, the extremal families. In this section we show that for large enough n , excluding **A**, **B** and **D** has the same effect on the size of a triple system as excluding all triangles. For finitely many cases $\text{ex}(n, \text{ABD})$ can exceed $\lfloor \frac{1}{8}n^2 \rfloor$, e.g., for $n = 4$ the complete triple system $K_4^3 := \binom{[4]}{3}$ is **ABD**-free, while $\lfloor \frac{1}{8}n^2 \rfloor = 2$. For $n = 7$ two K_4^3 sharing a vertex give $\text{ex}(7, \text{ABD}) \geq 8$ (and here equality holds) while $\lfloor \frac{1}{8}n^2 \rfloor = 6$. For $n = 16$ the family of 8 K_4^3 occupying the rows and columns of a 4×4 square lattice is **ABD**-free and has a completely different structure than $\mathcal{F}_{n,k}$. Denote this triple system by $\mathcal{Q}^{4 \times 4}$, and note that $|\mathcal{Q}^{4 \times 4}| = \lfloor \frac{1}{8}n^2 \rfloor = 32$.

Theorem 19.

$$\text{ex}(n, \text{ABD}) = \lfloor \frac{1}{8}n^2 \rfloor \text{ for } n \geq 8.$$

Moreover, for $n \geq 11$ the families $\mathcal{F}_{n,k}$ described in Section 2 are the only extremal systems, except that $\mathcal{Q}^{4 \times 4}$ is also extremal in the case $n = 16$.

For $n = 8, 9, 10$ here is the complete list of extremal families.

- $n = 8$: $\mathcal{F}_{8,2} = \{12x : 5 \leq x \leq 8\} \cup \{34x : 5 \leq x \leq 8\}$,
 $\mathcal{Q}_8^2 =$ two K_4^3 sharing a vertex (this triple system has an isolated vertex),
 $\mathcal{Q}_8^3 =$ two vertex disjoint K_4^3 .
 $n = 9$: $\mathcal{F}_{9,2} = \{12x : 5 \leq x \leq 9\} \cup \{34x : 5 \leq x \leq 9\}$,
 $\mathcal{Q}_9^2 =$ two K_4^3 on 1234 and 1567 with two additional triples 289, 589.
 $n = 10$: $\mathcal{F}_{10,2} = \{12x : 5 \leq x \leq 10\} \cup \{34x : 5 \leq x \leq 10\}$,
 $\mathcal{F}_{10,3} = \{12x : 7 \leq x \leq 10\} \cup \{34x : 7 \leq x \leq 10\} \cup \{56x : 7 \leq x \leq 10\}$,
 $\mathcal{Q}_{10}^3 =$ three K_4^3 sharing a vertex,
 $\mathcal{Q}_{10}^4 =$ three K_4^3 forming a path, $\{1234, 4567, 789X\}$.

10.1 Non-uniform triangle-free hypergraphs

Györi [17] proved his upper bound in the following more general form. Let $\mathcal{H}$ be a triangle-free multi-hypergraph on $n \geq 100$ vertices. Then

$$\sum_{E \in \mathcal{H}} (|E| - 2) \leq \lfloor \frac{1}{8}n^2 \rfloor. \quad (14)$$

In this formulation, $\mathcal{H}$ may contain edges of different sizes, and may contain multiple copies of the same set of vertices. Such multiple copies are considered distinct for the purpose of forming a triangle, and are counted with multiplicity in the summation in (14).

Given an ABD-free triple system $\mathcal{F}$ one can prove $|\mathcal{F}| \leq \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 100$ using (14) as follows. Let $\mathcal{Q} := \{Q_1, \dots, Q_q\}$ be all the vertex sets of $\mathbf{C}$ configurations in $\mathcal{F}$, i.e., $|Q_i| = 4$ and at least three triples within Q_i are in $\mathcal{F}$. Note that distinct Q_i can share at most one vertex, otherwise we get a $\mathbf{D}$ configuration. Let $\mathcal{H}$ be the multi-hypergraph having the following edges: two copies of each Q_i , $i = 1, \dots, q$ and one copy of each $F \in \mathcal{F}$ such that $F \not\subseteq Q_i$, $i = 1, \dots, q$. Note that

$$|\mathcal{F}| \leq 4|\mathcal{Q}| + |\mathcal{F} \setminus (\partial\mathcal{Q})| = \sum_{E \in \mathcal{H}} (|E| - 2). \quad (15)$$

Indeed, the inequality holds because each Q_i contains at most 4 triples in $\mathcal{F}$. The equality holds because each Q_i contributes $(4 - 2) + (4 - 2) = 4$ to the sum, and every other triple in $\mathcal{F}$ contributes $3 - 2 = 1$. Upon checking that $\mathcal{H}$ is triangle-free, (14) implies the desired bound $\lfloor \frac{1}{8}n^2 \rfloor$ for the right-hand side of (15).

