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ON STRONG REPRESENTATIONS OF GAMES BY SOCIAL CHOICE FUNCTIONS*

Ron HOLZMAN

C.O.R.E., Université Catholique de Louvain, 1348 Louvain-la-Neuve, Belgium

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Games in characteristic function form are used to model an allocation of decision power among individuals involved in a voting situation. The problem of strong representation is to find a strategically acceptable social choice function that entails the allocation of power prescribed by a given game. Within the class of non-weak characteristic function games, we fully characterize the games that admit a strong representation. We apply this result to Peleg's problem of strong representation of simple games. Our results indicate that a strong representation requires significantly more than has been recognized in the literature.

1. Introduction

The notion of the power of the various coalitions with respect to a given social choice function has often been used in the theory of social choice. It can be found already in Arrow's (1963) General Possibility Theorem: the notion of a decisive coalition is used to derive the existence of a dictator. Later literature has offered more extensive descriptions of the allocation of power entailed by social choice functions, using game theoretic objects as descriptive tools. Peleg (1978b) went one step further: he suggested to treat the game as the preliminary data of the problem, and to look for social choice functions (with desirable properties) that entail the allocation of power reflected by the given game. Problems of this type are representation problems. Their interest lies in the fact that they display the possibilities and limitations in designing social choice functions that allocate power in any prescribed form.

In this paper we focus on the representation problem considered by Ishikawa and Nakamura (1980). The game theoretic object they used is characteristic function games. In this context of social choice functions, where

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the possible outcomes are described by a finite set of alternatives A, a characteristic function game v is a specification of the feasible set of outcomes $v(S)$ (subset of A) for each coalition S (subset of the set N of individuals). Using Peleg's (1978a) notion of exact and strong consistency as the criterion for the acceptability of social choice functions on strategic grounds, they asked the following question: what conditions does v have to satisfy in order that it be possible to find an acceptable social choice function that entails the allocation of power prescribed by v? Such a social choice function is called a strong representation of the characteristic function game v.

Ishikawa and Nakamura give sufficient conditions and necessary conditions for the existence of strong representations, but these are hardly satisfactory. The necessary conditions reflect well-known core conditions, rather than the specific (and higher) demands of this problem.[1] The sufficient conditions are much stronger than the necessary conditions, and seem too crude. In this paper we formulate and prove necessary and sufficient conditions for the existence of strong representations of characteristic function games. This is a full solution of the problem, except that we restrict ourselves to games that do not endow any individual with veto power (such games are called non-weak).

Thus, the role of the present paper is not to suggest new notions or problems, but rather to solve a problem that has already been formulated in the literature. Nevertheless, we restate in section 2 all the necessary definitions and recall the conditions of Ishikawa and Nakamura. Following this, we state our own conditions, and indicate their sufficiency. This does not involve any new proof; it suffices to observe what is really required for Ishikawa and Nakamura's proof to go through.

Section 3 is the central part of the paper. Here we prove that our conditions are necessary for the existence of a strong representation of a non-weak characteristic function game. Thus, we obtain a full characterization of the family of games (within this class) that possess strong representations. The proof, although elementary, is quite complex and follows a rather unexpected line.

In section 4 we interpret the conditions and the characterization, discuss their relationship with other work in the area, and give some examples. We illustrate further the power of the characterization obtained by exhibiting its consequences for the problem of strong representations of simple games. This problem, analysed in Peleg (1978b), is solved here for the class of symmetric simple games. A solution for the class of all simple games is a more general result of this paper's characterization; it is obtained in a separate paper [Holzman (1984)].

[1]This feature is common to other work pertaining to exact and strong consistency. We elaborate on this in section 4.

2. Definitions and preliminary results

2.1. Choice, effectivity and games

The basic ingredients of the choice situations considered here are a set N of individuals and a set A of alternatives facing them. Both sets are assumed to be finite and to contain more than one element. The preferences of any individual over A may take the form of any linear order over A. L denotes the set of all linear orders over A. If we have $R^i \in L$ for each individual $i \in N$, we have a social profile of preferences, which we denote R^N. L^N denotes the set of all possible profiles.

Definition 2.1. A social choice function (SCF) is a function F: $L^N \to A$.

Thus, a SCF is a deterministic rule that assigns a chosen alternative to every possible profile of preferences. Given such a rule, coalitions (non-empty subsets of N) may or may not have the power to enforce certain outcomes. This notion is captured and made precise by the following definition of effectivity with respect to a SCF:

Definition 2.2. Let F be a SCF, let S be a coalition and let x be an alternative. We say that S is effective for x (with respect to F) if, for all $R^N \in L^N$, $[\max(R^S) = x \Rightarrow F(R^N) = x]$.[2]

Using this definition for a given SCF, we get information about the effectivity of the various coalitions for the various alternatives. The next definition introduces the game theoretic object that can be used to express all this information.

For a set X, $\mathscr{P}(X)$ denotes the set of all subsets of X, while $2^X = \mathscr{P}(X) - \{\phi\}$.

Definition 2.3. A characteristic function game is a function v: $2^N \to \mathscr{P}(A)$. v is monotonic if, for all S, $T \in 2^N$, $[S \subset T \Rightarrow v(S) \subset v(T)]$. In this case we write, for short, that v is a MCFG.[3]

Definition 2.4. Let F be a SCF. The characteristic function game associated with F, $v^* = v^*(F)$, is defined by $[x \in v^*(S) \Leftrightarrow S$ is effective for x (with respect to F)] for all $S \in 2^N$, $x \in A$.

[2]For $R \in L$, $\max(R)$ denotes the best element of A according to the order R. For a coalition S, we write $\max(R^S) = x$ if $\max(R^i) = x$ for all $i \in S$.

[3]Ishikawa and Nakamura required in their definition, in addition to the above, that $v(N) = A$. Although this seems very natural, it will not be satisfied by the characteristic function games associated with, e.g., imposed SCFs.

We remark that v^* is always monotonic. Conversely, given a MCFG v, we may ask whether or not there exists a SCF F such that $v=v^*(F)$. This is a representation problem. The notion of a strong representation is derived from this by adding a constraint on the admissible SCFs. This constraint is motivated by requirements of strategical stability, which are introduced in the next subsection.

2.2. Strong representations

When the individuals in N have to vote in order to choose an element of A, they are facing a game in strategic form. Each individual may report any order in L as his preference relation. An analysis of this interaction, under the assumptions that the individuals are completely informed and that coordination of strategies by coalitions is unrestricted, leads to the following notion of equilibrium. (For a coalition S, L^S denotes the set of all S-profiles, i.e., assignments of orders in L to the members of S; if $P^S \in L^S$ and $Q^N \in L^N$ then P^S, Q^{N-S} denotes the profile obtained from Q^N by substituting P^S for the components corresponding to the members of S.)

