On 2-colored graphs and partitions of boxes

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Abstract

We prove that if the edges of a graph G can be colored blue or red in such a way that every vertex belongs to a monochromatic k-clique of each color, then G has at least $4(k-1)$ vertices. This confirms a conjecture of Bucic et al. [2], and thereby solves the 2-dimensional case of their problem about partitions of discrete boxes with the k-piercing property. We also characterize the case of equality in our result.

1 Introduction

In this paper, a 2-*colored graph* will be a simple graph with edges colored blue or red. Bucic et al. [2] asked the following: Given an integer $k \geq 2$, what is the smallest possible number of vertices in a 2-colored graph having the property that every vertex belongs to a monochromatic k-clique of each color?

They gave the following construction, showing that $4(k-1)$ vertices suffice. First, for $k = 2$, take a 4-cycle with edges colored alternatingly. Now, for general k, blow up this 4-cycle, replacing each vertex by a monochromatic $(k-1)$-clique, with colors alternating along the 4-cycle (all edges between two adjacent $(k-1)$-cliques get the same color as the edge in the underlying 4-cycle). It is easy to verify that this 2-colored graph has the required property.

Bucic et al. [2] conjectured that this construction is optimal, and proved a lower bound of the form $(4 - o_k(1))k$ on the number of vertices in any 2-colored graph with the required property. Here we prove exact optimality.

Theorem 1 *Let $k \geq 2$ be an integer, and let $G = (V, E)$ be a 2-colored graph so that every vertex in V belongs to a monochromatic k-clique of each color. Then $|V| \geq 4(k-1)$.*

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Our proof, given in Section 2, combines counting arguments with a linear algebraic trick similar to one used by Tverberg [6]. In Section 3 we characterize the case of equality in Theorem 1. Perhaps surprisingly, for $k \geq 3$ it turns out that the above example on $4(k-1)$ vertices is not the only extremal one. In Section 4 we discuss some generalizations and reformulations of the above question. These involve, in particular, partitions of a box into sub-boxes and decompositions of a bipartite graph into complete bipartite subgraphs.

2 Proof of Theorem 1

Let $G = (V, E)$ be a 2-colored graph so that every vertex in V belongs to a monochromatic k-clique of each color. Instead of working directly with the graph, we store the information we have in the following form: We have a vertex-set V and two families $\mathcal{B} = \{B_1, \ldots, B_b\}$, $\mathcal{R} = \{R_1, \ldots, R_r\}$ of subsets of V satisfying:

$$|B_i| \geq k \text{ and } |R_j| \geq k \quad \forall i \in [b], j \in [r], \tag{1}$$

$$|B_i \cap R_j| \leq 1 \quad \forall i \in [b], j \in [r], \tag{2}$$

$$\bigcup_{i=1}^{b} B_i = \bigcup_{j=1}^{r} R_j = V. \tag{3}$$

Indeed, given G we can construct $\mathcal{B}$ and $\mathcal{R}$ as the families of vertex-sets of blue (resp. red) cliques witnessing that every vertex belongs to a monochromatic clique of each color.

Next, we claim that we can keep the same vertex-set V and possibly make some adjustments to the families $\mathcal{B}$ and $\mathcal{R}$, so that the following will be satisfied in addition to (1)–(3):

$$B_i \setminus \bigcup_{i' \neq i} B_{i'} \neq \emptyset \text{ and } R_j \setminus \bigcup_{j' \neq j} R_{j'} \neq \emptyset \quad \forall i \in [b], j \in [r], \tag{4}$$

$$|B_i \cap R_j| = 1 \quad \forall i \in [b], j \in [r]. \tag{5}$$

Indeed, if $B_i \subseteq \bigcup_{i' \neq i} B_{i'}$, say, then we can discard B_i from $\mathcal{B}$ while retaining properties (1)–(3). Iterating this operation we end up with families satisfying (1)–(4). At this point, if $B_i \cap R_j = \emptyset$ then we can choose a vertex $v \in B_i \setminus \bigcup_{i' \neq i} B_{i'}$ and replace R_j by $R_j \cup \{v\}$ while retaining properties (1)–(3). It may happen that this change causes a violation of (4), namely when we had $R_{j^*} \setminus \bigcup_{j' \neq j^*} R_{j'} = \{v\}$ for some $j^* \neq j$ before the change; in this case, after adding v to R_j we discard R_{j^*}. Iterating this operation we end up with families satisfying (1)–(5).

