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Gap Problems for Integer Part and Fractional Part Sequences

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Our main concern is with Beatty sequences, i.e., sequences of the form $\{\lfloor n\alpha+\gamma\rfloor : n=0, 1, ...\}$, where α, γ are real numbers ($\alpha \geqslant 1$ is called the modulus of the sequence). We look at the intersection of two Beatty sequences, and ask how many distinct gaps there are (a gap is the difference between two consecutive elements of the intersection). This problem turns out to be closely connected to two gap problems involving fractional part sequences of the form $\{\{n\alpha+\gamma\}: n=0, 1, ...\}$, introduced by Steinhaus and Slater, respectively. Our main results for the intersection problem and their connections to the other two problems are as follows. When one of the Beatty sequences is arithmetical (i.e., its modulus is an integer), we apply a three-gap theorem of Slater to show that there are at most three gaps; if there are three gaps, one of them is the sum of the other two. This result was obtained independently by Wolff and Pitman. When at least one of the Beatty sequences has a rational modulus, we ask for an upper bound on the number of gaps as a function of the denominator q of the rational modulus (or the smaller of the two denominators, if both moduli are rational). For $q \geqslant 2$, we show that the best upper bound is $q+3$. The upper bound follows from a recent result of Geelen and Simpson for a two-dimensional version of the Steinhaus problem, motivated by the current work. Finally, we prove that the intersection of two arbitrary Beatty sequences has finitely many gaps, by establishing a corresponding finiteness result for a two-dimensional version of the Slater problem. © 1995 Academic Press, Inc.

1. INTRODUCTION

For a real number x, we denote its integer part by $\lfloor x\rfloor$ and its fractional part by $\{x\}$. Thus, $\lfloor x\rfloor$ is the largest integer not exceeding x, and $\{x\} = x-\lfloor x\rfloor$. We denote the set of integers by $\mathbb{Z}$, the set of positive integers by $\mathbb{Z}_+$, and the set of nonnegative integers by $\mathbb{Z}_0$.

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Given a sequence of the form $\{n\alpha+\gamma : n\in\mathbb{Z}_0\}$, where α and γ are real numbers, we can apply to its terms either the integer part function or the fractional part function, thus obtaining two types of sequences which are of interest to us. Taking integer parts leads to a sequence of the form

$$S=\{s_n : n\in\mathbb{Z}_0\}, \qquad \text{where} \quad s_n=\lfloor n\alpha+\gamma\rfloor.$$

We always assume that $\alpha\geqslant 1$ (so that S is an increasing sequence of integers). Following tradition, we call such a sequence a *Beatty sequence*. Taking fractional parts leads to a sequence of the form

$$\Omega=\{\omega_n : n\in\mathbb{Z}_0\}, \qquad \text{where} \quad \omega_n=\{n\alpha+\gamma\}.$$

It is convenient to think of the terms of Ω as points on a circle of circumference 1. We call such a sequence a *fractional part sequence*. We write $S=S(\alpha,\gamma)$ or $\Omega=\Omega(\alpha,\gamma)$ to emphasize that the sequence is determined by two parameters: the *modulus* α and the *residue* γ. Beatty sequences have received considerable attention in the literature (see, e.g., the extensive bibliographies of [St] and [FMT]). Fractional part sequences, especially in the case of irrational α, are a classical topic of study.

We consider here a number of gap problems related to Beatty sequences and fractional part sequences, some known and some new. As a starting point, we recall a well-known gap result involving finite fractional part sequences, i.e., sequences of the form

$$\Omega_N=\{\omega_n : n=0,1,...,N-1\}, \qquad \text{where} \quad \omega_n=\{n\alpha+\gamma\} \text{ and } N\in\mathbb{Z}_+.$$

Steinhaus Problem. The points of Ω_N partition the circle of circumference 1 into intervals, the lengths of which are called *gaps*. How many distinct gaps can there be?

The answer is simple: at most three. Moreover, if three gaps occur, then one of them is the sum of the other two. This is true no matter what α we take (γ is of course immaterial) and what number N of steps we stop at. This result is known as the Steinhaus conjecture or the three-gap theorem. It was first proved by Sós [Só1, Só2], and independently by other people, e.g., [Su, Św].

The research reported in this paper started by considering another gap problem, concerning Beatty sequences. Suppose we have a Beatty sequence $S=S(\alpha,\gamma)$ and an *arithmetical sequence* $A=A(a,c)$, i.e.,

$$A=\{na+c : n\in\mathbb{Z}_0\}, \qquad \text{where} \quad a\in\mathbb{Z}_+,\ c\in\mathbb{Z}.$$

Intersection Problem. Let $n_1<n_2<\cdots$ be the indices of the subsequence of S formed by those terms s_{n_i} which appear also in A. The difference $n_{i+1}-n_i$ between two consecutive such indices of S is called an *S-gap*. How many distinct S-gaps can there be?

The answer, which we observed at first empirically, is strikingly similar to the answer to the Steinhaus problem: there are at most three S-gaps, and moreover, if three occur, then one of them is the sum of the other two. We were able to prove this result by reducing the intersection problem to yet another gap problem, which we formulated and solved. We found out later (thanks to Jane Pitman) that this "new" problem had actually been introduced and solved by Slater ([Sl1, Sl3]; see also [F]). Suppose we have a fractional part sequence $\Omega=\Omega(\alpha, \gamma)$ and a circular interval on the circle of circumference 1 of the form

$$I=I(\kappa, \mu)=[0, \kappa)+\mu, \qquad \text{where} \quad 0<\kappa\leqslant 1, \mu \text{ is any real,}$$
$$\text{and } + \text{ means circular translation (addition mod 1).}$$

Slater Problem. Let $n_1<n_2<\cdots$ be the indices of the subsequence of Ω formed by those terms ω_{n_i} which are in I. The difference $n_{i+1}-n_i$ between two consecutive such indices of Ω is called a *gap*. How many distinct gaps can there be?

The answer again is at most three, and if three occur, then one of them is the sum of the other two. The reduction of the intersection problem to the Slater problem is straightforward; we indicate it explicitly in Section 2, where we also derive additional information on the S-gaps (their independence, in a sense to be clarified, of the residues of S and A, and a sufficient condition for three gaps to exist). We should mention that, independent of our work, Wolff and Pitman [WP] have recently given a new proof of Slater's three-gap result, deducing from it the answer to the intersection problem. It seems that they did not obtain all the additional information mentioned above, but they got another kind of information concerning the order of appearance of the gaps (they form a "pre-Sturmian" sequence).

The connection between the Slater problem and the Steinhaus problem is not transparent, but it exists in the following sense. Given an instance of the Slater problem with a rational modulus $\alpha=p/q$ (in lowest terms), one can assign to it an equivalent instance of the Steinhaus problem with rational modulus $\alpha'=p'/q$, where $pp'\equiv 1 \pmod q$, and vice versa. This idea for showing the problems to be equivalent for rational moduli is due to Davenport (see [Sl3]). In this sense, our intersection problem is also indirectly related to the Steinhaus problem.

Since an arithmetical sequence is a special case of a Beatty sequence, it is natural to generalize the intersection problem to handle the intersection of two Beatty sequences. Let $S = S(\alpha, \gamma)$ and $T = T(\beta, \delta)$ be two Beatty sequences.

Generalized Intersection Problem. Let $m_1 < m_2 < \cdots$ be the sequence of terms which appear in both S and T. Let $n_1 < n_2 < \cdots$ and $k_1 < k_2 < \cdots$ be the corresponding indices of the subsequences of S and T, respectively; i.e., $m_i = s_{n_i} = t_{k_i}$. The differences $n_{i+1} - n_i$ are called *S-gaps*, the differences $k_{i+1} - k_i$ are called *T-gaps*, and the differences $m_{i+1} - m_i$ are called *gaps*. How many distinct S-gaps (T-gaps, gaps) can there be?