The above proof does not yield the extremal systems and only works for $n \geq 100$.

Definition 20. A (simple) hypergraph $\mathcal{H}$ is called a $\mathcal{Q}4$ -family on $[n]$ if there is a partition $\mathcal{H} = \mathcal{Q} \cup \mathcal{T} \cup \mathcal{E}$ such that

- $\mathcal{Q} \subseteq \binom{[n]}{4}$, $\mathcal{T} \subseteq \binom{[n]}{3}$, $\mathcal{E} \subseteq \binom{[n]}{2}$,
- $\mathcal{H}$ is triangle-free, and
- $|H \cap H'| \leq 1$ for any two distinct members $H, H' \in \mathcal{H}$.

Lemma 21. *Let $\mathcal{H}$ be a $Q4$ -family on $[n]$ with $\mathcal{Q} \neq \emptyset$ and suppose $n \geq 15$. Then*

$$8|\mathcal{Q}| + \frac{7}{2}|\mathcal{T}| + |\mathcal{E}| < 2\lfloor \frac{1}{8}n^2 \rfloor \quad (16)$$

except in the case when $\mathcal{T} = \mathcal{E} = \emptyset$ and $\partial\mathcal{Q}$ is the triple system $\mathcal{Q}^{4 \times 4}$.

This is a simplified version of an inequality from [17] where Győri considered $8|\mathcal{Q}| + 4|\mathcal{T}| + |\mathcal{E}|$. Although (16) does not seem to imply (14), and it does not apply to multi-hypergraphs, its proof is simpler and valid for $15 \leq n < 100$, too.

Proof. Let $\sigma(\mathcal{H}) := 8|\mathcal{Q}| + \frac{7}{2}|\mathcal{T}| + |\mathcal{E}|$, and $\sigma_3(n) := \max\{\sigma(\mathcal{H}) : \mathcal{H} \text{ is a } Q4\text{-family on } [n] \text{ with } \mathcal{Q} = \emptyset\}$. Since a complete bipartite graph can be considered as a $Q4$ -family we have $\sigma_3(n) \geq \lfloor \frac{1}{4}n^2 \rfloor$. As a first step we prove that for all $n \geq 0$

$$\frac{7}{2}|\mathcal{T}| + |\mathcal{E}| \leq \frac{1}{4}(n^2 + 5). \quad (17)$$

Indeed, $\sigma_3(0) = \sigma_3(1) = 0$, $\sigma_3(2) = 1 = \frac{1}{4}2^2$, $\sigma_3(3) = \frac{7}{2} = \frac{1}{4}(3^2 + 5)$. By inspection we can see that $\sigma_3(n)$ is maximized for $n = 4, 5, 6$ when $\mathcal{H}$ consists of a single triple and an edge, two triples sharing a vertex, or two disjoint triples joined by three disjoint pairs. This gives $\sigma_3(4) = \frac{9}{2} = \frac{1}{4}(4^2 + 2)$, $\sigma_3(5) = 7 = \frac{1}{4}(5^2 + 3)$, and $\sigma_3(6) = 10 = \frac{1}{4}(6^2 + 4)$.