Thus, we may assume that the set V and the two families $\mathcal{B} = \{B_1, \dots, B_b\}$, $\mathcal{R} = \{R_1, \dots, R_r\}$ of subsets of V satisfy (1)–(5). For every $v \in V$ we write

$$I_v = \{i \in [b] : v \in B_i\}, \quad J_v = \{j \in [r] : v \in R_j\}.$$

Note that the properties of V, $\mathcal{B}$ and $\mathcal{R}$ may be expressed in terms of the subsets I_v of $[b]$ and J_v of $[r]$, for $v \in V$, as follows: (1) says that every $i \in [b]$ is covered by the subsets I_v at least k times, and similarly for $[r]$ and the subsets J_v; (3) says that $I_v, J_v \neq \emptyset$; (4) says that for every $i \in [b]$ there is $v \in V$ such that $I_v = \{i\}$, and similarly for $[r]$ and J_v; (5) says that the product sets $I_v \times J_v$ partition $[b] \times [r]$.

Proposition 1 *If V, $\mathcal{B}$ and $\mathcal{R}$ satisfy (4) and (5) then $|V| \geq b + r - 1$.*

Proof We introduce for each $i \in [b]$ a variable x_i, and for each $j \in [r]$ a variable y_j (these variables take real values). By (5) we have the identity

$$\sum_{v \in v} (\sum_{i \in I_v} x_i)(\sum_{j \in J_v} y_j) = (\sum_{i=1}^{b} x_i)(\sum_{j=1}^{r} y_j). \tag{6}$$

Now we consider the following system of homogeneous linear equations:

$$\sum_{i \in I_v} x_i - \sum_{j \in J_v} y_j = 0, \quad v \in V, \tag{7}$$

$$\sum_{i=1}^{b} x_i = 0. \tag{8}$$

It suffices to show that the system has only the trivial solution, because this implies that the number of equations $|V| + 1$ is at least as large as the number of variables $b + r$. Let $(x_i)_{i \in [b]}, (y_j)_{j \in [r]}$ satisfy (7) and (8). By (7) we know that for each $v \in V$ there is a real number α_v so that $\sum_{i \in I_v} x_i = \sum_{j \in J_v} y_j = \alpha_v$. The identity (6) implies, using (8), that $\sum_{v \in V} \alpha_v^2 = 0$ and hence $\alpha_v = 0$ for all $v \in V$. Now, given $i \in [b]$ we can find by (4) some $v \in V$ such that $x_i = \sum_{i \in I_v} x_i = \alpha_v = 0$, and a similar argument shows that $y_j = 0$ for every $j \in [r]$, as required. □

Returning to the proof of Theorem 1, we may henceforth assume that $b+r \leq 4(k-1)$, otherwise $|V| \geq 4(k-1)$ follows from Proposition 1. We also know that $b \geq k$, because the sets I_v, $v \in R_1$, are k or more disjoint nonempty subsets of $[b]$; similarly $r \geq k$. Thus, the relevant domain for $b + r$ in the rest of the proof is

$$2k \leq b + r \leq 4(k-1). \tag{9}$$

Using (1) we have

$$\sum_{v \in V} |I_v| + |J_v| \geq k(b + r), \tag{10}$$

and using (5) we have

$$\sum_{v \in V} |I_v||J_v| = br. \tag{11}$$

Since $|I_v|$ and $|J_v|$ are nonzero by (3), their product is smallest (given their sum) when one of them is 1. Hence

$$|I_v||J_v| \geq |I_v| + |J_v| - 1 \quad \forall v \in V. \tag{12}$$

Using (10)–(12) we can write

$$\begin{aligned} |V| &= \sum_{v \in V} |I_v| + |J_v| - (|I_v| + |J_v| - 1) \\ &\geq k(b+r) - \sum_{v \in V}(|I_v| + |J_v| - 1) \\ &\geq k(b+r) - \sum_{v \in V} |I_v||J_v| \qquad (13) \\ &= k(b+r) - br \\ &\geq k(b+r) - \frac{(b+r)^2}{4}. \end{aligned}$$