The answer here is not so simple. We summarize below what we know about this problem (the content of Sections 3–5). But first we introduce some convenient terminology and notation, and present an example and some basic facts. We denote by $G_S(T)$, $G_T(S)$, and $G(S, T)$ the set of S-gaps, the set of T-gaps, and the set of gaps, respectively. We denote by $g_S(T)$, $g_T(S)$, and $g(S, T)$ the cardinalities of the respective sets. We use "index-gaps" as a generic name for S-gaps and T-gaps, to distinguish them from gaps. We say that an index-gap and a gap are *associated* with each other if there is a pair $(i, i+1)$ that gives rise to both of them.

EXAMPLE 1.1. Let $S = S(\frac{3}{2}, 0)$ and $T = T(\frac{11}{5}, 0)$. Then

$$S = \{0, 1, 3, 4, 6, ...\},$$

$$T = \{0, 2, 4, 6, 8, 11, 13, 15, 17, 19, 22, ...\}.$$

In other words, S consists of all numbers in $\mathbb{Z}_0$ whose residues mod 3 are 0 and 1, and T consists of all numbers in $\mathbb{Z}_0$ whose residues mod 11 are 0, 2, 4, 6, and 8. Hence $S \cap T$ has a period of 33, and its initial part is given by

$$S \cap T = \{0, 4, 6, 13, 15, 19, 22, 24, 28, 30, 33, ...\}.$$

We have therefore

$$G(S, T) = \{2, 3, 4, 7\} \qquad \text{and} \qquad g(S, T) = 4.$$

To determine $G_S(T)$, we pay attention to the indices in S of the elements of $S \cap T$. We have

$$S \cap T = \{s_0, s_3, s_4, s_9, s_{10}, s_{13}, s_{15}, s_{16}, s_{19}, s_{20}, s_{22}, ...\},$$

and therefore

$$G_S(T) = \{1, 2, 3, 5\} \qquad \text{and} \qquad g_S(T) = 4.$$

Rewriting $S \cap T$ from the point of view of T, we have

$$S \cap T = \{t_0, t_2, t_3, t_6, t_7, t_9, t_{10}, t_{11}, t_{13}, t_{14}, t_{15}, \ldots\}$$

and therefore

$$G_T(S) = \{1, 2, 3\} \quad \text{and} \quad g_T(S) = 3.$$

Note that the T-gap 1 is associated with two distinct gaps, $2 = t_3 - t_2$ and $3 = t_{10} - t_9$.

The following proposition further clarifies the relation between index-gaps and gaps.

PROPOSITION 1.2. *Let $S = S(\alpha, \gamma)$ and $T = T(\beta, \delta)$ be two Beatty sequences. Then*

(a) *if the S-gap v is associated with the gap u then $|v\alpha - u| < 1$.*

(b) *Each S-gap v is associated with at most two gaps.*

(c) *Each gap u is associated with at most two S-gaps; it is associated with two S-gaps only if $1 < \alpha < 2$.*

(d) *$g(S, T)/2 \leqslant g_S(T) \leqslant 2g(S, T)$; for $\alpha \geqslant 2$, $g_S(T) \leqslant g(S, T)$.*

The above remains true, of course, if the roles of S and T are interchanged and α is replaced by β.

Proof. (a) We have, for some i,

$$u = m_{i+1} - m_i = \lfloor n_{i+1}\alpha + \gamma \rfloor - \lfloor n_i\alpha + \gamma \rfloor,$$

whereas

$$v\alpha = (n_{i+1} - n_i)\,\alpha = (n_{i+1}\alpha + \gamma) - (n_i\alpha + \gamma).$$

Since $|(x - y) - (\lfloor x \rfloor - \lfloor y \rfloor)| = |(x - \lfloor x \rfloor) - (y - \lfloor y \rfloor)| < 1$ for all real x and y, we get $|v\alpha - u| < 1$.

(b) Follows from (a).

(c) By (a), for each S-gap v associated with u, the multiple $v\alpha$ must lie in the interval $(u - 1, u + 1)$. Since distinct multiples of α are a distance of at least α apart, and $\alpha \geqslant 1$, there can be at most two multiples in that interval. Furthermore, if $\alpha \geqslant 2$ there is room only for one. If $\alpha = 1$ then (a) implies $v = u$.

(d) Follows from (b) and (c). ∎

In Section 3 we present a two-dimensional version of the Slater problem (Slater himself introduced higher dimensional versions in the context of physical applications in [Sl2]). We show that the analysis of gaps in the generalized intersection problem can be reduced to this two-dimensional Slater problem. We introduce a framework for analyzing the latter, preparing the ground for further investigation in Sections 4 and 5.

In Section 4 we deal with the special case of the generalized intersection problem when at least one of the moduli is rational. We show that the number of gaps can be made arbitrarily large, even if both moduli are rational. We ask, however, for an upper bound on the number of gaps as a function of the denominator q of the rational modulus (or the smaller of the two denominators, if both moduli are rational). This is a natural generalization of the original intersection problem, which corresponds to $q=1$. For $q \geqslant 2$, we show that the best upper bound is $q+3$. These results apply to index-gaps as well as to gaps. We derive the upper bound by reducing our problem to a two-dimensional version of the Steinhaus problem and invoking a recent result of Geelen and Simpson [GS] for that problem. The two-dimensional Steinhaus problem was introduced (in collaboration with Simpson) with this application in mind, but seems very appealing in its own right.

Based on our observation that the number of gaps can be made arbitrarily large if one uses rational moduli with suitably large denominators, one may expect that infinitely many gaps can be realized using irrational moduli. We show, however, in Section 5 that the number of gaps (and therefore also the number of index-gaps) in the generalized intersection problem is always finite. In fact, we establish this by proving the finiteness of the number of gaps in the two-dimensional Slater problem.

2. The Intersection Problem

LEMMA 2.1. *Let $S=S(\alpha,\gamma)$ be a Beatty sequence and $A=A(a,c)$ be an arithmetical sequence. Then the sequence $n_1<n_2<\cdots$ that generates the S-gaps in the intersection problem for S and A is a tail of the sequence that generates the gaps in the Slater problem for $\Omega=\Omega(\alpha/a,\gamma/a)$ and $I=I(1/a,c/a)$.*

Proof. An index n is in the sequence corresponding to the intersection problem if and only if $\lfloor n\alpha+\gamma\rfloor = ka+c$ for some $k\in\mathbb{Z}_0$. This is equivalent to

$$k+\frac{c}{a}\leqslant n\frac{\alpha}{a}+\frac{\gamma}{a}<k+\frac{c}{a}+\frac{1}{a} \tag{2.1}$$

for some $k\in\mathbb{Z}_0$. On the other hand, an index n is in the sequence corresponding to the Slater problem for $\Omega=\Omega(\alpha/a,\gamma/a)$ and $I=I(1/a,c/a)$ if and only if it satisfies (2.1) for some $k\in\mathbb{Z}$. The latter sequence may contain terms for which the corresponding k is negative, but since k increases with n in (2.1), they must form a finite initial segment of the sequence; upon removing them, the remaining tail coincides with the former sequence. ∎

Having embedded the intersection problem into the Slater problem, we now recall Slater's result. Our statement of the result includes additional information on the gaps, which is evident from the proof in [Sl3], though not stated there explicitly.