Definition 2.5. Let F be a SCF and let $R^N \in L^N$. Let further $Q^N \in L^N$. We say that Q^N is a strong equilibrium point of the F-voting game at R^N unless there exist $S \in 2^N$ and $P^S \in L^S$ such that $F(P^S, Q^{N-S})$ is preferred in R^i to $F(Q^N)$ for all $i \in S$. If, moreover, $F(Q^N)=F(R^N)$, we say that Q^N is exact.

If Q^N is an exact and strong equilibrium point of the F-voting game at R^N (where F is the voting rule and R^N is the sincere profile), then playing Q^N will not create any incentive for deviation nor any distortion of the sincere outcome.

Definition 2.6. A SCF F is exactly and strongly consistent if, for each $R^N \in L^N$, there exists an exact and strong equilibrium point of the F-voting game at R^N.

We regard this requirement as the criterion for the acceptability of a SCF on strategic grounds. For more motivation of this definition, the reader is referred to Peleg (1978a) or Peleg (1984).

Definition 2.7. Let v be a MCFG. A SCF F is a strong representation of v if (i) F is exactly and strongly consistent, and (ii) $v^*(F)=v$.

For v to have a strong representation, it means that the allocation of power (in other words, the effectivity relation) prescribed in v can be implemented in terms of an acceptable SCF.

2.3. The conditions

In order to formulate conditions on MCFGs, in connection with the question of the existence of strong representations, it is convenient to use the terminology of blocking coalitions. Given a MCFG v, we say that a coalition S is blocking for an alternative x if $x \notin v(N-S)$. Since the blocking relation determines v, we may use it as an alternative form of v, via the following definition:

Definition 2.8. Let v be a MCFG. The blocking form of v consists of the collections $\mathscr{B}(x) \subset \mathscr{P}(N)$, for $x \in A$, defined by (i) $N \in \mathscr{B}(x)$ for all $x \in A$, and (ii) for $S \subsetneqq N [S \in \mathscr{B}(x) \Leftrightarrow x \notin v(N-S)]$.

Thus, $\mathscr{B}(x)$ is the collection of those coalitions that are blocking for x. It contains always, by definition, the total coalition. It may contain the empty set; this happens if $x \notin v(N)$. (In what follows, the empty set is not excluded; in particular, it is allowed to appear as a component in a partition of N.)

We introduce now Ishikawa and Nakamura's necessary conditions.

Definition 2.9. Let v be a MCFG. We say that v satisfies condition A [respectively condition B] if there exist no enumeration $x_1, \ldots, x_m$ of the set of alternatives A and partition $S_1, \ldots, S_m$ of the set of individuals N so that $S_i \notin \mathscr{B}(x_i)$ [resp. $S_i \in \mathscr{B}(x_i)$] for $i = 1, \ldots, m$.

Theorem 2.10 (Ishikawa and Nakamura). Let v be a MCFG satisfying $v(N) = A$. For v to have a strong representation, it is necessary that it satisfies conditions A and B.

The necessity of condition A can be explained as follows. If it is violated, a profile can be constructed in the spirit of the 'paradox of voting'; for this profile, every alternative is dominated; hence no alternative can be the outcome of a strong equilibrium point. The necessity of condition B is a simple feasibility argument: if everything can be blocked simultaneously, one is led to a contradiction.

In order to understand the sufficient conditions, we require the notion of a feasible elimination procedure. It was introduced by Peleg as a basis for the construction of exactly and strongly consistent SCFs, and was generalized by Ishikawa and Nakamura.

We use the following notation. For $R \in L$ and $A_0 \in 2^A$, $\min(R \mid A_0)$ denotes the worst element of A_0 according to the order R. For a coalition S, we write $\min(R^S \mid A_0) = x$ if $\min(R^i \mid A_0) = x$ for all $i \in S$.

Definition 2.11. Let v be a MCFG and let $R^N \in L^N$. A feasible elimination procedure at R^N (with respect to v) is a sequence $(x_1, S_1; \ldots; x_{m-1}, S_{m-1}; x_m)$ satisfying (i) $x_1, \ldots, x_m$ is an enumeration of A, (ii) $S_1, \ldots, S_{m-1}$ are pairwise disjoint subsets of N, and (iii) for $j = 1, \ldots, m-1$, $S_j \in \mathscr{B}(x_j)$ and $\min(R^{S_j} | \{x_j, \ldots, x_m\}) = x_j$. An alternative $x \in A$ is maximal at R^N (with respect to v) if there exists a feasible elimination procedure at R^N with $x_m = x$. $M(v, R^N)$ denotes the set of all maximal alternatives at R^N.

A maximal alternative, if it exists, is the outcome of successive elimination of alternatives that are regarded, each at the respective step of the procedure, as the worst remaining alternative by the members of a blocking set of individuals. It can be shown (and was in fact shown by Ishikawa and Nakamura, although not explicitly formulated) that if v satisfies condition B and F satisfies $F(R^N) \in M(v, R^N)$ for all $R^N \in L^N$, then F is a strong representation of v. For a construction of such F to be possible, one has to know that $M(v, R^N) \neq \phi$ for all $R^N \in L^N$. To guarantee this, Ishikawa and Nakamura introduced the following condition.

For a set X, $|X|$ denotes the cardinality of X.

Definition 2.12. We say that a MCFG v satisfies condition C if there exist no enumeration $x_1, \ldots, x_m$ of A and partition $S_0, S_1, \ldots, S_m$ of N so that $|S_0| \leqq m-2$ and for $i = 1, \ldots, m$ $S_i \notin \mathscr{B}(x_i)$.

They showed that, for v satisfying $v(N) = A$ and condition C, $M(v, R^N) \neq \phi$ for all $R^N \in L^N$, thus obtaining the following theorem.

Theorem 2.13 (Ishikawa and Nakamura). Let v be a MCFG satisfying $v(N) = A$. For v to have a strong representation, it is sufficient that it satisfy conditions B and C.

It is difficult to see, by looking directly at condition C, how it performs its role in the theorem. More insight will be gained by considering the string of conditions that we shall introduce here, to replace condition C. While the conjunction of our conditions is weaker than condition C,[4] they perform the same role in a more transparent way.

For a MCFG v, $\mathscr{B}_m(x)$ denotes the collection of all minimal sets in $\mathscr{B}(x)$, i.e., those sets in $\mathscr{B}(x)$ that do not have any proper subsets in $\mathscr{B}(x)$.