The latter is a decreasing function of $b + r$ in the domain (9), and is therefore bounded from below by its value at $b + r = 4(k - 1)$, which is $4(k - 1)$. This proves that $|V| \geq 4(k - 1)$, as required. □

3 Characterization of extremal graphs

If $G = (V, E)$ is a 2-colored graph having the property that every vertex in V belongs to a monochromatic k-clique of each color, then adding any edges to G (between existing vertices) and coloring them arbitrarily results in a graph with the same property. Therefore we can restrict attention to those graphs having this property which are *edge-critical*, in the sense that removing any edge entails the loss of this property.

Here is a construction of an edge-critical 2-colored graph on $4(k - 1)$ vertices, so that every vertex belongs to a monochromatic k-clique of each color, which generalizes the one from [2] described in the introduction. Let $k \geq 2$ be an integer, let X and Y be two disjoint sets of $2(k - 1)$ vertices each, and let $B(X, Y)$ and $R(X, Y)$ be two complementary $(k - 1)$-regular bipartite graphs on the bipartition (X, Y). Our graph $G = G(X, Y, B, R)$ has $X \cup Y$ as its vertex-set. It has the complete bipartite graph on (X, Y) as a subgraph, with edges in $B(X, Y)$ colored blue and edges in $R(X, Y)$ colored red. We refer to $B(X, Y)$ and $R(X, Y)$ as the blue and red graphs, respectively. In addition, any two vertices in X which have a common neighbor in the blue graph are joined by a blue edge in G, and any two vertices in Y which have a common neighbor in the red graph are joined by a red edge in G. It is easy to verify that this 2-colored graph has the required property and is edge-critical.

For $k = 2$ we have $|X| = |Y| = 2$ and the blue and red graphs must be two complementary perfect matchings, resulting in the 2-colored 4-cycle described in the introduction. But for higher values of k, we have more freedom in choosing $B(X,Y)$ and $R(X,Y)$. For example, consider $k = 3$, so $|X| = |Y| = 4$. We may choose $B(X,Y)$ and $R(X,Y)$ so that each of them is the disjoint union of two 4-cycles, resulting in the blown-up 4-cycle graph from the introduction. But we can also choose $B(X,Y)$ and $R(X,Y)$ to be 8-cycles, resulting in a new example, not isomorphic to the previous one.

Note that the construction described in the introduction corresponds to the following choice of $B(X,Y)$ and $R(X,Y)$: X is equi-partitioned into X_1 and X_2, Y is equi-partitioned into Y_1 and Y_2, $B(X,Y)$ consists of all edges between X_1 and Y_1 and between X_2 and Y_2, and $R(X,Y)$ consists of all edges between X_1 and Y_2 and between X_2 and Y_1. For this choice, the resulting graph $G(X,Y,B,R)$ induces blue cliques on X_1 and X_2 and red cliques on Y_1 and Y_2, and has a total of $2(k-1)(3k-4)$ edges. Among all graphs of the form $G(X,Y,B,R)$ for a given value of k, the latter uniquely minimizes the number of edges. To see this, observe that in the graph induced on X (and similarly for Y) each vertex must have degree at least $k-2$, and the only way to have these degrees equal to $k-2$ is by using X_1, X_2, Y_1, Y_2 as above.

The next result shows that all edge-critical extremal examples for Theorem 1 are of the form $G = G(X,Y,B,R)$, thus characterizing the case of equality in that theorem.

Theorem 2 *Let $k \geq 2$ be an integer, and let $|V| = 4(k-1)$. Let $G = (V,E)$ be a 2-colored graph so that every vertex in V belongs to a monochromatic k-clique of each color, and G is edge-critical with respect to this property. Then G is isomorphic to some $G(X,Y,B,R)$, where $B(X,Y)$ and $R(X,Y)$ are complementary $(k-1)$-regular bipartite graphs on (X,Y).*

Proof Let $G = (V,E)$ satisfy the assumptions of the theorem. In the case $k = 2$, it is easy to check directly that G must be a 4-cycle colored alternatingly, as claimed. We henceforth assume that $k \geq 3$.