THEOREM S. *Let $\Omega=\Omega(\beta,\delta)$ be a fractional part sequence and $I=I(\kappa,\mu)$ be a circular interval. Let*

$$v_1=\min\{v\in\mathbb{Z}_+ : \{v\beta\}<\kappa\},$$
$$v_2=\min\{v\in\mathbb{Z}_+ : \{v\beta\}>1-\kappa \text{ or } \{v\beta\}=0\}.$$

Then any gap in the Slater problem for Ω and I is either v_1 or v_2 or v_1+v_2. Moreover, if β is irrational, $\kappa\leqslant\frac{1}{2}$ and $\{v_1\beta\}+1-\{v_2\beta\}\neq\kappa$, then v_1, v_2, and v_1+v_2 are distinct and each of them appears as a gap infinitely many times.

COROLLARY 2.2. *Let $S=S(\alpha,\gamma)$ be a Beatty sequence and $A=A(a,c)$ be an arithmetical sequence. Then $g_S(A)\leqslant 3$. Moreover, there exist $v_1, v_2\in\mathbb{Z}_+$ which depend only on the moduli α and a (and not on the residues γ or c), so that $G_S(A)\subseteq\{v_1, v_2, v_1+v_2\}$. Furthermore, if α is irrational and $a\geqslant 2$, then v_1, v_2, and v_1+v_2 are distinct and each of them appears as an S-gap infinitely many times.*

Proof. Using Lemma 2.1, we consider the Slater problem for $\Omega=\Omega(\beta,\delta)$ and $I=I(\kappa,\mu)$, where $\beta=\alpha/a$, $\delta=\gamma/a$, $\kappa=1/a$, and $\mu=c/a$. From Theorem S, we obtain v_1 and v_2 which depend only on β and κ, which in turn depend only on α and a. By Lemma 2.1, any S-gap is also a gap in the Slater problem, and any gap in the Slater problem which appears infinitely often is also an S-gap which appears infinitely often. Thus, the first part of Theorem S implies that $G_S(A)\subseteq\{v_1, v_2, v_1+v_2\}$. Furthermore, if α is irrational and $a\geqslant 2$, then β is also irrational, $\kappa\leqslant\frac{1}{2}$, and $\{v_1\beta\}+1-\{v_2\beta\}$ is irrational, hence $\neq\kappa$; so we may apply the second part of Theorem S. ∎

We note that the residues γ and c play no role in determining the set $\{v_1, v_2, v_1+v_2\}$ which contains the S-gaps, but may determine which of these numbers are actually S-gaps. Indeed, if α is rational the residues may determine whether or not S and A intersect. If α is irrational then, by the last part of Corollary 2.2, the S-gaps are totally independent of the residues.

Corollary 2.2 focused on S-gaps. We want now to conclude from it similar statements about A-gaps and about gaps. Since A-gaps and gaps are in a one-to-one correspondence (w is an A-gap if and only if wa is a gap), it suffices to state the result for gaps.

COROLLARY 2.3. *Let $S = S(\alpha, \gamma)$ be a Beatty sequence and $A = A(a, c)$ be an arithmetical sequence. Then $g(S, A) \leqslant 3$. Moreover, there exist $u_1, u_2 \in \mathbb{Z}_+$ that depend on the moduli α and a (but not on the residues γ or c), so that $G(S, A) \subseteq \{u_1, u_2, u_1 + u_2\}$. Furthermore, assume that α is irrational and $a \geqslant 2$. Then $G(S, A) = \{u_1, u_2, u_1 + u_2\}$, and therefore $g(S, A) = 3$ except if $u_1 = u_2$, in which case $g(S, A) = 2$. This exceptional case occurs if and only if $1 - 1/\alpha < \{a/\alpha\} < 1/\alpha$, and then $G(S, A) = \{a, 2a\}$.*

Proof. If $a = 1$ then $G(S, A) = \{\lfloor \alpha \rfloor, \lceil \alpha \rceil\}$, where $\lceil \alpha \rceil$ denotes the smallest integer not less than α, and there is nothing to prove. We assume henceforth that $a \geqslant 2$. We observe first that each S-gap v is associated with a unique gap $u = u(v)$. Indeed, if v is associated with both u and u' then, using Proposition 1.2(a),

$$|u - u'| \leqslant |u - v\alpha| + |v\alpha - u'| < 2;$$

since both u and u' are multiples of a, we must have $u = u'$. Thus, $g(S, A) \leqslant 3$ follows immediately from $g_S(A) \leqslant 3$.

Using the notation introduced in the proof of Corollary 2.2, we know from the definitions of v_1 and v_2 in Theorem S, that there exist $k_1, k_2 \in \mathbb{Z}$ so that

$$0 \leqslant v_1 \beta - k_1 < \kappa, \qquad -\kappa < v_2 \beta - k_2 \leqslant 0. \tag{2.2}$$

Upon multiplying by a, these inequalities become

$$0 \leqslant v_1 \alpha - k_1 a < 1, \qquad -1 < v_2 \alpha - k_2 a \leqslant 0. \tag{2.3}$$

We define $u_i = k_i a$, $i = 1, 2$. Then (2.3) shows that if v_i is an S-gap, the associated gap must be u_i. Adding up the two inequalities (2.3) shows that if $v_1 + v_2$ is an S-gap, the associated gap must be $u_1 + u_2$. It follows therefore from Corollary 2.2 that $G(S, A) \subseteq \{u_1, u_2, u_1 + u_2\}$, with equality in case α is irrational and $a \geqslant 2$.

It remains to characterize the circumstances when $u_1 = u_2$ although $v_1 \neq v_2$. If this is the case, we must have in (2.2) $k_1 = k_2 = k$, $v_1 = v_2 + 1$, and $\beta < 2\kappa$. When $\beta < 2\kappa$, the multiples of β on the circle of circumference 1 visit the open interval of length 2κ centered at 0 every time they turn around 0. We must therefore have $k=1$ in (2.2), which becomes, upon dividing by $-\beta$,

$$0 \geqslant \frac{a}{\alpha} - v_1 > -\frac{1}{\alpha}, \qquad \frac{1}{\alpha} > \frac{a}{\alpha} - v_2 \geqslant 0.$$

This implies that $1 - 1/\alpha < \{a/\alpha\} < 1/\alpha$. Conversely, if $1 - 1/\alpha < \{a/\alpha\} < 1/\alpha$ then two consecutive multiples of β lie in $(1-\kappa, 1+\kappa)$; they must be $v_2\beta$ and $v_1\beta$, and $k_1 = k_2 = 1$. Hence $u_1 = u_2 = a$ and $u_1 + u_2 = 2a$. ∎

3. The Two-Dimensional Slater Problem

The two-dimensional counterpart of the circle is the torus. We use the notation $\{(x, y)\}$, for $(x, y) \in \mathbb{R}^2$, to mean the equivalence class of (x, y) mod $\mathbb{Z}^2$. We refer to the points of the torus $\mathbb{T}^2$, whenever convenient, as the equivalence classes of $\mathbb{R}^2$ mod $\mathbb{Z}^2$. A *two-dimensional fractional part sequence* $\Theta = \Theta_m(\zeta, \eta)$, where ζ and η are real numbers and $m \in \mathbb{Z}$, has the form

$$\Theta = \{\theta_n : n \in \mathbb{Z}, n \geqslant m\}, \qquad \text{where} \quad \theta_n = \{n(\zeta, \eta)\}.$$

A rectangle $R = R(\kappa, \lambda; \mu, \nu)$ on $\mathbb{T}^2$, where $0 < \kappa$, $\lambda \leqslant 1$ and μ, ν are real numbers, has the form

$$R = \{\{(x, y)\} : \mu - \kappa < x \leqslant \mu, \nu - \lambda < y \leqslant \nu\}.$$

Two-Dimensional Slater Problem. Let $n_1 < n_2 < \cdots$ be the indices of the subsequence of Θ formed by those terms θ_{n_i} which are in R. The difference $n_{i+1} - n_i$ between two consecutive such indices of Θ is called a *gap*. How many distinct gaps can there be? (The set of gaps is denoted by $G(\Theta, R)$, its cardinality by $g(\Theta, R)$.)

