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The majority action on infinite graphs: strings and puppets

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Abstract

We consider the following dynamic process on the 0–1 colourings of the vertices of a graph. The initial state is an arbitrary colouring, and the state at time $t+1$ is determined by assigning to each vertex the colour of the majority of its neighbours at time t (in case of a tie, the vertex retains its own colour at time t). It is known that if the graph is finite then the process either reaches a fixed colouring or becomes periodic with period two. Here we show that an infinite (locally finite) graph displays the same behaviour locally, provided that the graph satisfies a certain condition which, roughly speaking, imposes an upper bound on the growth rate of the graph. Among the graphs obeying this condition are some that are most common in applications, such as the grid graph in two or more dimensions. We also extend the analysis to more general dynamic processes, and compare our results to the seminal work of Moran in this area. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The graphs considered in this paper are simple, undirected, connected and locally finite. A *2-colouring* of a graph G is a map $x : V \to \{0,1\}$, where V is the set of vertices of G. The set of all 2-colourings is denoted by 2^V. Let $N(v)$ denote the neighbourhood of the vertex v, i.e., the set of vertices joined to v by an edge, and let $d(v)$ denote the degree of v (the cardinality of $N(v)$). The *majority action* on the

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2-colourings of G is the operator $M : 2^V \to 2^V$ defined by

$$(Mx)(v) = \begin{cases} 1 & \text{if } \sum_{u \in N(v)} x(u) > \dfrac{d(v)}{2}, \text{ or} \\ & \text{if } \sum_{u \in N(v)} x(u) = \dfrac{d(v)}{2} \text{ and } x(v) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Given an initial 2-colouring $x \in 2^V$, we may consider the infinite *string* of 2-colourings defined by $x^0 = x$ and $x^{t+1} = M(x^t)$ for $t \geqslant 0$. We are interested in the behaviour of this sequence.

This model (or close variants thereof) has proved to be useful in a large variety of domains including immunologic system research, interaction between cells, pattern recognition, etc. For a detailed survey see [1]. The underlying graph depends on the application; typical examples are cycles, paths, planar grid graphs or higher-dimensional analogues. Note that the latter examples are 'naturally' infinite graphs. In the study of this model, one is primarily interested in describing the steady states or periodic behaviour of the system.

It is known (e.g. [2,4]) that if the graph G is finite, then under the majority action (or even some more general actions as will be mentioned below), any string $\{x^t\}$ reaches a period of length one or two. That is, for all $x \in 2^V$, there exists $t \in \mathbb{N}$ such that $x^{t+2} = x^t$. To prove this, one usually associates an energy functional (Lyapunov functional) with the 2-colourings. Such functionals are nonincreasing and strictly decrease if and only if $M^2(x) \neq x$. The finiteness of the system implies that the energy reaches a quiescent state and the result follows at once.

In [3] there is a generalization of this property to locally finite connected graphs under the majority action M.

Definition 1.1. The graph G has the *period-two-property* ($p2p$) if for every $x \in 2^V$ the following holds: if for some $t \geqslant 0$ there exists $T > 0$ such that $x^{t+T} = x^t$, then $x^{t+2} = x^t$.

Moran gives a sufficient condition for a graph to have the p2p. In order to formulate it, we define the growth of a graph. Let $\rho(u, v)$ denote the distance between vertices u and v in the shortest-path metric induced by G. Let

$$B_n(v) = \{u \in V \mid \rho(u, v) \leqslant n\},$$
$$b_n(v) = |B_n(v)|.$$

It is easy to show (see [3, Proposition 1.4]) that if u, v are two vertices in a locally finite connected graph, then

$$\limsup_{n \in N} b_n(u)^{1/n} = \limsup_{n \in N} b_n(v)^{1/n}.$$

Definition 1.2. The *growth* of a locally finite connected graph G is

$$g(G) = \limsup_{n \in N} b_n(v)^{1/n},$$

where v is any vertex of G.

For example, the planar grid graph whose set of vertices is $\mathbb{Z}^2$ and whose edges are the pairs $(x_1, y_1), (x_2, y_2)$ such that either $x_1 = x_2$ and $|y_1 - y_2| = 1$, or $y_1 = y_2$ and $|x_1 - x_2| = 1$, has growth 1. The infinite binary tree has growth 2.

For every integer m, denote by m^- the greatest even integer denote strictly less than m. (E.g., $6^- = 4$, $7^- = 6$.)

Theorem. (Moran [3, Theorem 2]). *Let G be a connected graph and let $d \geqslant 3$ be an integer. Suppose that all the vertices of G have degree at most d, and that $g(G) < 1 + 2/d^-$. Then G has the $p2p$.*

Moran presents examples of graphs either with degrees bounded by d and growth exactly $1 + 2/d^-$, or with growth 1 and unbounded degrees, which violate the p2p. In fact, these graphs admit any period for appropriate initial 2-colourings.