For $n \geq 7$ let $T_1, \dots, T_\ell$ be a maximal family of pairwise disjoint triples from $\mathcal{T}$, let $L = \cup T_i$, $|L| = 3\ell$, and $\bar{L} := [n] \setminus L$. The case $\ell = 0$ is obvious, so we may suppose that $\ell \geq 1$. Make a tally of the pairs in $G := (\partial\mathcal{T}) \cup \mathcal{E}$. Every vertex $x \in [n] \setminus T_i$ has at most one G -neighbor in T_i , so G can have at most $3\ell + 3\binom{\ell}{2}$ edges in L , $\ell(n - 3\ell)$ edges joining L and $\bar{L}$. Since G is triangle-free on $\bar{L}$ it can have at most $\frac{1}{4}(n - 3\ell)^2$ edges in it. Define weights $w(e)$ on the edges of G as follows. The weight $w(e) = \frac{7}{6}$ if e lies in L , $w(e) = \frac{5}{4}$ if e joins L and $\bar{L}$, and finally $w(e) = 1$ for edges in $\bar{L}$. The sum of the weights of the three pairs in $T \in \mathcal{T}$ is $3 \times \frac{7}{6}$ if $T \subseteq L$, it is $\frac{7}{6} + 2 \times \frac{5}{4}$ if $|T \cap L| = 2$, it is $2 \times \frac{5}{4} + 1$ if $|T \cap L| = 1$, and there is no $T \in \mathcal{T}$ such that $|T \cap L| = 0$. So every triangle T in G belongs to $\mathcal{T}$ and receives at least $\frac{7}{2}$ total weight. Hence

$$\begin{aligned} \frac{7}{2}|\mathcal{T}| + |\mathcal{E}| &\leq \frac{7}{6} \left(3\ell + 3\binom{\ell}{2} \right) + \frac{5}{4}\ell(n - 3\ell) + \frac{1}{4}(n - 3\ell)^2 \\ &= \frac{1}{4}(n^2 + \ell^2 - n\ell + 7\ell). \end{aligned} \quad (18)$$

The polynomial $p_n(\ell) := \ell^2 - n\ell + 7\ell$ attains its maximum over $0 \leq \ell \leq \lfloor \frac{n}{3} \rfloor$ at $\ell = 0$ or $\ell = \lfloor \frac{n}{3} \rfloor$, so it is at most 4 for $n = 7$, at most 2 for $n = 8$, at most 3 for $n = 9$, and at most 0 for $n \geq 10$. This completes the proof of inequality (17).

Next, consider the case $|\mathcal{Q}| \geq 1$ and let $Q_1, \dots, Q_k$ be a maximal family of pairwise disjoint quadruples from $\mathcal{Q}$, let $K = \cup Q_i$, $|K| = 4k$, and $\overline{K} := [n] \setminus K$. Define the hypergraph $\mathcal{H}_3$ on the vertex set $\overline{K}$ as

- the triples of $\mathcal{T}$ and the pairs of $\mathcal{E}$ contained in $\overline{K}$, and
- the triples of the form $Q \cap \overline{K}$ when $Q \in \mathcal{Q}$, $|Q \cap K| = 1$, and
- the pairs of the form $Q \cap \overline{K}$ when $Q \in \mathcal{Q}$, $|Q \cap K| = 2$, and
- the pairs of the form $T \cap \overline{K}$ when $T \in \mathcal{T}$, $|T \cap K| = 1$.

This is a Q_4 -family without any quadruple, so (17) gives

$$\sigma(\mathcal{H}_3) \leq \frac{1}{4}((n - 4k)^2 + 5).$$

Let $\partial_2 \mathcal{H}$ be the set of pairs that are contained in some edge of $\mathcal{H}$. Define the graph G as the set of pairs in $\partial_2 \mathcal{H}$ meeting K . As before, G can have at most $6k + 4\binom{k}{2}$ edges in K and $k(n - 4k)$ edges joining K and $\overline{K}$. Define weights $w(e)$ on the edges of G as follows. The weight $w(e) = \frac{4}{3}$ if e lies in K , $w(e) = \frac{3}{2}$ if e joins K to $\overline{K}$. Define weights on the members of $\mathcal{H}_3$ as needed, $w(S) = \frac{7}{2}$ for triples in $\mathcal{H}_3$ and $w(e) = 1$ for pairs in $\mathcal{H}_3$.