A comparison with the one-dimensional version reveals three differences. First, we dropped the residue in the fractional part sequence, in order to simplify the notation. This entails no loss of generality, because the problem for the sequence defined by $\{n(\zeta, \eta) + (\gamma, \delta)\}$ and $R = R(\kappa, \lambda; \mu, \nu)$ is equivalent to the problem for the sequence $\{n(\zeta, \eta)\}$ and $R = R(\kappa, \lambda; \mu - \gamma, \nu - \delta)$. Second, we allowed the indices of the fractional part sequence to start from an arbitrary integer. Third, we defined a rectangle using intervals which are open on the left and closed on the right, and not vice versa. These two modifications are inessential, and were done to facilitate the following embedding of the generalized intersection problem.

Lemma 3.1. *Let $S = S(\alpha, \gamma)$ and $T = T(\beta, \delta)$ be two Beatty sequences. Then the sequence $m_1 < m_2 < \cdots$ that generates the gaps in the generalized* *intersection problem for S and T is a tail of the sequence that generates the gaps in the two-dimensional Slater problem for* $\Theta = \Theta_{m_1}(1/\alpha, 1/\beta)$ *and* $R = R(1/\alpha, 1/\beta; \gamma/\alpha, \delta/\beta)$.

Proof. Our proof follows an approach originated by Skolem and Bang [Sk1, Sk2, B], who found necessary and sufficient conditions for the intersection of two Beatty sequences to be nonempty and also answered some related questions. A number $m \in \mathbb{Z}$ is in $S \cap T$ if and only if there exist $n, k \in \mathbb{Z}_0$ so that

$$m \leqslant n\alpha + \gamma < m + 1 \qquad \text{and} \qquad m \leqslant k\beta + \delta < m + 1,$$

or equivalently

$$\frac{\gamma}{\alpha} - \frac{1}{\alpha} < \frac{m}{\alpha} - n \leqslant \frac{\gamma}{\alpha} \qquad \text{and} \qquad \frac{\delta}{\beta} - \frac{1}{\beta} < \frac{m}{\beta} - k \leqslant \frac{\delta}{\beta}. \tag{3.1}$$

On the other hand, an index $m \geqslant m_1$ of $\Theta = \Theta_{m_1}(1/\alpha, 1/\beta)$ satisfies $\theta_m \in R = R(1/\alpha, 1/\beta; \gamma/\alpha, \delta/\beta)$ if and only if (3.1) holds for some $n, k \in \mathbb{Z}$. Since n, k increase with m in (3.1), the indices m for which n or k is negative must form a finite initial segment of the subsequence corresponding to the Slater problem; upon removing them, the remaining tail coincides with $S \cap T$. ∎

For the analysis of the two-dimensional Slater problem, we introduce the following concepts.

DEFINITION 3.2. Let ζ, η be real numbers, and let $\theta_n = \{n(\zeta, \eta)\}$ for $n \in \mathbb{Z}_+$. Let $0 < \kappa, \lambda \leqslant 1$, and let $d \in \mathbb{Z}_+$. An *admissible rectangle* for d (given the parameters ζ, η, κ, λ) is a rectangle of the form $R_d = R(\kappa, \lambda; \mu_d, \nu_d)$ for some real μ_d, ν_d, so that $\{(0, 0)\}$ and θ_d are in R_d. A *suitable rectangle* for d is an admissible rectangle R_d which satisfies $\theta_{d'} \notin R_d$ for all $d' \in \mathbb{Z}_+$, $d' < d$.

The above definition is motivated by the following lemma, which is used in the next two sections of the paper.

LEMMA 3.3. *Let* $\Theta = \Theta_m(\zeta, \eta)$ *be a two-dimensional fractional part sequence, let* $R = R(\kappa, \lambda; \mu, \nu)$ *be a rectangle on* $\mathbb{T}^2$, *and let* $n \in \mathbb{Z}$, $n \geqslant m$, *and* $d \in \mathbb{Z}_+$. *Then* n *and* $n + d$ *are two consecutive terms in the sequence that generates the gaps in the corresponding two-dimensional Slater problem, if and only if* $R_d = R(\kappa, \lambda; \mu - n\zeta, \nu - n\eta)$ *is a suitable rectangle for d. In particular every gap d has a suitable rectangle* R_d.

Proof. Since R_d is obtained from R by a negative translation of $\{(n\zeta, n\eta)\}$, we have $\theta_{d'} \in R_d$ if and only if $\theta_{n+d'} \in R$, for all d'. Thus $\{(0, 0)\} \in R_d$ amounts to $\theta_n \in R$ (setting $d' = 0$) and $\theta_d \in R_d$ amounts to

$\theta_{n+d} \in R$ (setting $d' = d$); so admissibility of R_d amounts to $\theta_n, \theta_{n+d} \in R$. Furthermore, $\theta_{d'} \notin R_d$ for all $d' \in \mathbb{Z}_+$, $d' < d$, is equivalent to $\theta_{n+d'} \notin R$ for all $d' \in \mathbb{Z}_+$, $d' < d$. ∎

4. The Rational Case

We saw in Section 2 that the number of gaps and the number of index-gaps in the intersection problem are bounded above by 3. This is no longer true in the generalized intersection problem, as seen, e.g., in Example 1.1. The following family of examples shows that the number of gaps and the number of index-gaps can be made arbitrarily large.

Example 4.1. Let $q \geqslant 2$ be an integer, and let

$$S = S\left(\frac{2q+9}{q+4}, 0\right) \quad \text{and} \quad T = T\left(\frac{2q+1}{q}, 0\right).$$

We determine the gaps in this example using the notions introduced in the previous section. Instead of working with the points $\theta_n = \{n((q+4)/(2q+9), q/(2q+1))\}$ on $\mathbb{T}^2$, as indicated by Lemma 3.1, it is convenient to work with the points $P_n \equiv n(q+4, q)$, where $\equiv$ means congruence mod $2q+9$ in the first coordinate and mod $2q+1$ in the second. Dividing $\mathbb{Z}^2$ by this congruence relation, we may assume that the points P_n range over

$$L = \{(i, j) \in \mathbb{Z}^2 : |i| \leqslant q+4, |j| \leqslant q\}.$$

Indeed, since $(q+4, 2q+9) = (q, 2q+1) = (2q+9, 2q+1) = 1$ (where $(\cdot, \cdot)$ denotes the greatest common divisor), it is clear that each point of L is P_n for infinitely many $n \in \mathbb{Z}_0$. It follows from Lemmas 3.1 and 3.3 that a number $d \in \mathbb{Z}_+$ is a gap (i.e., $d \in G(S, T)$) if and only if there exists a $(q+4) \times q$ rectangular subgrid of L containing $(0, 0)$ and P_d but not any $P_{d'}$ with $d' \in \mathbb{Z}_+$, $d' < d$. We use the adjectives "admissible" and "suitable" for such rectangular subgrids by obvious analogy with Definition 3.2. As admissibility requires that $P_d = (i, j)$ with $|i| \leqslant q+3$, $|j| \leqslant q-1$, we restrict attention to this relevant part of L, depicted in Fig. 1. In the text below, we argue why the numbers d for which a suitable rectangular subgrid exists are those marked in the figure.