We wish to emphasize that the p2p only excludes the existence of periods longer than two; it does not guarantee that a string $\{x^t\}$ ultimately reaches a period. To illustrate this point, consider the graph having the integers as its vertices and the pairs of successive integers as its edges. This graph, which is really the simplest infinite graph, admits nonperiodic behaviour. For example, if

$$x(n) = \begin{cases} 1 & \text{if } n \text{ is odd or } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then the colour of any vertex n will alternate between 0 and 1 until stabilizing on the colour 1 after $|n| - 1$ steps. We see that globally no period is reached, but locally the colour of every vertex is eventually constant. This phenomenon is captured by the following concept:

Definition 1.3. The graph G has the *local period-two-property* ($lp2p$) at the vertex v if for every initial 2-colouring $x \in 2^V$ there exists $t_0 = t_0(v, x)$ such that $x^{t+2}(v) = x^t(v)$ for every $t \geqslant t_0$. The graph G is a pointwise ultimately periodic with period two (*puppet*) if it has the lp2p at every vertex.

Note that if G is a puppet and $x^{t+T} = x^t$ for some $t \geqslant 0$, $T > 0$, then obviously $x^{t+2} = x^t$, and hence a puppet satisfies the p2p. But as illustrated above the concept of a puppet tells us more about the string of colourings $\{x^t\}$ than does the p2p.

Our main result, Theorem 1.4 (proved in the next section), gives a sufficient condition for a graph to be a puppet. Our condition covers all the graphs satisfying Moran's condition (see Corollary 1.5) as well as a variety of other graphs. Given an infinite, locally finite connected graph G and a vertex v of it, we let

$$S_n(v) = \{u \in V \mid \rho(u,v) = n\},$$

$$d_n(v) = \max_{u \in S_n(v)} d(u),$$

$$D_n(v) = \sum_{u \in S_n(v)} d(u).$$

The condition in the following theorem makes sense if $d_j(v_0) \geqslant 3$ for all $j \in \mathbb{N}$; essentially this entails no loss of generality (see Section 3.1).

Theorem 1.4. *Let G be an infinite, locally finite connected graph. Suppose there exists a vertex $v_0 \in V$ satisfying*

$$\sum_i \frac{D_i(v_0)}{\prod_{j=1}^{i-1}(1 + 2/d_j(v_0)^-)} < \infty. \tag{1.2}$$

Then G is a puppet.

Corollary 1.5. *Let G be a connected graph and let $d \geqslant 3$ be an integer. Suppose that all the vertices of G have degree at most d, and that $g(G) < 1 + 2/d^-$. Then G is a puppet.*

Proof. By the period-two-property of finite graphs, every finite graph is a puppet (see Fig. 1). Thus, we assume that G is infinite. Let v_0 be an arbitrary vertex of G. We assume w.l.o.g. (see Section 3.1) that $d_j(v_0) \geqslant 3$ for all $j \in \mathbb{N}$, and show that the conditions of the corollary imply condition (1.2). Indeed, since $g(G) < 1 + 2/d^-$, we may choose $q < 1$ and $n_0 \in \mathbb{N}$ such that $b_i(v_0)^{1/i}/(1 + 2/d^-) \leqslant q$ for all $i \geqslant n_0$. The convergence of the series in (1.2) follows from:

$$\sum_{i \geqslant n_0} \frac{D_i(v_0)}{\prod_{j=1}^{i-1}(1 + 2/d_j(v_0)^-)} \leqslant \sum_{i \geqslant n_0} \frac{b_i(v_0)d}{(1 + 2/d^-)^{i-1}} = d\left(1 + \frac{2}{d^-}\right) \sum_{i \geqslant n_0} \frac{b_i(v_0)}{(1 + 2/d^-)^i}$$

$$\leqslant d\left(1 + \frac{2}{d^-}\right) \sum_{i \geqslant n_0} q^i < \infty. \qquad \square$$

In the literature on finite graphs, the period-two-property has been established for more general dynamic processes. This includes the case when weights are assigned to the edges and majority is replaced by a required threshold, as well as the case when there are more than two colours. The latter was also treated in [3] for infinite graphs. Although for the sake of simplicity we focus on the majority dynamics in this paper, the ideas can be adapted to handle the more general situations (see Sections 3.3 and 3.4). It turns out that our conditions for an infinite graph to be a puppet are quite robust with respect to variations of the dynamic process under consideration.