Definition 2.14. Let v be a MCFG, and let k be an integer, $0 \leqq k \leqq m-2$, where $m = |A|$. We say that v satisfies condition $D(k)$ if there exist no enumeration $x_1, \ldots, x_m$ of A and partition $S_1, \ldots, S_m$ of N so that (i) $S_i \in \mathscr{B}_m(x_i)$, $i = 1, \ldots, k$ and (ii) $S_j \notin \mathscr{B}(x_j)$, $j = k+1, \ldots, m$.

[4]This is not difficult to see. It will be shown in section 4.

Theorem 2.15. *Let v be a MCFG. For v to have a strong representation, it is sufficient that it satisfy conditions B and $\bigwedge_{k=0}^{m-2} D(k)$.*

Proof. We shall prove that if v satisfies condition $D(k)$ for every $k=0,\ldots,m-2$ then $M(v,R^N)\neq\phi$ for all $R^N\in L^N$. (Given this, the theorem follows in the same way as in Ishikawa and Nakamura's paper; see above.) The argument is that, given $R^N\in L^N$, a feasible elimination procedure $(x_1,S_1;\ldots;x_{m-1},S_{m-1};x_m)$ can be constructed step by step, where at step k we pick an alternative x_k, distinct from all x_i, $i<k$, and a set S_k, disjoint from all S_i, $i<k$, such that $S_k\in\mathscr{B}_m(x_k)$ and $\min(R^{S_k}|(A-\{x_1,\ldots,x_{k-1}\}))=x_k$. Indeed, to see that this construction can be carried out, assume that k steps went through, so we have $(x_1,S_1;\ldots;x_k,S_k)$ with the appropriate properties; k is any integer between 0 and $m-2$ (the case $k=0$ takes care of the initial step, and the case $k=m-2$ is the last to worry about, since after $k=m-1$ steps we already have the whole procedure). For each $x\in A-\{x_1,\ldots,x_k\}$, let

$$S_x=\left\{s\in N: s\notin\bigcup_{i=1}^{k} S_i, \min(R^s|(A-\{x_1,\ldots,x_k\}))=x\right\}.$$

If it were the case that $S_x\notin\mathscr{B}(x)$ for each $x\in A-\{x_1,\ldots,x_k\}$, the sets $S_1,\ldots,S_k$ together with the sets S_x, $x\in A-\{x_1,\ldots,x_k\}$, would yield a counter-example to condition $D(k)$. Therefore there exists $x\in A-\{x_1,\ldots,x_k\}$ such that $S_x\in\mathscr{B}(x)$; we can take this x as x_{k+1}, and a subset of S_x that is in $\mathscr{B}_m(x)$ as S_{k+1}, to carry on the construction. $\square$

We remark that when the sufficient conditions of Theorem 2.15 are satisfied, a strong representation F can be constructed that has two further desirable properties: it is monotonic (if x is chosen and the preferences change in favor of x, then x is still chosen), and it is faithful (F displays all the symmetries among individuals in v). As these properties are away from the focus of this paper, we omit a formal statement and proof of this fact.

3. The characterization theorem

In this section we shall prove that the sufficient conditions for the existence of a strong representation (Theorem 2.15), are also necessary conditions, when we restrict ourselves to the class of non-weak games. We shall discuss this restriction in the last paragraph of the paper.

Definition 3.1. Let v be a MCFG. An individual $i\in N$ is a veto player if $i\in S$ for every coalition S such that $v(S)=A$. v is non-weak if there are no veto players.

Theorem 3.2. *Let v be a non-weak MCFG. For v to have a strong representation, it is necessary and sufficient that it satisfy conditions B and $\bigwedge_{k=0}^{m-2} D(k)$.*

This is our characterization theorem. The sufficiency of the conditions was established in Theorem 2.15. The necessity of conditions B and $D(0)$ is a result of Ishikawa and Nakamura cited above. [See Theorem 2.10. Notice that condition $D(0)$ is exactly condition A. Notice also that the assumption $v(N)=A$ is satisfied by any non-weak MCFG, since if $v(N)\neq A$ then no S satisfies $v(S)=A$ and therefore all the individuals are veto players.]

Thus, we have to prove the necessity of the conditions $D(k)$, $k=1,\dots,m-2$. That this is true is rather surprising, since these conditions were introduced to suit the needs of elimination procedures, while here we have to derive them from the existence of any strong representation. The proof being quite complex, we shall first outline its structure.

The necessity of condition $D(1)$ reflects a deep analysis of the requisites of exact and strong consistency. This forms the kernel of the proof (Lemma 3.14). It would seem that the analysis would become hopelessly complicated if one attempted to deal with conditions $D(k)$, $k>1$, in the same manner. Induction saves the day. It exempts us from the need to work further with social choice functions; instead, we work directly with the conditions and establish implications between them. However, in order for the induction argument to go through, it is necessary to strengthen the induction hypothesis. This strengthening involves working with conditions $D^*(k)$, that are similar to the $D(k)$, but are expressed in terms of sets of alternatives rather than only single alternatives. Effectivity functions enable us to express these things, so we start by introducing them. (As the following definitions are, mostly, natural generalizations of similar definitions given in section 2, the reader should refer to that section for comparison and interpretation.)

Definition 3.3. An effectivity function is a function E: $2^N \to \mathscr{P}(2^A)$. E is monotonic if (i) for all S, $T\in 2^N$, $[S\subset T \Rightarrow E(S)\subset E(T)]$, and (ii) for all $S\in 2^N$ and C, $D\in 2^A$, $[C\in E(S)$ and $C\subset D]\Rightarrow D\in E(S)$. E is maximal if, for all $S\in 2^N-\{N\}$ and $C\in 2^A-\{A\}$, either $C\in E(S)$ or $A-C\in E(N-S)$.

Definition 3.4. Let F be a SCF. We say that a coalition S is effective for a subset C of A (with respect to F) if, whenever R^N is a profile in which every member of S prefers every element of C to every element of $A-C$, $F(R^N)\in C$. The effectivity function associated with F, $E^*=E^*(F)$, is defined by $[C\in E^*(S)\Leftrightarrow S$ is effective for C (with respect to F)], for all $S\in 2^N$, $C\in 2^A$.[5]

[5]For simplicity's sake, we do not introduce two other effectivity functions that can be associated with a SCF, namely the α- and β-effectivity functions. We can afford this simplification since the three notions coincide for exactly and strongly consistent SCFs; see Peleg (1984, Corollary 4.1.29).