The first multiple of $(q+4, q)$ is $P_1 = (q+4, q)$, which falls outside of the relevant subgrid. Next comes $P_2 \equiv 2(q+4, q) \equiv (-1, -1)$; this is the first multiple inside the relevant subgrid, so any $(q+4) \times q$ rectangular subgrid that includes it and $(0, 0)$ is suitable. The next point is $P_3 = (q+3, q-1)$, for which the nonnegative quadrant of the relevant subgrid is suitable.

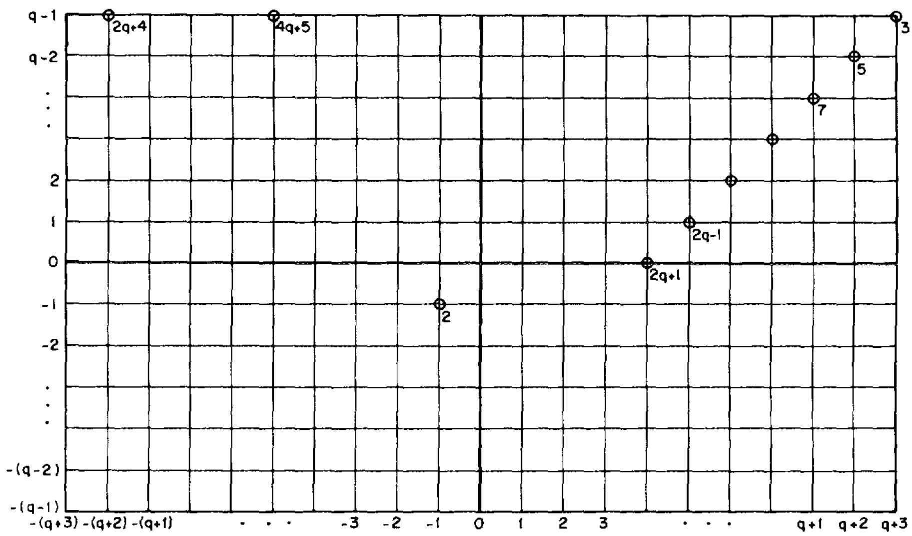

FIG. 1. Determining the gaps in Example 4.1. (A grid point (i, j) marked d means that $P_d = (i, j)$ and it has a suitable rectangular subgrid, hence d is a gap.)

Then we have $P_4 \equiv 2P_2 \equiv (-2, -2)$ which has no suitable rectangular subgrid, because any admissible one contains P_2. Next, $P_5 = (q+2, q-2)$ has a suitable rectangular subgrid, e.g., $\{-1, 0, ..., q+2\} \times \{0, ..., q-1\}$. Then comes $P_6 \equiv 3P_2 \equiv (-3, -3)$ which has no suitable rectangular subgrid, again because of P_2. We continue like this, accepting the odd values $d = 3$, 5, 7, ..., $2q-1$, $2q+1$ and rejecting the even values $d = 4$, 6, 8, ..., $2q$, $2q+2$ (the last two because P_{2q} and P_{2q+2} fall outside of the relevant subgrid).

Suppose now that $d > 2q+1$ and $I \times J$ is a suitable rectangular subgrid for d. Thus, I is a set of $q+4$ consecutive integers in $\{-(q+3), ..., 0, ..., q+3\}$ and J is a set of q consecutive integers in $\{-(q-1), ..., 0, ..., q-1\}$. Since $0 \in J$ but $P_{2q+1} = (4, 0) \notin I \times J$ (by suitability), we must have $4 \notin I$. Hence $I \subset \{-(q+3), ..., 3\}$. We claim that $J = \{0, ..., q-1\}$. Suppose this is false. Then $-1 \in J$ and, since $-1 \in I$, we have $P_2 = (-1, -1) \in I \times J$, contradicting suitability. Hence we have $P_d \in I \times J \subset \{-(q+3), ..., 3\} \times \{0, ..., q-1\}$. This permits us to reject any $d > 2q+1$ for which $P_d \notin \{-(q+3), ..., 3\} \times \{0, ..., q-1\}$.

Thus, we reject $P_{2q+3} = (3, -1)$. Next comes $P_{2q+4} = (-(q+2), q-1)$ which is accepted (a suitable rectangular subgrid is, e.g., $\{-(q+3), ..., 0\} \times \{0, ..., q-1\}$). From now on, if $I \times J$ is a suitable rectangular subgrid for $d > 2q+4$, we must have $P_d \in I \times J \subset \{-(q+1), ..., 3\} \times \{0, ..., q-1\}$. Since, as can be verified, $P_d \notin \{-(q+1), ..., 3\} \times \{0, ..., q-1\}$ for all $2q+5 \leqslant d \leqslant 4q+4$, all these values of d are rejected. Then comes $P_{4q+5} = (-(q-2), q-1)$ which is accepted (a suitable rectangular subgrid is, e.g., $\{-q, ..., 3\} \times \{0, ..., q-1\}$). For $d > 4q+5$, the I-side of a suitable rectangular subgrid would have to exclude $-(q-2)$, hence be contained in $\{-(q-3), ..., 3\}$. But this interval is too short, so no $d > 4q+5$ possesses a suitable rectangular subgrid. Thus, we conclude that

$$G(S, T) = \{2, 3, 5, ..., 2q+1, 2q+4, 4q+5\} \quad \text{and} \quad g(S, T) = q+3.$$

To find the index-gaps, we note that by Proposition 1.2(c) each gap u is associated with a unique S-gap v and a unique T-gap w, which are easily determined by $|v\alpha - u| < 1$ and $|w\beta - u| < 1$. The gap $u = 2$ is associated with the S-gap $v = 1$ and the T-gap $w = 1$. The same is true of the gap $u = 3$. Beyond that, we get distinct v's (and distinct w's) for distinct u's. We obtain

$$G_S(T) = \{1, 2, ..., q, q+2, 2q+2\} \quad \text{and} \quad g_S(T) = q+2;$$

similarly,

$$G_T(S) = \{1, 2, ..., q, q+1, 2q+1\} \quad \text{and} \quad g_T(S) = q+2.$$

It follows from Example 4.1 that there is no finite upper bound on the number of gaps (or of index-gaps) which holds uniformly for all pairs of Beatty sequences with rational moduli. We may, however, ask for an upper bound that depends on the denominators of the rational moduli. To make our goal precise, we introduce the following notation. For a positive real number x, we let

$$d(x) = \begin{cases} q & \text{if } x \text{ is rational, } x = p/q \text{ in lowest terms,} \\ \infty & \text{if } x \text{ is irrational.} \end{cases}$$

For $q \in \mathbb{Z}_+$, we define

GIP(q) = the set of all instances of the generalized intersection problem of the form $(S = S(\alpha, \gamma),\ T = T(\beta, \delta))$ with $d(\alpha) \geqslant d(\beta) = q$,

$\rho(q) = \sup\{g(S, T)\colon (S, T) \in \mathrm{GIP}(q)\}$,

$\sigma(q) = \sup\{g_S(T)\colon (S, T) \in \mathrm{GIP}(q)\}$,

$\tau(q) = \sup\{g_T(S)\colon (S, T) \in \mathrm{GIP}(q)\}$.

Our purpose is to show that $\rho(q)$, $\sigma(q)$, and $\tau(q)$ are finite and to describe their behavior as functions of q. The value of these functions at $q = 1$ was shown (in Section 2) to be 3.