As in the proof of Theorem 1, we associate with G two families $\mathcal{B} = \{B_1, \ldots, B_b\}$, $\mathcal{R} = \{R_1, \ldots, R_r\}$ of subsets of V satisfying (1)–(3). Clearly, the blue edges of G are those pairs $\{u,v\}$ contained in some B_i, and the red edges are those pairs $\{u,v\}$ contained in some R_j (by edge-criticality, there can be no other edges in G). In the main part of the proof below, we assume that V, $\mathcal{B}$, $\mathcal{R}$ satisfy (4) and (5) as well; at the end of the proof we will justify this assumption. We also use the notations I_v and J_v for $v \in V$ as introduced in the proof of Theorem 1. According to that proof, the only values of $b+r$ which may result in $|V| = 4(k-1)$ are $4(k-1)$ and $4(k-1)+1$ (if $b+r < 4(k-1)$ then (13) forces $|V|$ to be larger, and if $b+r > 4(k-1)+1$ then Proposition 1 does that).

Case 1 $b+r=4(k-1)$

Because $|V|=4(k-1)$, (13) must hold as an equality. This implies that (10) and (12) hold as equalities, and $b=r=2(k-1)$. Equality in (10) means that every B_i and every R_j is of size exactly k. Equality in (12) means that for every $v\in V$, at least one of I_v, J_v is a singleton. For $j\in[r]$, the sets I_v, $v\in R_j$, partition $[b]$ into k nonempty subsets. This implies that $|I_v|\le k-1$, and similarly $|J_v|\le k-1$, for every $v\in V$. Therefore $|I_v||J_v|\le k-1$ for every $v\in V$, but since $\sum_{v\in V}|I_v||J_v|=4(k-1)^2$ we must have equality for every $v\in V$. This means that we can partition V into two sets:

$$X=\{v\in V: |I_v|=k-1, |J_v|=1\},\quad Y=\{v\in V: |I_v|=1, |J_v|=k-1\}.$$

As $\sum_{v\in V}|I_v|=kb=2k(k-1)$, we must have $|X|=|Y|=2(k-1)$.

Now, consider a vertex $v\in X$. There is $j\in[r]$ such that $v\in R_j$. Since the sets I_u, $u\in R_j$, partition $[b]$ into k subsets, one of which is I_v of size $k-1$, all other I_u must be singletons, so that $R_j\setminus\{v\}\subseteq Y$. This accounts for $k-1$ red edges from v into Y. As this holds for every $v\in X$, and similarly every $v\in Y$ must have at least $k-1$ blue edges into X, the complete bipartite graph on (X,Y) must appear in G and be colored so that the blue graph $B(X,Y)$ and the red graph $R(X,Y)$ are both $(k-1)$-regular. The above also implies that the neighbors of every $v\in X$ in the red graph must form a red clique in Y, and the neighbors of every $v\in Y$ in the blue graph must form a blue clique in X. This shows that $G(X,Y,B,R)$ is contained in G, and as G is edge-critical, they must coincide.

Case 2 $b+r=4(k-1)+1$

We will show that this case cannot occur. Consider the mapping from $\mathcal{B}\cup\mathcal{R}$ into V defined as follows. To each B_i we assign, using (4), an element u of B_i which belongs to no other $B_{i'}$; if among the possible choices of u for a given B_i there is one which belongs to more than one of the sets R_j, we assign to B_i such a u. Similarly, to each R_j we assign an element u of R_j which belongs to no other $R_{j'}$, with priority to such u which belongs to more than one of the sets B_i. As $|\mathcal{B}\cup\mathcal{R}|>|V|$, the mapping is not injective, so we can find some $u\in V$ which was assigned to some B_i and to some R_j.

Assume w.l.o.g. that $b\ge 2(k-1)+1$. For the set R_j just found, there is no vertex v such that $|I_v|>1$ and $J_v=\{j\}$; indeed, if there were such v it would be given priority as the vertex assigned to R_j, over the actual assignment of u which belongs to a unique B_i. It follows that we can partition R_j into two sets:

$$S=\{v\in R_j: |I_v|=1\},\quad T=\{v\in R_j: |I_v|, |J_v|\ge 2\}.$$

Write $|R_j|=k+\ell$, where $\ell\ge 0$, and $|S|=s$.