Definition 3.5. Let E be an effectivity function. A SCF F is a strong representation of E if (i) F is exactly and strongly consistent, and (ii) $E^*(F)=E$.

An important notion defined with respect to effectivity functions is the notion of the core, which we recall now.

Definition 3.6. Let E be an effectivity function and let $R^N \in L^N$. For $x \in A$, $B \subset A-\{x\}$ and $S \subset N$, we shall write $B\,\mathrm{dom}(R^N,S)x$ if $B \in E(S)$ and every member of S prefers in R^N every element of B to x. The core of A with respect to E and R^N, $C(E,R^N)$, is the set of alternatives x for which $B\,\mathrm{dom}(R^N,S)x$ holds for no $B \subset A-\{x\}$ and $S \subset N$. E is stable if $C(E,R^N) \neq \phi$ for all $R^N \in L^N$.

In the following Proposition we state, for future reference, some basic properties of the effectivity functions that are associated with exactly and strongly consistent SCFs (in other words, those that have a strong representation). Proofs can be found in section 4.1. of Peleg (1984).

Proposition 3.7. Let E be an effectivity function. If E has a strong representation then E is monotonic, maximal and stable. Any strong representation F of E satisfies $F(R^N) \in C(E,R^N)$ for all $R^N \in L^N$.

The last preparation, before we can start to formulate the conditions $D^*(k)$ for effectivity functions, is to introduce for them too the blocking form.

Definition 3.8. Let E be an effectivity function. The blocking form of E consists of the collections $\mathscr{B}(C) \subset \mathscr{P}(N)$, for $C \in \mathscr{P}(A)$, defined by (i) $N \in \mathscr{B}(C)$ for all $C \in \mathscr{P}(A)$, and (ii) for $S \subsetneq N[S \in \mathscr{B}(C) \Leftrightarrow C \notin E(N-S)]$.

In the next definition, as well as later on, we shall formulate conditions on an effectivity function by referring directly to its blocking form.

Definition 3.9. Let E be an effectivity function. We say that E satisfies condition $D^*(0)$ if there exist no partitions $C_1,\ldots,C_p$ of A and $S_1,\ldots,S_p$ of N so that $S_i \notin \mathscr{B}(C_i)$ for $i=1,\ldots,p$.

This condition generalizes condition $D(0)$ by allowing for arbitrary partitions of the alternatives, not only into singletons. As explained above, we shall need these more general statements in our proof. In the first step, we shall show that the same argument that leads to the necessity of condition $D(0)$ can be applied to obtain the necessity of condition $D^*(0)$.

Lemma 3.10. Any stable effectivity function satisfies condition $D^*(0)$.

Proof. Assume that E is a stable effectivity function, but $C_1,\dots,C_p$ and $S_1,\dots,S_p$ are partitions for which condition $D^*(0)$ is violated. Clearly $p \geqq 2$, since by definition $N \in \mathscr{B}(A)$. We shall obtain a contradiction by considering a profile R^N of the following form:

$$\begin{array}{cccccc}
S_1 & S_2 & \dots & S_{p-1} & S_p \\
\hline
C_2 & C_3 & & C_p & C_1 \\
C_3 & C_4 & & C_1 & C_2 \\
\vdots & \vdots & \dots & \vdots & \vdots \\
C_p & C_1 & & C_{p-2} & C_{p-1} \\
C_1 & C_2 & & C_{p-1} & C_p
\end{array}$$

(In this scheme, column j describes the preferences of each member of S_j. In any column, all elements of a set are preferred to all elements of a set depicted below it. The preferences within the sets C_i are immaterial. The construction is a generalized version of the 'paradox of voting'.)

We shall show that $C(E,R^N)=\phi$, which contradicts stability. Let $x \in C_1$. Then $C_p \operatorname{dom}(R^N, N-S_p)x$, by the assumption $S_p \notin \mathscr{B}(C_p)$ and the scheme above; hence $x \notin C(E,R^N)$. By the symmetry in the assumptions and the cyclic construction of R^N, it is obvious that the same argument shows that $C_i \cap C(E,R^N)=\phi$ for all i; hence $C(E,R^N)=\phi$. □

Corollary 3.11. Let E be an effectivity function. If E has a strong representation then E satisfies condition $D^*(0)$.

Proof. By Proposition 3.7 and Lemma 3.10. □

We shall formulate now the conditions $D^*(k)$ for $k=1,\dots,m-2$. When dealing with them, we assume of course that the number of alternatives m is at least 3. The generalization of $D(k)$ to $D^*(k)$ is not as straightforward as that of $D(0)$ to $D^*(0)$ above; we shall later establish the relevant implications between the $D^*(k)$ and the $D(k)$. In part of the formulation of $D^*(k)$ we stick to single alternatives. For them, we retain the notations $\mathscr{B}(x)$ [instead of $\mathscr{B}(\{x\})$] and $\mathscr{B}_m(x)$.

Definition 3.12. Let E be an effectivity function, and let k be an integer, $1 \leqq k \leqq m-2$, where $m=|A|$. We say that E satisfies condition $D^*(k)$ if there exist no $x_1,\dots,x_k \in A$, C_1, $C_2 \in \mathscr{P}(A)$ and $S_1,\dots,S_k$, T_1, $T_2 \in \mathscr{P}(N)$ so that

(i) $\{x_1\},\dots,\{x_k\}$, C_1, C_2 is a partition of A,
(ii) $S_1,\dots,S_k$, T_1, T_2 is a partition of N,
(iii) $S_i \in \mathscr{B}_m(x_i)$, $i=1,\dots,k$, and
(iv) $T_j \notin \mathscr{B}(C_j)$, $j=1,2$.

The next step will be to prove the necessity of condition $D^*(1)$. In the proof we shall refer several times to the following remark:

Remark 3.13. Let E be an effectivity function and let F be a strong representation of E. Let $S \in 2^N - \{N\}$ and $C \in 2^A - \{A\}$ satisfy $S \in \mathscr{B}(C)$. Let R^N be a profile in which every member of S prefers every element of $A-C$ to every element of C. Then $F(R^N) \notin C$.

Proof. From $S \in \mathscr{B}(C)$ and $S \neq N$ we know that $C \notin E(N-S)$. By Proposition 3.7 E is maximal, hence $A-C \in E(S)$. Since $E=E^*(F)$, this means that S is effective for $A-C$ (with respect to F); hence $F(R^N) \in A-C$. □

Lemma 3.14. Assume that E is an effectivity function such that no singleton belongs to any of the collections $\mathscr{B}(x)$. If E has a strong representation then E satisfies condition $D^*(1)$.