It is easy to check that the total weight in each member of $Q \in \mathcal{Q}$ is at least 8, it is at least $\frac{7}{2}$ for the members of $\mathcal{T}$, and it is at least 1 for pairs in $\mathcal{E}$. Indeed, if $Q \subseteq K$ then it gets $6 \times \frac{4}{3} = 8$, if $|Q \cap K| = 3$ then it gets $3 \times \frac{4}{3} + 3 \times \frac{3}{2}$, if $|Q \cap K| = 2$ then it gets $\frac{4}{3} + 4 \times \frac{3}{2} + 1$, and if $|Q \cap K| = 1$ then it gets $3 \times \frac{3}{2} + \frac{7}{2}$. If $T \in \mathcal{T}$ and $T \subseteq K$ then it gets $3 \times \frac{4}{3}$, if $|T \cap K| = 2$ then it gets $\frac{4}{3} + 2 \times \frac{3}{2}$, if $|T \cap K| = 1$ then it gets $2 \times \frac{3}{2} + 1$, and if $T \cap K = \emptyset$ then its weight is $\frac{7}{2}$. Finally, the weight of the (relevant) pairs is at least 1. Hence

$$8|\mathcal{Q}| + \frac{7}{2}|\mathcal{T}| + |\mathcal{E}| \leq \sum_{e \in G} w(e) + \sigma_3(n - 4k) \quad (19)$$

$$\begin{aligned} &\leq \frac{4}{3} \left(6k + 4\binom{k}{2} \right) + \frac{3}{2}k(n - 4k) + \frac{1}{4}((n - 4k)^2 + 5) \quad (20) \\ &= \frac{1}{4}n^2 + \frac{1}{12}(8k^2 - 6kn + 64k + 15). \end{aligned}$$

Here $2k(4k - n + 32 - 2n) + 15 < -12$ for $n \geq 18$ (and $1 \leq k \leq \lfloor \frac{n}{4} \rfloor$), it is less than -3 for $n = 17$, and it is negative for $n = 16$ and $k = 1, 2, 3$.

For the case $n = 16$ and $k = 4$ we have $K = V(G)$ and G has 24 edges in $Q_1, \dots, Q_4$ and at most $4 \times \binom{4}{2} = 24$ further edges. Recall that the weight of each edge is $\frac{4}{3}$. So every $Q \in \mathcal{Q}$ gets weight 8, $T \in \mathcal{T}$ gets weight 4, and a pair $E \in \mathcal{E}$ has weight $\frac{4}{3}$. So $\sigma(\mathcal{H}) \leq 48 \times \frac{4}{3} = 64$, as claimed. Here equality can hold only if $\mathcal{T}$ and $\mathcal{E}$ are both empty, and $|\mathcal{Q}| = 8$. Then $\mathcal{Q}$ must have the lattice structure.

Finally, in the case $n = 15$ we use (19) directly. By (18) we have $\sigma_3(15 - 4) \leq \frac{1}{4}11^2$ so in the case $(n, k) = (15, 1)$ instead of (20) we have $\sigma(\mathcal{H}) \leq \frac{1}{4}n^2 + \frac{1}{12}(8k^2 - 6kn + 64k)$ which

is less than $2\lfloor \frac{1}{8}n^2 \rfloor$. Similarly, in the case $(n, k) = (15, 2)$ using $\sigma_3(15 - 2 \times 4) \leq \frac{1}{4}(7^2 + 4)$ leads to $\sigma(\mathcal{H}) < 2\lfloor \frac{1}{8}n^2 \rfloor$.

In the case $(n, k) = (15, 3)$ we define another weight function on the edges of $\partial_2\mathcal{H}$. We assign weight $\frac{4}{3}$ to each pair. With this weighting every member of $\mathcal{Q}$ receives a total weight 8, a triple in $\mathcal{T}$ gets 4 and a pair from $\mathcal{E}$ gets $\frac{4}{3}$. The graph $\partial_2\mathcal{H}$ has 18 edges covered by Q_1, Q_2 , and Q_3 , there are at most 12 edges between these three quadruples, there are at most 3×3 edges from $\overline{K}$ to K , and at most $\binom{|\overline{K}|}{2} = 3$ edges in $\overline{K}$. Hence $|\partial_2\mathcal{H}| \leq 42$ and $\sigma(\mathcal{H}) \leq 56 = 2\lfloor \frac{1}{8}n^2 \rfloor$. However, here equality can hold only if all the 42 pairs of $\partial_2\mathcal{H}$ belong to members of $\mathcal{Q}$. But it is not possible to add to Q_1, Q_2, Q_3 four more quadruples that form with them a Q_4 -family. This completes the proof of the lemma. $\square$

10.2 Proof of Theorem 19

Let $\mathcal{F}$ be an ABD-free triple system on $[n]$ with $|\mathcal{F}| \geq \lfloor \frac{1}{8}n^2 \rfloor$, $n \geq 8$. Our aim is to prove that $|\mathcal{F}| = \lfloor \frac{1}{8}n^2 \rfloor$ and $\mathcal{F}$ is among the families prescribed earlier. We start as in the beginning of Subsection 10.1.