Example 4.1 shows that $\rho(q) \geqslant q+3$ for $q \geqslant 2$. A similar analysis applied to the pair of Beatty sequences $S = S((6q+7)/(2q+2), 0)$, $T = T((3q-1)/q, 0)$, where $q \geqslant 2$, shows that $g(S, T) = g_S(T) = g_T(S) = q+3$ in this example (we omit the details). Thus,

$$\rho(q), \sigma(q), \tau(q) \geqslant q+3 \quad \text{for} \quad q \geqslant 2. \tag{4.1}$$

In order to establish upper bounds on the values of our functions, we introduce corresponding functions for the two-dimensional Slater problem and for the two-dimensional Steinhaus problem (to be formulated below) and relate them to our functions ρ, σ, τ. Consider the two-dimensional Slater problem for the sequence $\Theta = \Theta_m(\zeta, \eta)$ and the rectangle $R = R(\kappa, \lambda; \mu, \nu)$ on $\mathbb{T}^2$. If ζ is rational, say $\zeta = r/s$ in lowest terms, then the requirement that $\theta_n = \{n(\zeta, \eta)\} \in R$ amounts, in the first coordinate, to demanding that nr be congruent mod s to one of the integers i such that $\mu - \kappa < i/s \leqslant \mu$. Let

$$l_1 = l_1(\Theta, R) = \left| \left\{ i \in \mathbb{Z} : \mu - \kappa < \frac{i}{s} \leqslant \mu \right\} \right|. \tag{4.2}$$

If ζ is irrational, we let $l_1 = \infty$. We define l_2 similarly, by looking at the second coordinate. For $q \in \mathbb{Z}_+$, we define

TDSlP(q) = the set of all instances (Θ, R) of the two-dimensional Slater problem with $\min\{l_1, l_2\} = q$,

$$\pi(q) = \sup\{ g(\Theta, R) \colon (\Theta, R) \in \mathrm{TDSlP}(q)\}.$$

It follows from Lemma 3.1 that

$$\rho(q) \leqslant \pi(q) \qquad \text{for all} \quad q \in \mathbb{Z}_+. \tag{4.3}$$

A *fractional part matrix* $\Phi = \Phi(q_1, q_2; \xi_1, \xi_2, \varepsilon)$, where $q_1, q_2 \in \mathbb{Z}_+$ and ξ_1, ξ_2, ε are real numbers, is an array of the form

$$\Phi = \{\varphi_{ij} \colon i = 0, 1, ..., q_1 - 1, j = 0, 1, ..., q_2 - 1\},$$

where

$$\varphi_{ij} = \{ i\xi_1 + j\xi_2 + \varepsilon \}.$$

Two-Dimensional Steinhaus Problem. The points of Φ partition the circle of circumference 1 into intervals, the lengths of which are called *gaps*. How many distinct gaps can there be? (The set of gaps is denoted by $G(\Phi)$, its cardinality by $g(\Phi)$.)

For $q \in \mathbb{Z}_+$, we define

TDStP(q) = the set of all instances $\Phi = \Phi(q_1, q_2; \xi_1, \xi_2, \varepsilon)$ of the two-dimensional Steinhaus problem with $\min\{q_1, q_2\} = q$,

$$\phi(q) = \sup\{ g(\Phi) \colon \Phi \in \mathrm{TDStP}(q)\}.$$

We remark that TDStP(1) is just the one-dimensional Steinhaus problem, so $\phi(1) = 3$.

LEMMA 4.2. *Let $\Theta = \Theta_m(r_1/s_1, r_2/s_2)$ be a two-dimensional fractional part sequence, with $r_1, r_2 \neq 0$ and $(r_1, s_1) = (r_2, s_2) = (s_1, s_2) = 1$. Let $R = R(\kappa, \lambda; \mu, \nu)$ be a rectangle on $\mathbb{T}^2$, and suppose that $\Theta \cap R \neq \emptyset$. Let $l_1, l_2 \in \mathbb{Z}_+$ be defined as in (4.2). Then there exists a fractional part matrix $\Phi = \Phi(l_1, l_2; \xi_1, \xi_2, \varepsilon)$ so that the following holds: n appears in the sequence that generates the gaps in the two-dimensional Slater problem for Θ and R if and only if $n \geqslant m$ and $\{n/(s_1 s_2)\}$ is one of the points of Φ.*

Proof. The condition $\theta_n = \{n(r_1/s_1, r_2/s_2)\} \in R$ can be rewritten in the following form: there exist integers i, j such that $\mu - \kappa < i/s_1 \leqslant \mu$ and $\nu - \lambda < j/s_2 \leqslant \nu$, and

$$nr_1 \equi i \pmod{s_1}, \qquad nr_2 \equiv j \pmod{s_2}. \tag{4.4}$$

Let r_k^{-1} satisfy $r_k r_k^{-1} \equiv 1 \pmod{s_k}$, $k = 1, 2$. Let u_1, u_2 be integers satisfying $1 = u_1 s_1 + u_2 s_2$. Then (4.4) is equivalent to

$$n \equiv i r_1^{-1} u_2 s_2 + j r_2^{-1} u_1 s_1 \pmod{s_1 s_2},$$

which can be rewritten as

$$\left\{\frac{n}{s_1 s_2}\right\} = \left\{i \frac{r_1^{-1} u_2}{s_1} + j \frac{r_2^{-1} u_1}{s_2}\right\}. \tag{4.5}$$

Noting that i, j vary over intervals of l_1 and l_2 consecutive integers respectively, we see that the right-hand side of (4.5) varies over the points of $\Phi = \Phi(l_1, l_2; \xi_1, \xi_2, \varepsilon)$, where $\xi_1 = r_1^{-1} u_2/s_1$, $\xi_2 = r_2^{-1} u_1/s_2$, and ε is chosen to offset the fact that these intervals of integers may not begin at 0. ∎

If (Θ, R) and Φ satisfy the conclusion of Lemma 4.2, then clearly $g(\Theta, R) = g(\Phi)$. Thus, if $(\Theta, R) \in \text{TDSlP}(q)$ satisfies the assumptions of Lemma 4.2, we have $g(\Theta, R) \leqslant \phi(q)$. We claim that this is true even if (Θ, R) does not satisfy the assumptions of Lemma 4.2. Indeed, we may assume that $\Theta = \Theta_0(\zeta, r_2/s_2)$ with $r_2 \neq 0$ and $(r_2, s_2) = 1$, and $l_1 \geqslant l_2 = q$. Given any finite initial segment $n_1 < n_2 < \cdots < n_t$ of the sequence that generates $G(\Theta, R)$, we can produce a sufficiently good approximation $\zeta' = r_1/s_1$ of ζ so that $\Theta' = \Theta'_0(r_1/s_1, r_2/s_2)$ satisfies the required assumptions and gives rise to the same initial segment. This shows that $g(\Theta, R) \leqslant \phi(q)$ for all $(\Theta, R) \in \text{TDSlP}(q)$, and therefore

$$\pi(q) \leqslant \phi(q) \qquad \text{for all} \quad q \in \mathbb{Z}_+. \tag{4.6}$$

We have reduced the gap problem in the generalized intersection problem, via the two-dimensional Slater problem, to the two-dimensional Steinhaus problem. For index-gaps, we give a reduction directly to the two-dimensional Steinhaus problem.