Proof. Let E be as assumed, and suppose that condition $D^*(1)$ is violated. Let, accordingly, $\{x\}$, C_1, C_2 and S, T_1, T_2 be partitions of A and N respectively so that $S \in \mathscr{B}_m(x)$ and $T_j \notin \mathscr{B}(C_j)$, $j=1,2$.

We observe that $1<|S|<|N|$. The first inequality follows from the assumption of the Lemma. For the second inequality, suppose that $S=N$. Pick an individual $i \in N$. Then $N-\{i\} \notin \mathscr{B}(x)$ since N is minimal in $\mathscr{B}(x)$, and $\{i\} \notin \mathscr{B}(A-\{x\})$ by the assumption of the lemma and the monotonicity of E (known from Proposition 3.7). These two facts constitute together a violation of condition $D^*(0)$, contradicting Corollary 3.11.

Let $S^{(1)}$, $S^{(2)}$ be a partition of S into two non-empty coalitions. We shall consider a profile R^N of the following form:

$S^{(1)}$	$S^{(2)}$	T_1	T_2
C_2	C_1	x	x
C_1	C_2	C_2	C_1
x	x	C_1	C_2

(For the conventions for interpreting such a scheme, see the proof of Lemma 3.10.)

Let F be a strong representation of E. By Remark 3.13 $F(R^N) \neq x$. W.l.o.g., $F(R^N) \in C_1$. Let Q^N be an exact and strong equilibrium point of the F-voting game at R^N (see Definitions 2.5 and 2.6). We distinguish two possible cases.

Case I. There exist $i \in S$ and $y \in A-\{x\}$ such that xQ^iy (i.e., i prefers x to y in Q^N).

We claim that $T_1 \cup T_2 \in \mathscr{B}(A-\{x,y\})$. Otherwise, condition $D^*(0)$ is violated for the partitions $A-\{x,y\}$, $\{x\}$, $\{y\}$ of A and $T_1 \cup T_2$, $S-\{i\}$, $\{i\}$ of N [indeed, $S-\{i\} \notin \mathscr{B}(x)$ as $S \in \mathscr{B}_m(x)$, and $\{i\} \notin \mathscr{B}(y)$ by the assumption of the lemma].

We construct a $T_1 \cup T_2$-profile $P^{T_1 \cup T_2}$ as follows: every individual in $T_1 \cup T_2$ ranks x as best and y as second-best (the rest of the ranking is immaterial). Denote $P^N = P^{T_1 \cup T_2}, Q^S$ (i.e., P^N is obtained from Q^N when the members of $T_1 \cup T_2$ deviate to $P^{T_1 \cup T_2}$). By Remark 3.13 $F(P^N) \notin A-\{x,y\}$. Also, $\{x\} \operatorname{dom}(P^N, T_1 \cup T_2 \cup \{i\})y$, so by Proposition 3.7 $F(P^N) \neq y$. The only remaining possibility is $F(P^N)=x$, but this shows that the deviation is profitable for the members of $T_1 \cup T_2$.

Case II. For all $i \in S$, the worst alternative in Q^i is x.

We claim that $S^{(1)} \cup T_1 \in \mathscr{B}(C_1)$. Otherwise, condition $D^*(0)$ is violated for the partitions C_1, $\{x\}$, C_2 of A and $S^{(1)} \cup T_1$, $S^{(2)}$, T_2 of N.

We construct a $S^{(1)} \cup T_1$-profile $P^{S^{(1)} \cup T_1}$ as follows: every individual in $S^{(1)} \cup T_1$ ranks every alternative in C_2 above x and every alternative in C_1 below x (the ranking within the sets C_j is immaterial). Denote $P^N = P^{S^{(1)} \cup T_1}$, $Q^{S^{(2)} \cup T_2}$. By Remark 3.13 $F(P^N) \notin C_1$. Also, $C_2 \operatorname{dom}(P^N, S \cup T_1)x$ [recall that $T_2 \notin \mathscr{B}(C_2)$], so $F(P^N) \neq x$. Hence $F(P^N) \in C_2$, but this shows that the deviation is profitable for the members of $S^{(1)} \cup T_1$. $\square$

Having proved the necessity of condition $D^*(1)$, we proceed to obtain the necessity of all the $D^*(k)$ by induction.

Lemma 3.15. Let E be a monotonic and maximal effectivity function. If E satisfies condition $D^*(1)$ then E satisfies $\bigwedge_{k=1}^{m-2} D^*(k)$.

Proof. For any integer k, $2 \leqq k \leqq m-2$, we shall show that if E satisfies conditions $D^*(1)$ and $D^*(k-1)$ then it satisfies condition $D^*(k)$.

Indeed, assume that condition $D^*(k)$ is violated for the partitions $\{x_1\},\ldots,\{x_k\}$, C_1, C_2 of A and $S_1,\ldots,S_k$, T_1, T_2 of N. We observe first that $S_i \neq \phi$ for $i=1,\ldots,k$. Otherwise, say $S_k=\phi$. Then condition $D^*(k-1)$, that E is assumed to satisfy, is violated for the partitions $\{x_1\},\ldots,\{x_{k-1}\}$, $C_1 \cup \{x_k\}$, C_2 and $S_1,\ldots,S_{k-1}$, T_1, T_2. [$T_1 \notin \mathscr{B}(C_1 \cup \{x_k\})$ follows from $T_1 \notin \mathscr{B}(C_1)$ by monotonicity.]

From the non-emptiness of the S_i it follows that $S_k \cup T_1 \in 2^N - \{N\}$. Hence maximality requires that either (i) $S_k \cup T_1 \notin \mathscr{B}(\{x_k\} \cup C_1)$, or (ii) $N-(S_k \cup T_1) \notin \mathscr{B}(A-(\{x_k\} \cup C_1))$. If (i) holds then condition $D^*(k-1)$ is violated for the partitions $\{x_1\},\ldots,\{x_{k-1}\}$, $\{x_k\} \cup C_1$, C_2 and $S_1,\ldots,S_{k-1}$, $S_k \cup T_1$, T_2. If (ii) holds then condition $D^*(1)$ is violated for the partitions $\{x_k\}$, C_1, $A-(\{x_k\} \cup C_1)$ and S_k, T_1, $N-(S_k \cup T_1)$. $\square$

We proceed now to the last step in the proof of the main theorem. This step consists of going back from the conditions $D^*(k)$ to the original conditions $D(k)$.