Due to the size of R_j, the difference between the two sides of (10) is at least ℓ. Hence the first inequality in (13) holds with a slack of at least ℓ.

For each $v \in T$, the difference between the two sides of (12) is at least

$$2(|I_v| + |J_v| - 2) - (|I_v| + |J_v| - 1) = |I_v| + |J_v| - 3 \geq |I_v| - 1.$$

Since $|T| = k + \ell - s$, the second inequality in (13) holds with a slack of at least $\sum_{v \in T} |I_v| - (k + \ell - s)$. The sets I_v, $v \in R_j$, partition $[b]$, and therefore $\sum_{v \in T} |I_v| = b - s \geq 2(k-1) + 1 - s$, so the slack in the second inequality in (13) is at least $k - \ell - 1$.

Adding up the two slacks, we obtain

$$\begin{aligned} |V| &\geq k(b+r) - br + k - 1 \\ &\geq k(4(k-1)+1) - 2(k-1)(2(k-1)+1) + k - 1 \\ &= 4(k-1) + 1, \end{aligned}$$

which contradicts our assumption on $|V|$.

It remains to address the possibility that the families $\mathcal{B}$, $\mathcal{R}$ associated with G do not satisfy (4) and (5). In this case, by performing the steps indicated in the proof of Theorem 1, we obtain modified families $\mathcal{B}'$, $\mathcal{R}'$ which do satisfy (4) and (5) as well as (1)–(3). By the foregoing proof, $\mathcal{B}'$ and $\mathcal{R}'$ must be as described in Case 1 above, and the graph corresponding to them is isomorphic to some $G(X, Y, B, R)$. In particular, all sets in $\mathcal{B}' \cup \mathcal{R}'$ are of size k exactly. It follows that in passing from $\mathcal{B}$, $\mathcal{R}$ to $\mathcal{B}'$, $\mathcal{R}'$, the step of adding a vertex to a set could never occur. Thus, the only steps performed were deletions of sets. Therefore the original graph G contains a graph of the form $G(X, Y, B, R)$, and by edge-criticality they must coincide. □

4 Generalizations and reformulations

4.1 More than two colors

It is natural to generalize the question treated here to t-colored graphs, i.e., simple graphs with edges colored in one of t colors.

Question 1 (Bucic et al. [2]) *Given integers $k, t \geq 2$, what is the smallest possible number of vertices in a t-colored graph having the property that every vertex belongs to a monochromatic k-clique of each color?*

Bucic et al. noted that their construction for $t = 2$ on $4(k-1)$ vertices can be adapted to one for general t using $2t(k-1)$ vertices. In fact, our more general construction in Section 3 can also be adapted as follows. Let $X_1, \ldots, X_t$ be t disjoint sets of $2(k-1)$ vertices each. For any pair of colors i, j, take the complete bipartite graph on (X_i, X_j) and color its edges i or j so that both color graphs are $(k-1)$-regular. In addition, for each color i, any two vertices in X_i which have a common neighbor in the color i graph (in any X_j, $j \neq i$) are joined by an edge colored i. This yields an edge-critical graph with the required property.

Regarding optimality, we note that the above construction is not optimal for $k = 2$ and $t > 2$. For example, 4 vertices suffice for $k = 2$, $t = 3$. But it may be optimal for higher values of k. As observed by Bucic et al. [2], their proof of an asymptotic lower bound for the case of two colors extends to general t, yielding a lower bound of $(2t - o_k(1))k$ in Question 1. Unfortunately, it seems that our proof of the exact lower bound does not extend to general t.

While the above generalization looks interesting in its own right, the intended application of Bucic et al. [2] suggests a different generalization. This will be explained in the following subsections.