Let $\mathcal{Q} := \{Q_1, \dots, Q_q\}$ be all the vertex sets of $\mathbf{C}$ configurations in $\mathcal{F}$, i.e., $|Q_i| = 4$ and at least three triples within Q_i are in $\mathcal{F}$. Define $\mathcal{F}_3$ as the set of triples from $\mathcal{F}$ having three own pairs, and let $\mathcal{F}_2 := \mathcal{F} \setminus (\partial\mathcal{Q} \cup \mathcal{F}_3)$. Each member of $\mathcal{F}_2$ has exactly two own pairs, the set of these are denoted by $\mathcal{E}$. Then the family $\mathcal{H} = \mathcal{Q} \cup \mathcal{F}_3 \cup \mathcal{E}$ is a Q_4 -family.

If $\mathcal{Q} = \emptyset$ then $\mathcal{F}$ is ABCD-free and Theorem 1 can be applied. So from now on, we may suppose that $q \geq 1$. In case of $n \geq 15$ we can apply Lemma 21. We get

$$2|\mathcal{F}| \leq 8|\mathcal{Q}| + 2|\mathcal{F}_3| + |\mathcal{E}| \leq \sigma(\mathcal{H}) < 2\lfloor \frac{1}{8}n^2 \rfloor$$

unless $\mathcal{F} = \mathcal{Q}^{4 \times 4}$. So the proof is complete for $n \geq 15$.

From now on we may suppose that $n \leq 14$ (and $q \geq 1$). These are finitely many cases so one can check them with a computer. For completeness we sketch a case by case check in the next subsection. $\square$

10.2.1 The case $8 \leq n \leq 14$

Starting with $Q_1 = 1234$ and adding new quadruples one by one it can be seen that there are 16 possibilities for $\mathcal{Q}$. The list is presented below (also see Figure 4). κ denotes $|\cup \mathcal{Q}|$, $U := \cup \mathcal{Q}$, and i' denotes the element $10 + i$.

1. $q = 1, \kappa = 4, \mathcal{Q}_1 = \{1234\}$
2. $q = 2, \kappa = 7, \mathcal{Q}_2 = \{1234, 1567\}$
3. $q = 2, \kappa = 8, \mathcal{Q}_3 = \{1234, 5678\}$
4. $q = 3, \kappa = 10, \mathcal{Q}_4 = \{1234, 1567, 1890'\}$
5. $q = 3, \kappa = 10, \mathcal{Q}_5 = \{1234, 4567, 7890'\}$