Lemma 3.16. Let E be a monotonic and maximal effectivity function. Define a MCFG v_E *by* $[x \in v_E(S) \Leftrightarrow \{x\} \in E(S)]$, *for all* $S \in 2^N$ *and* $x \in A$. *If E satisfies* $\bigwedge_{k=0}^{m-2} D^*(k)$ *then* v_E *satisfies* $\bigwedge_{k=0}^{m-2} D(k)$.

Proof. We remark first that in passing from E to v_E we lose the information concerning effectivity for sets of more than one alternative. However, for single alternatives E and v_E carry the same information. Thus, the collections $\mathscr{B}(x)$ defined with respect to E coincide with those defined with respect to v_E, so we do not distinguish between them.

The satisfaction of condition $D(0)$ follows directly from the satisfaction of condition $D^*(0)$. Assume now that $1 \leqq k \leqq m-2$ and that we have a counterexample to condition $D(k)$ consisting of the enumeration $x_1, \ldots, x_m$ of A and the partition $S_1, \ldots, S_m$ of N. So $S_i \in \mathscr{B}_m(x_i)$, $i = 1, \ldots, k$, and $S_j \notin \mathscr{B}(x_j)$, $j = k+1, \ldots, m$.

As there exists some $i \in \{1, \ldots, m\}$ with $S_i \neq \phi$, there exists a subset of indices $I \subset \{k+1, \ldots, m\}$ with $|I| = m-k-1$ and $\bigcup_{i \in I} S_i \neq N$. W.l.o.g., $I = \{k+1, \ldots, m-1\}$ satisfies this. We denote $T = \bigcup_{i=k+1}^{m-1} S_i$, so $T \neq N$.

We show now that $T \notin \mathscr{B}(\{x_{k+1}, \ldots, x_{m-1}\})$. If $T = \phi$ then $S_{k+1} = \phi$, so $\phi \notin \mathscr{B}(x_{k+1})$, hence by monotonicity $\phi \notin \mathscr{B}(\{x_{k+1}, \ldots, x_{m-1}\})$. Thus, we may assume that $T \neq \phi$, so $T \in 2^N - \{N\}$. Now, if $T \in \mathscr{B}(\{x_{k+1}, \ldots, x_{m-1}\})$ then maximality requires that $N - T \notin \mathscr{B}(A - \{x_{k+1}, \ldots, x_{m-1}\})$. But this entails a violation of condition $D^*(0)$ for the partitions $\{x_{k+1}\}, \ldots, \{x_{m-1}\}$, $A - \{x_{k+1}, \ldots, x_{m-1}\}$ and $S_{k+1}, \ldots, S_{m-1}$, $N - T$.

So we know that $T \notin \mathscr{B}(\{x_{k+1}, \ldots, x_{m-1}\})$. This, in turn, entails a violation of condition $D^*(k)$ for the partitions $\{x_1\}, \ldots, \{x_k\}$, $\{x_{k+1}, \ldots, x_{m-1}\}$, $\{x_m\}$ and $S_1, \ldots, S_k$, T, S_m. □

Having proved all the ingredients, we are ready to sum up.

Proof of Theorem 3.2. As indicated above, what was to be proved in this section is the necessity of the conditions $D(k)$, $k = 1, \ldots, m-2$, for the existence of a strong representation of a non-weak MCFG.

So, let v be a non-weak MCFG and let F be a strong representation of v. We assume $m \geqq 3$, otherwise there is nothing to prove. Let $E = E^*(F)$. F is a strong representation of E, so by Corollary 3.11 E satisfies condition $D^*(0)$. As v is non-weak, E satisfies the assumption of Lemma 3.14 [indeed, $\{i\} \in \mathscr{B}(x)$ would mean that $x \notin v(S)$ for all $S \subset N - \{i\}$, making i a veto player]. Hence E satisfies condition $D^*(1)$. E is monotonic and maximal (Proposition 3.7), hence we know from Lemma 3.15 that E satisfies $\bigwedge_{k=1}^{m-2} D^*(k)$. Given that it also satisfies $D^*(0)$, we conclude from Lemma 3.16 that v_E satisfies $\bigwedge_{k=0}^{m-2} D(k)$. But clearly $v = v_E$, so we are done. □

4. Discussion and application

The following interpretation may help understand conditions B and $\bigwedge_{k=0}^{m-2} D(k)$, and thus Theorem 3.2. Let v be a MCFG, given in blocking form by the collections $\mathscr{B}(x)$, $x \in A$. Let us perceive the relation $S \in \mathscr{B}(x)$ as saying that it is in the power of the members of S to prevent the outcome x (this perception is formalized in Remark 3.13). The grand coalition N is in a position to determine the outcome, so it is in its power to prevent all outcomes except one.

Consider now a partition $S_1,\dots,S_m$ of N together with an enumeration $x_1,\dots,x_m$ of A. We may regard this as a 'division of labour': S_i is assigned the task of preventing the outcome x_i. For any given division of labour, some tasks can be carried out and others cannot [according as $S_i \in \mathscr{B}(x_i)$ or not]. We can observe the 'feasibility coefficient' of the given division of labour: the number of tasks that can be carried out. Thus, with any division of labour we associate a feasibility coefficient, which may take integer values between 0 and m, and is a measure of what the division of labour can accomplish.

Let us take now the view (often cited to motivate superadditivity requirements for games) that the grand coalition should be able to achieve, when acting together, anything that can be accomplished by any division of labour. Since N cannot prevent all outcomes (simultaneously), this view requires that the feasibility coefficient of any division of labour should be at most $m-1$. This is exactly what condition B requires.

Let us take now the view (that is, in a certain sense, the converse of the above superadditivity argument) that any division of labour should be able to accomplish the full power of the grand coalition. Actually, we have to be more careful. Call a division of labour 'inefficient' if one of the tasks can be carried out by a proper subset of the coalition to which it is assigned [i.e., there exist i and $T_i \subsetneqq S_i$ with $T_i \in \mathscr{B}(x_i)$]. Otherwise, call it 'efficient'. Restricting the above requirement to efficient divisions of labour, and recalling that N can prevent all outcomes except one, we require: the feasibility coefficient of any efficient division of labour should be at least $m-1$. This is exactly what the conjunction $\bigwedge_{k=0}^{m-2} D(k)$ requires [indeed, condition $D(k)$ rules out efficient divisions of labour with feasibility coefficient k].

Thus, conditions B and $\bigwedge_{k=0}^{m-2} D(k)$ may be viewed, under this interpretation, as a superadditivity requirement and its converse. From this we are led to the following interpretation of our characterization theorem: when allocations of power that do not endow individuals with veto power are considered, the requirement that the allocation of power be implementable in terms of a (strategically) acceptable SCF amounts to a superadditivity requirement and its converse.