LEMMA 4.3. *Let $S = S(p_1/q_1, \gamma)$ and $T = T(p_2/q_2, \delta)$ be two Beatty sequences with $(p_1, q_1) = (p_2, q_2) = (p_1, p_2) = 1$. Then there exist a fractional part matrix $\Phi = \Phi(q_1, q_2; \xi_1, \xi_2, \varepsilon)$ and a number $n_0 \in \mathbb{Z}_0$ so that the following holds: n appears in the sequence that generates the S-gaps in the generalized intersection problem for S and T if and only if $n \geqslant n_0$ and $\{n/(q_1 p_2)\}$ is one of the points of Φ.*

Proof. Let T^* be the two-way infinite sequence obtained by allowing the indices in T to range over $\mathbb{Z}$, i.e., $T^* = \{\lfloor kp_2/q_2 + \delta \rfloor : k \in \mathbb{Z}\}$. For a sequence of this form, it is shown in [Si, Theorems 1 and 3] that there exist integers d and r so that an integer m appears in T^* if and only if $m \equiv jr + d$ (mod p_2) for some $j = 0, 1, ..., q_2 - 1$. Thus, the condition $s_n = \lfloor np_1/q_1 + \gamma \rfloor \in T^*$ can be rewritten as follows: there exist integers j, l with $0 \leqslant j < q_2$ so that

$$jr + d + lp_2 \leqslant n\frac{p_1}{q_1} + \gamma < jr + d + lp_2 + 1. \tag{4.7}$$

Clearly, there is $n_0 \in \mathbb{Z}_0$ so that $s_n \in T$ if and only if $n \geqslant n_0$ and $s_n \in T^*$. Subtracting $jr + d + lp_2$ and multiplying by q_1, (4.7) becomes

$$0 \leqslant np_1 - (d - \gamma)\, q_1 - jrq_1 - lq_1 p_2 < q_1. \tag{4.8}$$

Now, there is $l \in \mathbb{Z}$ so that (4.8) holds if and only if

$$np_1 - \lceil (d - \gamma)\, q_1 \rceil - jrq_1 \equiv i \pmod{q_1 p_2} \tag{4.9}$$

for some $i = 0, 1, ..., q_1 - 1$. Let p_1^{-1} satisfy $p_1 p_1^{-1} \equiv 1 \pmod{q_1 p_2}$. Then (4.9) is equivalent to

$$n \equiv ip_1^{-1} + jrq_1 p_1^{-1} + \lceil (d - \gamma)\, q_1 \rceil p_1^{-1} \pmod{q_1 p_2}.$$

Letting $\xi_1 = p_1^{-1}/(q_1 p_2)$, $\xi_2 = rp_1^{-1}/p_2$, and $\varepsilon = \lceil (d - \gamma)\, q_1 \rceil p_1^{-1}/(q_1 p_2)$, we get the required result. ∎

In similar manner to the derivation of (4.6) from Lemma 4.2, we can conclude from Lemma 4.3 via an approximation argument that

$$\sigma(q), \tau(q) \leqslant \phi(q) \qquad \text{for all} \quad q \in \mathbb{Z}_+. \tag{4.10}$$

Collecting the inequalities (4.1), (4.3), (4.6), and (4.10), we see that all our functions are bounded below by $q + 3$ (for $q \geqslant 2$), and any upper bound on $\phi(q)$ is also an upper bound on the values of the other functions at q. We deduce the first upper bound from a theorem of Chung and Graham [CG] dealing with a generalization of the Steinhaus problem (see also a short proof of the theorem in [L]).

THEOREM CG. *Let* $q, N_1, N_2, ..., N_q \in \mathbb{Z}_+$ *and let* $\alpha, \gamma_1, \gamma_2, ..., \gamma_q$ *be real numbers. Let* $\Omega^i_{N_i} = \{\omega^i_n : n = 0, 1, ..., N_i - 1\}$ *where* $\omega^i_n = \{n\alpha + \gamma_i\}$, $i = 1, ..., q$. *Then the points of* $\bigcup_{i=1}^q \Omega^i_{N_i}$ *partition the circle of circumference* 1 *into intervals of at most* $3q$ *distinct lengths.*

Since any instance $\Phi \in \mathrm{TDStP}(q)$ can be presented in the form $\bigcup_{i=1}^q \Omega^i_{N_i}$, we conclude that $\phi(q) \leqslant 3q$ for all $q \in \mathbb{Z}_+$. The $3q$ upper bound in Theorem CG is sharp. However, in the two-dimensional Steinhaus problem we have more structure on the set of points partitioning the circle, which suggests a lower upper bound for $\phi(q)$.

In an early draft of this paper, where we studied only the functions ρ, σ, and τ, we proved that the values of these functions at q are bounded above by $q + C$, for some constant C. We conjectured that the actual value is $q + 3$ for all $q \geqslant 2$, and verified this for $q = 2$. Later, in an attempt to resolve this conjecture, we introduced together with R. J. Simpson the two-dimensional Steinhaus problem and made the corresponding conjecture that $\phi(q) = q + 3$ for all $q \geqslant 2$. We observed that our earlier results for ρ, σ, and τ (the $q + C$ upper bound and the exact answer for $q = 2$) carry over by essentially the same arguments to give corresponding results for ϕ. These results have now become obsolete, since Geelen and Simpson [GS], taking a different approach, obtained the following exact result (confirming the conjecture).

THEOREM GS. $\phi(q) = q + 3$ *for all* $q \geqslant 2$.

We summarize the consequences of Theorem GS and the inequalities proved in this section as follows.

THEOREM 4.4. *Let* $\chi \in \{\rho, \sigma, \tau, \pi, \phi\}$. *Then* $\chi(1) = 3$ *and* $\chi(q) = q + 3$ *for* $q \geqslant 2$.

We conclude the section with a comment. An important feature of the above upper bounds is that they show that the number of gaps can be kept low by controlling just one of the two relevant parameters ($d(\alpha)$ and $d(\beta)$ in the generalized intersection problem, l_1 and l_2 in the two-dimensional Slater problem, and q_1 and q_2 in the two-dimensional Steinhaus problem). One might hope for better results by exercising control on both relevant parameters. However, Example 4.1 indicates that there is not much to be gained this way: although the larger parameter, $q + 4$, is only slightly bigger than q, the number of gaps is the maximum achievable given that one of the parameters is q.

5. FINITENESS OF THE NUMBER OF GAPS

We have seen that if the moduli of the two Beatty sequences in the generalized intersection problem are rational numbers whose denominators are allowed to grow unbounded, then the number of gaps, though finite, may be arbitrarily large (see Example 4.1). On the basis of this evidence, one might guess that if the moduli are permitted to be irrational there may be infinitely many gaps. The purpose of this section is to deny this intuition. We prove that the number of gaps in the two-dimensional Slater problem is always finite and deduce from this a corresponding result for the generalized intersection problem. We remark that Slater [Sl2] proved the finiteness of the number of gaps $n_{i+1} - n_i$ for a class of problems of the form $\{n(\zeta_1, \zeta_2, ..., \zeta_d)\} \in C$, where C is a closed convex region on the d-dimensional torus. However, he confined attention to the case when $\zeta_1, \zeta_2, ..., \zeta_d$, and 1 are linearly independent over the rationals, which corresponds to Case 1 in our proof below.

THEOREM 5.1. *Let $\Theta = \Theta_m(\zeta, \eta)$ be a two-dimensional fractional part sequence, and let $R = R(\kappa, \lambda; \mu, \nu)$ be a rectangle on $\mathbb{T}^2$. Then $g(\Theta, R)$ is finite. Moreover, there exists a number $M < \infty$ that depends only on ζ, η, κ, and λ so that $g(\Theta, R) \leqslant M$.*

Proof. Let $\mathscr{R}(\kappa, \lambda)$ be the family of all rectangles on $\mathbb{T}^2$ of the form $R' = R(\kappa, \lambda; \mu', \nu')$, i.e., all translates of R. Our basic plan for the proof is the following. Suppose we can show that for some $M \in \mathbb{Z}_+$ the set $\{\theta_n : n = 1, 2, ..., M\}$, where $\theta_n = \{n(\zeta, \eta)\}$, intersects every $R' \in \mathscr{R}(\kappa, \lambda)$. Then it follows that no $d > M$ can be a gap, because by Lemma 3.3 every gap d has a suitable rectangle $R_d \in \mathscr{R}(\kappa, \lambda)$ which contains no $\theta_{d'}$ with $d' \in \mathbb{Z}_+$, $d' < d$. So we are able to conclude that $g(\Theta, R) \leqslant M$. In trying to execute this plan, we distinguish two main cases.