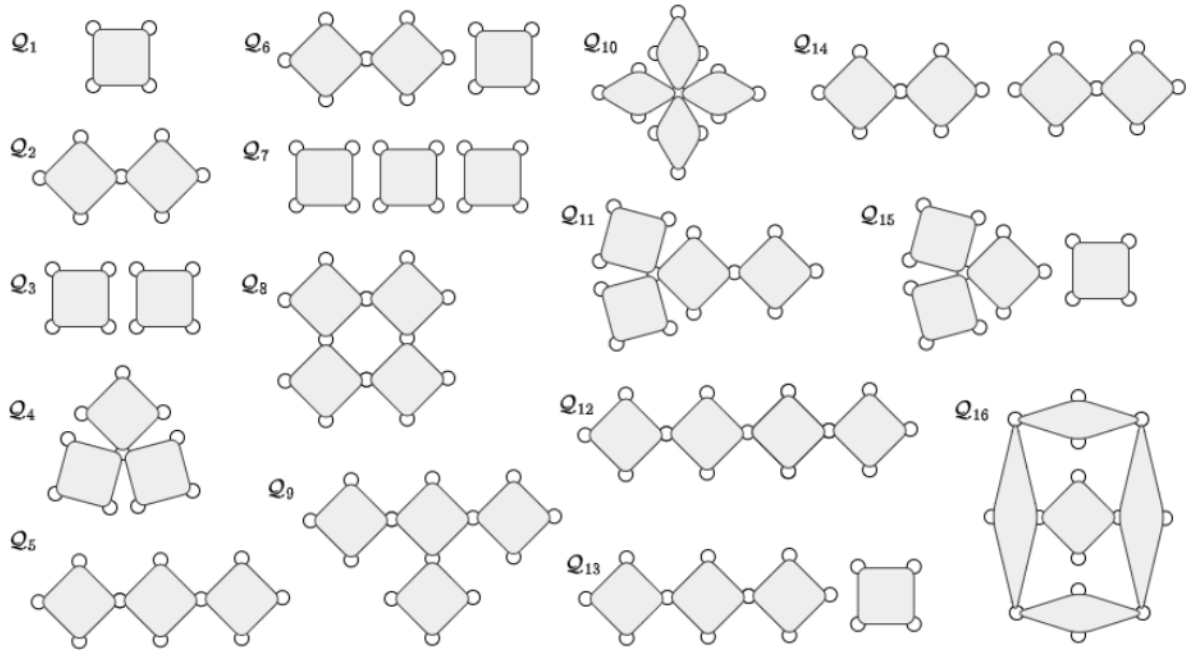


Figure 4: The 16 possibilities for $\mathcal{Q}$ on 14 vertices

6. $q = 3, \kappa = 11, \mathcal{Q}_6 = \{1234, 1567, 890'1'\}$
7. $q = 3, \kappa = 12, \mathcal{Q}_7 = \{1234, 5678, 90'1'2'\}$
8. $q = 4, \kappa = 12, \mathcal{Q}_8 = \{1234, 5678, 1590', 261'2'\}$
9. $q = 4, \kappa = 13, \mathcal{Q}_9 = \{1234, 5678, 90'1'2', 1593'\}$
10. $q = 4, \kappa = 13, \mathcal{Q}_{10} = \{1234, 1567, 1890', 11'2'3'\}$
11. $q = 4, \kappa = 13, \mathcal{Q}_{11} = \{1234, 1567, 1890', 81'2'3'\}$
12. $q = 4, \kappa = 13, \mathcal{Q}_{12} = \{1234, 4567, 7890', 0'1'2'3'\}$
13. $q = 4, \kappa = 14, \mathcal{Q}_{13} = \{1234, 5678, 890'1', 1'2'3'4'\}$
14. $q = 4, \kappa = 14, \mathcal{Q}_{14} = \{1234, 1567, 890'1', 82'3'4'\}$
15. $q = 4, \kappa = 14, \mathcal{Q}_{15} = \{1234, 5678, 590'1', 52'3'4'\}$
16. $q = 5, \kappa = 14, \mathcal{Q}_{16} = \{1234, 5678, 90'1'2', 1593', 260'4'\}.$

In all cases we define a graph G by selecting two own pairs from each triple in $\mathcal{F} \setminus (\partial \mathcal{Q})$. This G is triangle-free. Then we find an upper bound α for the possible number of edges

of G in U , a bound β for the number of G -edges joining U to $[n] \setminus U$, and a bound γ for the number of G -edges in $[n] \setminus U$. In particular, we have $\gamma \leq \lfloor \frac{1}{4}(n - \kappa)^2 \rfloor$. Then

$$|\mathcal{F}| \leq 4|\mathcal{Q}| + \frac{1}{2}|E(G)| \leq 4q + \lfloor \frac{1}{2}(\alpha + \beta + \gamma) \rfloor. \quad (21)$$

In almost all cases (21) immediately gives $|\mathcal{F}| < \lfloor \frac{1}{8}n^2 \rfloor$.

— Case of $\mathcal{Q}_1$. $\alpha = 0$, $\beta \leq n - 4$, $\gamma \leq \frac{1}{4}(n - 4)^2$, so $|\mathcal{F}| \leq 4 + \frac{1}{2}(\alpha + \beta + \gamma) \leq \frac{1}{8}(n^2 - 4n + 32)$. This is less than $\lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 9$. In case of $n = 8$ equality must hold, G is a 4-cycle on $[8] \setminus Q_1$ plus four edges to Q_1 . But this cannot be realized by triples.