In this form, our result is reminiscent of the following result: for maximal effectivity functions, stability (see Definition 3.6) is equivalent to superadditivity and subadditivity.[6] However, this similarity might be misleading, by giving the impression that the existence of a strong representation amounts just to core stability. Indeed, an important consequence of our work is that the existence of a strong representation requires significantly more than core stability. For characteristic function games, core stability[7] amounts to condition A, as shown by Ishikawa and Nakamura (1980). This condition A is just $D(0)$, one end point of our string of conditions $D(k)$, $k=0,\ldots,m-2$. In the domain of effectivity functions, already the canonical example of a stable effectivity function, namely the one derived from the proportional veto function [see Moulin (1981)], taken for the case of 3 alternatives and 4 individuals, violates condition $D^*(1)$ and therefore (Lemma 3.14) has no strong representation.

The difference between the requirement of core stability and that of the existence of a strong representation may be explained as follows. While stability guarantees that we can always choose an outcome upon which no coalition can improve on its own (irrespective of what the others do), in voting situations coalitions may still improve the outcome for their members by exploiting the actual votes of the others. This type of exploitation is well illustrated in the proof of Lemma 3.14, where the improvement is constructed in each of the cases taking into account the votes of individuals outside the deviating coalition. Our result, compared as above with results on stability, points to the significance of the difference between the notion of stability embodied in the core and the more demanding notion of strong equilibrium underlying the concept of a strong representation.

Another consequence of this work that sheds light on the concept of a strong representation is the following. Assuming conditions B and $\bigwedge_{k=0}^{m-2} D(k)$, feasible elimination procedures were shown in section 2 to supply a strong representation. In section 3, with the assumption of non-weakness, these same conditions were shown to be necessary for the existence of any strong representation. Therefore, whenever a non-weak MCFG can be strongly represented, this can be done using the method of elimination.[8] In other words, replacing the general requirement of exactly and strongly

[6]We do not state here the definitions of these two properties. One direction of this equivalence is due to Abdou (1981), the other is due to Peleg (1983).

[7]The definition of this notion for MCFGs is analogous to Definition 3.6, except that only domination by single alternatives is considered.

[8]The same is true on the level of effectivity functions (this is an outcome of the conditions for effectivity functions established in section 3). When interpreting this fact, however, one should bear in mind two things. First, it is not claimed (and it is not in general true) that every stable outcome for a given allocation of power can be reached by elimination; it is the allocation itself, not its stable outcomes, that can be implemented through the method of elimination. Second, we do not consider in this paper implementation by mechanisms that are not SCFs; if such mechanisms (game forms) are allowed, the class of implementable allocations of power does increase.

consistent choice by the actual method of choice by elimination entails no loss of generality in terms of the attainable (non-weak) allocations of power.

As we have emphasized the strength of the conditions B and $\bigwedge_{k=0}^{m-2} D(k)$, one might get the impression that they are hardly ever satisfied. This, however, is not true. The verification of these conditions is, generally speaking, a difficult task. But when the MCFG has many symmetries, this task is greatly simplified. We shall give two examples satisfying the conditions. The first provides a whole class of MCFGs where all individuals are treated symmetrically. The second provides a specific MCFG where individuals are treated according to their weights. We define the MCFGs directly in blocking form.

Example 4.1. Let the set of individuals N and the set of alternatives A be given, and denote $|N|=n$. Let there be assigned an integer b_x, $0 \leqq b_x \leqq n$, to every $x \in A$, in such a way that $\sum_{x \in A} b_x = n+1$. Define $\mathscr{B}(x)=\{S \subset N: |S| \geqq b_x\}$. The verification of conditions B and $\bigwedge_{k=0}^{m-2} D(k)$ is immediate. We remark that non-weakness is satisfied as soon as $\min_{x \in A} b_x \geqq 2$.

Example 4.2. Let $N=\{1,\dots,7\}$ and let $|A|=3$. Let there be assigned weights w_i to the individuals as follows: $w_1=2$, $w_i=1$ for $i=2,\dots,7$. For $S \subset N$, denote $w(S)=\sum_{i \in S} w_i$. Define $\mathscr{B}(x)=\{S \subset N: w(S) \geqq 3\}$ for all $x \in A$. This is a non-weak MCFG that satisfies conditions B and $\bigwedge_{k=0}^{1} D(k)$, as one can easily check.

We proceed now to discuss the relation between our work and other papers in the area. The most directly related paper is Ishikawa and Nakamura (1980), whose results we reviewed in section 2. They dealt with the same problem as we do here. By the above discussion it should be clear that their necessary conditions (A and B) are much weaker than our conditions B and $\bigwedge_{k=0}^{m-2} D(k)$; they did not go beyond core stability. Their sufficient conditions (B and C) are stronger than our conditions B and $\bigwedge_{k=0}^{m-2} D(k)$. Indeed, from a counterexample to condition $D(k)$ one can construct a counterexample to condition C by extracting one individual out of each of the k minimal blocking coalitions[9] and collecting them in S_0. To see that condition C is strictly stronger than $\bigwedge_{k=0}^{m-2} D(k)$, one can check that the MCFG of Example 4.2 violates condition C.

Two papers, Oren (1981) and Polishchuk (1978), dealt with the symmetric case where the collections $\mathscr{B}(x)$ are defined by numbers b_x (see Example 4.1). While our conditions B and $\bigwedge_{k=0}^{m-2} D(k)$ applied to the symmetric case yield the equality $\sum_{x \in A} b_x = n+1$, Oren proved only that the inequalities

[9]To be able to do this, we must know that ϕ does not belong to any of the collections $\mathscr{B}(x)$. This amounts to assuming $v(N)=A$, an assumption that Ishikawa and Nakamura made throughout.

$0 \leqq \sum_{x \in A} b_x - (n+1) \leqq m-2$ are necessary for the existence of a strong representation. Then, assuming that in fact $\sum_{x \in A} b_x = n+1$ (an assumption that is now justified, by our results, for the non-weak case), he gave a characterization of the class of all strong representations. Polishchuk did prove that $\sum_{x \in A} b_x = n+1$ must hold if there exists a strong representation and $\min_{x \in A} b_x \geqq 2$, but he did so only in the special case $m=3$. Thus we see that even for the relatively simple symmetric case, previously known necessary conditions are much weaker than those found generally in this paper.

We shall show now how our result can be applied to solve the problem of the existence of strong representations of simple games. This problem was introduced and analysed in Peleg (1978b), and we recall here the definitions.