4.2 Partition of a box into sub-boxes

A set of the form $A = A_1 \times \cdots \times A_d$, where $A_1, \ldots, A_d$ are finite sets with $|A_i| \geq 2$, is called a d-dimensional discrete box. A set of the form $B = B_1 \times \cdots \times B_d$, where $B_i \subseteq A_i$ for all $i \in [d]$, is a sub-box of A; it is said to be nontrivial if $\emptyset \neq B_i \neq A_i$ for all $i \in [d]$. It is easy to partition a d-dimensional discrete box into 2^d nontrivial sub-boxes, by cutting each A_i into two parts. The following theorem answered a question of Kearnes and Kiss [4].

Theorem 3 (Alon et al. [1]) *Let A be a d-dimensional discrete box, and let $\{B^1, \ldots, B^m\}$ be a partition of A into m nontrivial sub-boxes. Then $m \geq 2^d$.*

Instead of requiring the sub-boxes $B^1, \ldots, B^m$ to be nontrivial, one may equivalently require that every axis-parallel line (i.e., set of the form $\{(x_1, \ldots, x_d) \in A : x_j = a_j \ \forall j \in [d] \setminus \{i\}\}$) intersects at least two of them. This observation led Bucic et al. [2] to consider families of sub-boxes $\{B^1, \ldots, B^m\}$ with the k-piercing property, namely: every axis-parallel line intersects at least k sub-boxes in the family. Generalizing the question of Kearnes and Kiss, they asked the following.

Question 2 (Bucic et al. [2]) *Let $d \geq 1$ and $k \geq 2$ be integers, and let $A = A_1 \times \cdots \times A_d$ be a d-dimensional discrete box with all $|A_i|$ sufficiently large. What is the smallest possible number m of sub-boxes in a partition $\{B^1, \ldots, B^m\}$ of A having the k-piercing property?*

They denoted the answer to Question 2 by $p_{\text{box}}(d, k)$. The case $k = 2$ is solved by Theorem 3: $p_{\text{box}}(d, 2) = 2^d$. For larger k, it is natural to consider first the 2-dimensional case ($d = 1$ is trivial). Here, cutting each A_i into k parts gives a construction with $m = k^2$ sub-boxes. But Bucic et al. [2] showed that in fact $m = 4(k - 1)$ is enough. Their construction is illustrated in Figure 1.

Bucic et al. conjectured that this construction is optimal, that is, $p_{\text{box}}(2, k) = 4(k-1)$. They observed that this is the case if one restricts attention to sub-boxes which

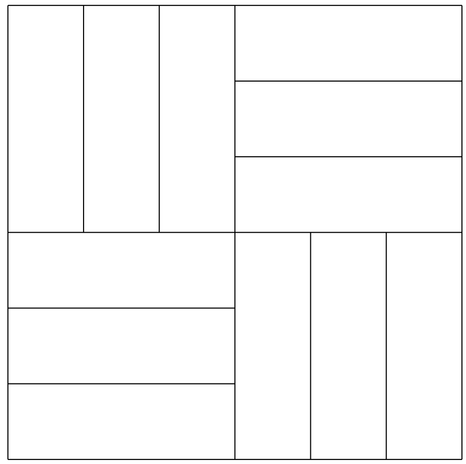

Figure 1: A k-piercing partition of a 2-dimensional box, showing that $p_{\text{box}}(2,k) \le 4(k-1)$. Each quarter of the box consists of $k-1$ parallel sub-boxes.

are bricks, i.e., products of intervals. In an attempt to prove optimality among partitions into general sub-boxes, they associated with any such partition of a 2-dimensional box a 2-colored graph as follows: the vertices are the sub-boxes in the partition, and two sub-boxes are joined by a blue (resp. red) edge if there is a horizontal (resp. vertical) line which intersects both of them. The k-piercing property implies that every vertex belongs to a monochromatic k-clique of each color. This led them to ask for the minimum number of vertices in such a graph. Note that the 2-colored graph with $4(k-1)$ vertices constructed by them (and presented in the introduction) corresponds to the partition shown in Figure 1.

The asymptotic lower bound that Bucic et al. [2] obtained for the question about 2-colored graphs enabled them to deduce that $p_{\text{box}}(2,k) \ge (4-o_k(1))k$. Our full solution of the question (Theorem 1) allows us to confirm their conjecture: $p_{\text{box}}(2,k) = 4(k-1)$. In fact, since the reduction described in the previous paragraph does not depend on the sub-boxes being a covering of the given box, but only on their disjointness, we have the following more general statement.