Case 1. ζ, η, and 1 are linearly independent over the rationals.

In this case, as is well known, the sequence $\{\theta_n : n \in \mathbb{Z}_+\}$ is dense in $\mathbb{T}^2$, so in particular it intersects every $R' \in \mathscr{R}(\kappa, \lambda)$. It follows from a compactness argument (e.g., by considering a partition of the torus into finitely many rectangles in $\mathscr{R}(\alpha, \beta)$, with $\alpha \leqslant \kappa/2$, $\beta \leqslant \lambda/2$) that some finite initial subsequence already intersects every $R' \in \mathscr{R}(\kappa, \lambda)$. Thus the basic plan works.

Case 2. ζ, η, and 1 are linearly dependent over the rationals.

If ζ or η is rational, the theorem follows from the finiteness of $\pi(q)$—see Section 4. Hence we assume that ζ and η are irrational. Then there exists a unique (up to proportionality) linear dependence of ζ, η, and 1 with rational coefficients, which can be expressed in the form

$$a\zeta + b\eta = c,$$

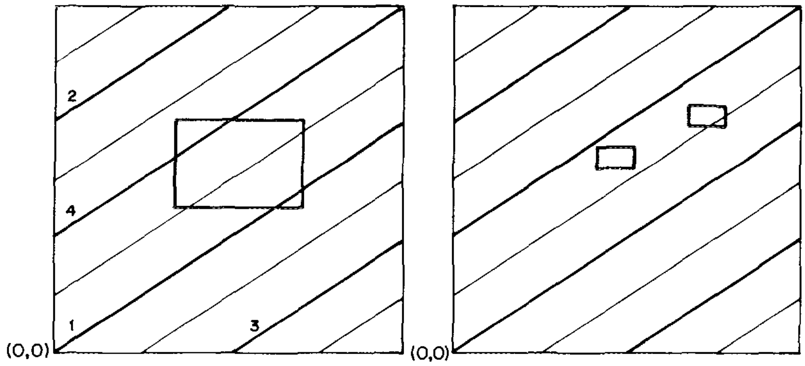

FIG. 2. Case 2 in the proof of Theorem 5.1. (The set L is drawn for $a=4$, $b=-6$. The four line segments drawn in bold together form the circle C_0, when traversed from left to right in the order of their numbering. The remaining five line segments form the circle C_1. The figure on the left illustrates Subcase 2.1, the one on the right illustrates Subcase 2.2.)

where $a, b, c \in \mathbb{Z}$, $a, b \neq 0$, and $(a, b, c) = 1$. Given this relation, it is well known that the sequence $\{\theta_n : n \in \mathbb{Z}_+\}$ is contained and dense in the set

$$L = \{\{(x, y)\} \in \mathbb{T}^2 : ax + by \in \mathbb{Z}\}.$$

Let $d = (a, b)$. Then $L = \bigcup_{i=0}^{d-1} C_i$, where

$$C_i = \{\{(x, y)\} : ax + by \equiv i \pmod{d}\}.$$

Each C_i is a circle on the torus, non-parallel to the axes. The C_i's are all parallel and equally spaced (see Fig. 2). The intersection of a rectangle in $\mathscr{R}(\kappa, \lambda)$ and the set L is a (possibly empty) finite union of intervals. We denote by $\psi(R')$ the length of a longest component of $R' \cap L$ ($\psi(R') = 0$ if $R' \cap L = \varnothing$). We distinguish two subcases.

Subcase 2.1. $\psi(R') > 0$ for every $R' \in \mathscr{R}(\kappa, \lambda)$.

A simple continuity and compactness argument shows that $\psi(R')$ attains a minimum $\psi_0 > 0$ over its domain $\mathscr{R}(\kappa, \lambda)$. The sequence $\{\theta_n : n \in \mathbb{Z}_+\}$ is dense in L, and by a one-dimensional analogue of the argument in Case 1, some finite initial subsequence of it already intersects every interval of length ψ_0 on L, hence every $R' \in \mathscr{R}(\kappa, \lambda)$. Thus the basic plan works.

Subcase 2.2. $\psi(R') = 0$ for some $R' \in \mathscr{R}(\kappa, \lambda)$.

In this case the basic plan cannot work. However, the assumption of this subcase implies a simple form for $R \cap L$; it must be one of the following:

(i) empty,

(ii) a point,

(iii) a half-open interval,

(iv) an open interval,

(v) a closed interval.

Indeed, given that one translate of the rectangle lies entirely between two adjacent line segments of L (possibly touching one or both at a corner), it is impossible for another translate to meet two line segments (if it just touches them, the intersection is at most a point, because only one corner of R belongs to R).

We are interested in the gaps between consecutive indices n of Θ such that $\theta_n \in R$. Since $\Theta \subseteq L$, we can replace here R by $R \cap L$. In Cases (i) and (ii), there is at most one index n such that $\theta_n \in R \cap L$ (using, in Case (ii), irrationality), so there are no gaps. In the other cases, $R \cap L$ is an interval I, say on C_i. Hence we need to consider only indices n such that $\theta_n \in C_i$. If $\theta_n = \{n(\zeta, \eta)\} = \{(x, y)\}$, we have $ax + by \equiv an\zeta + bn\eta \equiv nc \pmod d$; thus, letting c^{-1} satisfy $cc^{-1} \equiv 1 \pmod d$, we have $\theta_n \in C_i$ if and only if $n \equiv ic^{-1} \pmod d$. So the subsequence Θ^i of Θ formed by the terms $\theta_n \in C_i$ has the form $\{\theta_k, \theta_{k+d}, \theta_{k+2d}, ...\}$. Hence Θ^i can be viewed, up to rescaling, as a one-dimensional fractional part sequence on C_i. In Case (iii) we can apply Slater's theorem (Theorem S in Section 2) to show that there are at most three gaps for Θ^i and I, hence also for Θ and R. In Cases (iv) and (v) the same conclusion holds, by the following observation based on Slater's proof: if we replace the half-open interval in Theorem S with an open or a closed interval, it is still true that any gap is either v_1 or v_2 or $v_1 + v_2$ (for the case of a closed interval, the strict inequalities in the definitions of v_1 and v_2 should be made weak). Thus in all variants of Subcase 2.2 we have $g(\Theta, R) \leqslant 3$. ∎

COROLLARY 5.2. *Let $S = S(\alpha, \gamma)$ and $T = T(\beta, \delta)$ be two Beatty sequences. Then $g(S, T)$, $g_S(T)$, and $g_T(S)$ are finite. Moreover, there exists a number $N < \infty$ that depends only on the moduli α and β so that $g(S, T)$, $g_S(T)$, $g_T(S) \leqslant N$.*

Proof. Let $M < \infty$ be an upper bound as guaranteed in Theorem 5.1, corresponding to $\zeta = \kappa = 1/\alpha$ and $\eta = \lambda = 1/\beta$. Then by Lemma 3.1, $g(S, T) \leqslant M$. By Proposition 1.2(d) the number $N = 2M$ is an upper bound for $g_S(T)$ and $g_T(S)$ as well. ∎

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