Definition 4.3. A simple game is a pair $G=(N,\mathscr{W})$, where $\mathscr{W} \subset 2^N$ is the collection of winning coalitions, satisfying, for all S, $T \in 2^N$, $[S \in \mathscr{W}$ and $S \subset T] \Rightarrow T \in \mathscr{W}$. G is weak if $\bigcap_{S \in \mathscr{W}} S \neq \phi$, non-weak otherwise. G is symmetric if there exists a positive integer w such that, for all $S \in 2^N$, $[S \in \mathscr{W} \Leftrightarrow |S| \geqq w]$. In this case, we shall write $G=(n,w)$, where $n=|N|$.

Definition 4.4. Let F be a SCF. The simple game associated with F, $G^* = G^*(F) = (N, \mathscr{W}^*)$, is defined by $[S \in \mathscr{W}^* \Leftrightarrow S$ is effective for every $x \in A$ (with respect to F)], for all $S \in 2^N$.

Definition 4.5. Let $G=(N,\mathscr{W})$ be a simple game. A SCF $F: L^N \rightarrow A$ is a strong representation of G of order m if (i) F is exactly and strongly consistent, (ii) $G^*(F)=G$, and (iii) $|A|=m$.

As $G^*(F)$ contains less information about F than $v^*(F)$, in looking for a strong representation of a simple game we have more degrees of freedom than in doing so for a characteristic function game. The relationship between the two problems is clarified by the following definition and remark:

Definition 4.6. Let $G=(N,\mathscr{W})$ be a simple game and let $v: 2^N \rightarrow \mathscr{P}(A)$ be a MCFG. We say that G is induced by v if $\mathscr{W} = \{S: v(S)=A\}$. For a simple game $G=(N,\mathscr{W})$ and a set of alternatives A, $\mathscr{I}_A(G)$ denotes the collection of all MCFGs $v: 2^N \rightarrow \mathscr{P}(A)$ that induce G.

Remark 4.7. Let G be a simple game and let m be an integer $\geqq 2$. Let A be a set of m alternatives. Then G has a strong representation of order m if and only if there exists a MCFG $v \in \mathscr{I}_A(G)$ that has a strong representation.

This remark follows directly from the definitions, since for every F, $G^*(F)$ is induced by $v^*(F)$. Using Theorem 3.2, we obtain the following:

Corollary 4.8. Let G be a non-weak simple game and let m be an integer $\geqq 2$. *Let A be a set of m alternatives. Then G has a strong representation of order m if and only if there exists a MCFG* $v \in \mathscr{J}_A(G)$ *that satisfies conditions B and* $\bigwedge_{k=0}^{m-2} D(k)$.

While this corollary gives an answer to the question of the existence of a strong representation for every non-weak G and every $m \geqq 2$, this is an indirect answer, as it involves the verification of conditions for various MCFGs rather than for G itself. It is possible to translate this indirect answer into a direct one, but this requires a lot more work; we do this in a separate paper [Holzman (1984)]. However, in the special case of symmetric simple games it is easy to perform this translation, so we shall do it here.

Theorem 4.9. Let $G=(n,w)$ *be a symmetric simple game with* $w<n$, *and let m be an integer* $\geqq 2$. *Then G has a strong representation of order m if and only if* $m \leqq (n+1)/(n-w+1)$.

Proof. As the assumption $w<n$ expresses non-weakness in the symmetric case, Corollary 4.8 applies here. Let A be a set of m alternatives. Denote $b=n-w+1$. We have to show that there exists a MCFG $v \in \mathscr{J}_A(G)$ that satisfies conditions B and $\bigwedge_{k=0}^{m-2} D(k)$ if and only if $mb \leqq n+1$.

If $mb \leqq n+1$, we can assign integers b_x to $x \in A$ in such a way that $\sum_{x \in A} b_x = n+1$ and $\min_{x \in A} b_x = b$. Then the MCFG constructed as in Example 4.1 satisfies the requirements. To prove the converse, assume that $mb \geqq n+2$. We can construct a partition $S_1, \ldots, S_m$ of N where $|S_i| \leqq b$ for $i=1,\ldots,m$ and the inequality is strict in at least two places. For any $v \in \mathscr{J}_A(G)$ and any $x \in A$, no coalition with less than b members belongs to $\mathscr{B}(x)$. Indeed, if $|T|<b$ then $N-T \in \mathscr{W} = \{S: v(S)=A\}$, hence by the definition of the blocking form of v $T \notin \mathscr{B}(x)$. Therefore, for any $v \in \mathscr{J}_A(G)$, for an arbitrary enumeration $x_1, \ldots, x_m$ of A and for the above partition $S_1, \ldots, S_m$ of N, we have for all i either $S_i \notin \mathscr{B}(x_i)$ or $S_i \in \mathscr{B}_m(x_i)$ with the former being the case in at least two places. This being a violation of some $D(k)$, $0 \leqq k \leqq m-2$, we conclude that no $v \in \mathscr{J}_A(G)$ satisfies the conditions. □

We recall that Peleg (1978b) proved the 'if' part of Theorem 4.9. In the opposite direction, however, he proved only that the inequality $m \leqq (n-1)/(n-w)$ must hold when G has a strong representation of order m. This upper bound was obtained by the argument that a strong representation requires core stability. Thus the gap between the exact result of Theorem 4.9 and Peleg's upper bound reflects again the fact that a strong representation requires more than core stability. This gap may be sizeable: if $G=(10,9)$ then the interval where strong representations exist is $2 \leqq m \leqq 5$, while Peleg's upper bound is 9.

Finally, a remark about the assumption of non-weakness. Peleg (1978b) proved that a weak simple game has a strong representation of every order $m \geqq 2$. This leads us to two observations. First, the application of the characterization obtained in Theorem 3.2 affords a full solution of the problem of the existence of strong representations for simple games, because the weak case has already been solved. Second, the non-weakness assumption in Theorem 3.2 cannot be dispensed with. Indeed, if Theorem 3.2 were true without this assumption then Theorem 4.9 would be true even when $w=n$, which would contradict Peleg's result for weak simple games.

From a normative point of view, the non-weakness assumption seems justifiable for large societies (one individual should not be in a position to overrule the common wish of everyone else). When the society is small, however, non-weakness is normatively less persuasive and formally incompatible with strategical stability. Indeed, core stability requires that condition A be satisfied, which can happen in the non-weak case only if $n>m$ (more individuals than alternatives); strong representation requires in the non-weak case $n>2m-2$ [as follows from $\bigwedge_{k=0}^{m-2} D(k)$]. This suggests an investigation of strong representations of weak MCFGs, which is not undertaken in this paper.

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