Corollary 1 *Let $k \ge 2$ be an integer, let A be a 2-dimensional discrete box, and let $\{B^1, \dots, B^m\}$ be a family of m disjoint sub-boxes of A having the k-piercing property. Then $m \ge 4(k-1)$.*

The question of determining $p_{\text{box}}(d,k)$ when both d and k are greater than 2 remains wide open. Bucic et al. [2] attempted a reduction to colored graphs similar to the above, but it led to a less natural and less tractable question than in the case $d = 2$. Their best bounds for general d and k are of the form $e^{\Omega(\sqrt{d})}k \le p_{\text{box}}(d,k) \le 15^{d/2}k$ (of course, when k is small relative to d, the bounds $2^d = p_{\text{box}}(d,2) \le p_{\text{box}}(d,k) \le k^d$ may be better).

4.3 Decomposition of a bipartite graph into complete bipartite subgraphs

A well-studied parameter of a graph $G = (V, E)$ is the minimum number of edge-disjoint complete bipartite subgraphs of G which cover the edge-set E. The best known result is that of Graham and Pollak [3], saying that any such decomposition of the complete graph $G = K_n$ must consist of at least $n - 1$ complete bipartite subgraphs. For more general results about decomposition of an arbitrary graph G, see e.g. Kratzke et al. [5]. The case when G itself is complete bipartite is of course uninteresting, because there is a decomposition into one subgraph. But it becomes interesting under some constraints on the decomposition, as we will see below.

A 2-dimensional discrete box $A = A_1 \times A_2$ (discussed in the previous subsection) may be viewed as the edge-set of a complete bipartite graph on (A_1, A_2). A partition of A into sub-boxes is then a decomposition of a complete bipartite graph into complete bipartite subgraphs. We can restate Corollary 1 from this point of view, as follows.

Corollary 2 *Let $k \geq 2$ be an integer. Let $G = (A_1, A_2, E)$ be a bipartite graph, and let $\{G^i = (B^i_1, B^i_2, E^i)\}_{i \in [m]}$ be a decomposition of G into m complete bipartite subgraphs. Assume that every vertex in A_1 (resp. A_2) belongs to B^i_1 (resp. B^i_2) for at least k values of $i \in [m]$. Then $m \geq 4(k - 1)$.*

Proposition 1 may also be reformulated in this terminology, as follows.

Corollary 3 *Let $G = (A_1, A_2, E)$ be a complete bipartite graph, and let $\{G^i = (B^i_1, B^i_2, E^i)\}_{i \in [m]}$ be a decomposition of G into m complete bipartite subgraphs. Assume that for every vertex x in A_1 (resp. A_2) there is $i \in [m]$ such that $B^i_1 = \{x\}$ (resp. $B^i_2 = \{x\}$). Then $m \geq |A_1| + |A_2| - 1$.*

Indeed, Tverberg's [6] proof of Graham and Pollak's theorem inspired the proof of Proposition 1.

This point of view on partition problems for 2-dimensional discrete boxes suggests a generalization to higher dimensions expressed in terms of d-partite hypergraphs. In particular, the following question asks for a d-partite version of Corollary 2.

Question 3 *Let $d, k \geq 2$ be integers. Let $H = (A_1, \ldots, A_d, E)$ be a complete d-partite hypergraph, and let $\{H^i = (B^i_1, \ldots, B^i_d, E^i)\}_{i \in [m]}$ be a decomposition of H into m complete d-partite subhypergraphs. Assume that for every $\ell \in [d]$ and for every $(d-1)$-tuple of vertices $x_j \in A_j$, $j \in [d] \setminus \{\ell\}$, there are at least k values of $i \in [m]$ such that $x_j \in B^i_j$ for all $j \in [d] \setminus \{\ell\}$. If all $|A_j|$ are sufficiently large, what is the smallest possible number m of subhypergraphs in such a decomposition?*

This is a reformulation of Question 2, so the answer is the same $p_{\text{box}}(d, k)$ investigated by Bucic et al. [2]. Hopefully, this interpretation of the question may suggest a useful approach, but we were unable to extend the methods of this paper to handle it.

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