— Case of $\mathcal{Q}_2$. $\alpha = 0$, $\beta \leq 2(n - 7)$, $\gamma \leq \lfloor \frac{1}{4}(n - 7)^2 \rfloor$. So $|\mathcal{F}| < \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 11$ by (21). In case of $n = 10$ we have that $\gamma = 2$ and $\beta = 6$, but this cannot be realized. In case of $n = 9$ we obtain $\mathcal{Q}_9^2$. The case of $n = 8$ is obvious, $\mathcal{F} = \partial\mathcal{Q}$ ($= \mathcal{Q}_8^2$).

— Case of $\mathcal{Q}_3$. For $n \geq 11$ we have $\alpha \leq 4$, $\beta \leq 2(n - 8)$, so $|\mathcal{F}| \leq \frac{1}{8}(n^2 - 8n + 80) < \lfloor \frac{1}{8}n^2 \rfloor$. In case of $n = 8$ we have $\mathcal{F} = \partial\mathcal{Q}$ ($= \mathcal{Q}_8^3$). If $n = 9$ then only one more triple can be added to $\mathcal{Q}_8^3$, and in case of $n = 10$ at most two more can be added. These are too few, so we are done.

— Case of $\mathcal{Q}_4$. Here $\alpha = 0$, $\beta \leq 3(n - 10)$ so (21) gives $|\mathcal{F}| < \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 11$. In case of $n = 10$ we have $\mathcal{F} = \partial\mathcal{Q}$ ($= \mathcal{Q}_{10}^3$).

— Case of $\mathcal{Q}_5$. Here $\alpha = 3$, $\beta \leq 3(n - 10)$ so (21) gives $|\mathcal{F}| \leq \frac{1}{8}(n^2 - 8n + 88) < \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 12$. In case of $n = 10$ we have $\mathcal{F} = \partial\mathcal{Q}$ ($= \mathcal{Q}_{10}^4$). In case of $n = 11$ one can add at most one new triple to $\mathcal{Q}_5$, so we have $|\mathcal{F}| \leq 13$.

— Case of $\mathcal{Q}_6$. Here $\alpha = 7$, $\beta \leq 3(n - 11)$ so (21) gives $|\mathcal{F}| \leq \frac{1}{8}(n^2 - 10n + 113) < \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 12$. In case of $n = 11$ we have $\mathcal{F} = \partial\mathcal{Q}$ so $|\mathcal{F}| \leq 12$.

— Case of $\mathcal{Q}_7$. Here $\alpha = 12$, $\beta \leq 3(n - 12)$ so (21) gives $|\mathcal{F}| \leq \frac{1}{8}(n^2 - 12n + 144) < \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 13$. In case of $n = 12$ one can add at most four disjoint new triples to $\mathcal{Q}_7$, so we have $|\mathcal{F}| \leq 16$.

— Case of $\mathcal{Q}_8$. Here $\alpha = 4$, $\beta \leq 4(n - 12)$ so (21) gives $|\mathcal{F}| \leq \frac{1}{8}(n^2 - 8n + 96) < \lfloor \frac{1}{8}n^2 \rfloor$ for $n \geq 13$. In case of $n = 12$ one cannot add any new triple to $\mathcal{Q}_8$, so we have $|\mathcal{F}| \leq 16$.

In the rest of the cases $n \in \{13, 14\}$ and $|U| \geq 13$. Let η (or η_n) be the size of $\mathcal{F} \setminus (\partial\mathcal{Q})$. Since $|\mathcal{Q}|$ is at least 4 we have more restrictions, it is easier to give an upper bound for η . In the cases $\mathcal{Q}_9$ – $\mathcal{Q}_{15}$ we have to show that $\eta_{13} < 21 - 16 = 5$, and $\eta_{14} < 8$. In the case $\mathcal{Q}_{16}$ we need $\eta_{14} < 4$.

— Case of $\mathcal{Q}_9$. No more triples meeting 1593', so $\eta_{13} \leq 3$ and $\eta_{14} \leq 6$.

— Case of $\mathcal{Q}_{10}$. Obviously $\eta_{13} = 0$ and $\eta_{14} = 0$.

— Case of $\mathcal{Q}_{11}$. $\eta_{13} = 0$ and $\eta_{14} \leq 2$.

— Case of $\mathcal{Q}_{12}$. $\eta_{13} = 0$ and $\eta_{14} \leq \binom{4}{2}$.

— Case of $\mathcal{Q}_{13}$. No more triples meeting 890'1', so $\eta \leq 3$.

— Cases of $\mathcal{Q}_{14}$ and $\mathcal{Q}_{15}$. $\eta = 0$.

— Case of $\mathcal{Q}_{16}$. $\eta \leq 2$. □

11 Excluding B, C and D

For **BCD**-free families we know the upper bound

$$\text{ex}(n, \mathbf{BCD}) \leq \text{ex}(n, \mathbf{CD}) = \lfloor \frac{1}{4}(n-1)^2 \rfloor.$$

The extremal construction for $\text{ex}(n, \mathbf{CD})$ – of the form $\mathcal{F}_T$ for some tournament T – does contain **B** configurations, so this bound cannot be exactly matched. But we do have an asymptotically matching construction.

Theorem 22.

$$\text{ex}(n, \mathbf{BCD}) = \frac{1}{4}n^2(1 - o(1)).$$

Our construction uses Wilson's well-known decomposition theorem, which we recall now.

Theorem 23 (Wilson [24]). *For every fixed graph H there exists $n_0 = n_0(H)$ such that the following holds for all $n \geq n_0$:*

There exists a decomposition of the edge set $E(K_n)$ of the complete graph on n vertices into copies of H if and only if the number of edges of H divides $\binom{n}{2}$, and the greatest common divisor of the vertex degrees in H divides $n - 1$.

Proof of Theorem 22. For every integer $m \geq 2$ we construct a graph H_m to which we will apply Theorem 23. It has m^2 vertices, partitioned into $\binom{m}{2}$ pairs $A_1, \dots, A_{\binom{m}{2}}$ and a set B of size m . The graph induced on each $A_i \cup B$, $1 \leq i \leq \binom{m}{2}$, is complete, and these are all the edges of H_m . Thus

- $|E(H_m)| = \binom{m}{2}(2m+1) + \binom{m}{2} = \binom{m}{2}(2m+2) = m(m-1)(m+1)$.
- $\gcd(\{\deg_{H_m}(v) : v \in V(H_m)\}) = \gcd(m+1, m^2-1) = m+1$.

Let $n \geq n_0(H_m)$ be such that $2m(m-1)(m+1)$ divides $n-1$. Then both divisibility conditions in Theorem 23 are satisfied, so we can find a decomposition of $E(K_n)$ into $\frac{1}{m(m-1)(m+1)}\binom{n}{2}$ copies of H_m .

In each copy of H_m in the decomposition, we place $\binom{m}{2}m$ triples of the form $A_i \cup \{b\}$ where $1 \leq i \leq \binom{m}{2}$ and $b \in B$. Within each copy, this triple system is triangle-free: it actually follows the construction presented before Theorem 1. Let $\mathcal{H}_m^n$ be the system formed by all these triples in all copies of H_m . Because the triples in a given copy use only edges of that copy of H_m , and distinct copies are edge-disjoint, triples in distinct copies can share at most one vertex. This immediately implies that $\mathcal{H}_m^n$ is **CD**-free. Suppose there is a **B** configuration in $\mathcal{H}_m^n$. Then the two triples in it that share an edge must come from the same copy of H_m and be of the form $A_i \cup \{b\}$, $A_i \cup \{b'\}$ for some $b \neq b' \in B$. The third triple contains $\{b, b'\}$, which is an edge of the same copy of H_m . Thus the third

triple could only come from this same copy, but no triple in the construction contains two vertices of B . We have shown that $\mathcal{H}_m^n$ is **BCD**-free. The number of triples in it is

$$|\mathcal{H}_m^n| = \binom{m}{2} m \cdot \frac{1}{m(m-1)(m+1)} \binom{n}{2} = \frac{1}{2} \frac{m}{m+1} \binom{n}{2}.$$

When m and n both go to infinity, with n satisfying the above conditions with respect to m , namely $n \geq n_0(H_m)$ and $2m(m-1)(m+1) \mid n-1$, we have $|\mathcal{H}_m^n| = \frac{1}{4}n^2(1 - o(1))$. For sufficiently large n that does not satisfy the divisibility condition, we can find an n' that does, $n - 2m(m-1)(m+1) < n' < n$, and take the triple system $\mathcal{H}_m^{n'}$, still of the required asymptotic size. $\square$

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