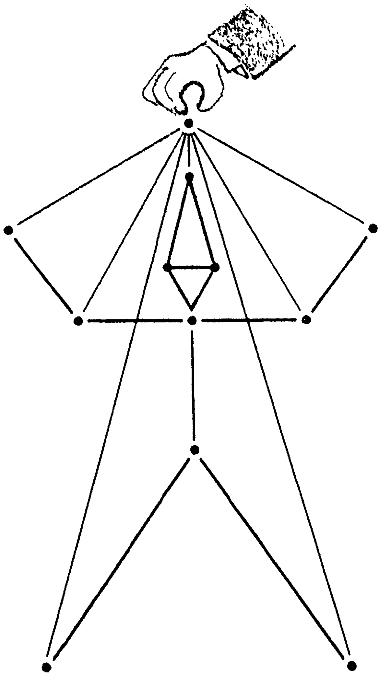

Fig. 1. A finite graph is a puppet.

2. Proof of the main theorem

For the reader's convenience, we begin by giving a proof of the period-two-property for finite graphs. Let G be a finite graph. For $x \in 2^V$ and $t \geqslant 0$, we define

$$E^t(x) = \sum_{u,v \in V} a(u,v)\,|x^{t+1}(u) - x^t(v)|,$$

where

$$a(u,v) = \begin{cases} 1 & \text{if } u, v \text{ are neighbours, or} \\ & \text{if } u = v \text{ and } d(u) \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$

and the summation is over all ordered pairs of vertices.

Claim 2.1. *$E^t(x)$ is a Lyapunov functional, i.e.,*

$$E^t(x) \leqslant E^{t-1}(x)$$

for all $x \in 2^V$ and $t \geqslant 1$.

Proof. Let us rewrite $E^t(x)$ and $E^{t-1}(x)$ in the form:

$$E^t(x) = \sum_{u\in V}\left(\sum_{v\in V} a(u,v)\left|x^{t+1}(u) - x^t(v)\right|\right),$$

$$E^{t-1}(x) = \sum_{u\in V}\left(\sum_{v\in V} a(u,v)\left|x^{t-1}(u) - x^t(v)\right|\right).$$

It suffices to show that for every $u \in V$

$$\sum_{v\in V} a(u,v)\left|x^{t+1}(u) - x^t(v)\right| \leqslant \sum_{v\in V} a(u,v)\left|x^{t-1}(u) - x^t(v)\right|. \tag{2.1}$$

This holds with equality when $x^{t+1}(u) = x^{t-1}(u)$. So, assume that $x^{t+1}(u) \neq x^{t-1}(u)$. Put

$$A_1 = \left\{v \in N(u) \,|\, x^t(v) \neq x^{t+1}(u)\right\},$$

$$A_2 = \left\{v \in N(u) \,|\, x^t(v) \neq x^{t-1}(u)\right\} = N(u)\backslash A_1$$

and (2.1) becomes

$$|A_1| + a(u,u)|x^{t+1}(u) - x^t(u)| \leqslant |A_2| + a(u,u)|x^{t-1}(u) - x^t(u)|. \tag{2.2}$$

By the definitions of the majority action (1.1) and of $a(u,u)$ we deduce that (i) when $d(u)$ is odd, $|A_1| < |A_2|$ and $a(u,u) = 0$; (ii) when $d(u)$ is even, either $|A_1| < |A_2|$, or $|A_1| = |A_2|$ and $x^{t+1}(u) = x^t(u)$, and $a(u,u) = 1$. In either case we have strict inequality in (2.2). □

In fact, the proof just given shows that equality in Claim 2.1 holds if and only if every $u \in V$ satisfies $x^{t+1}(u) = x^{t-1}(u)$. Now, $\{E^t(x)\}$ is a nonincreasing sequence of nonnegative integers which stabilizes eventually, forcing $x^{t+1} = x^{t-1}$ for t large enough. Thus, the period-two-property holds.

Proof of Theorem 1.4. Let G be an infinite, locally finite connected graph, and let v_0 be a vertex satisfying (1.2). To simplify the notation, we shall henceforth omit v_0 from it. Thus, B_n, S_n, d_n and D_n will stand for $B_n(v_0), S_n(v_0), d_n(v_0)$ and $D_n(v_0)$, respectively.

For every integer $l \geqslant 1$, we construct a functional $E_l^t(x)$ which is obtained from the functional $E^t(x)$ used in the finite case by truncation to the ball of radius l around v_0, and a modification of the definition of $a(u,u)$ which affects only vertices of even degree on the boundary of that ball. The precise definition is: for $x \in 2^V$ and $t \geqslant 0$,

$$E_l^t(x) = \sum_{u,v\in B_l} a_l(u,v)|x^{t+1}(u) - x^t(v)|,$$

where

$$a_l(u,v) = \begin{cases} 1 & \text{if } u, v \text{ are neighbours, or} \\ & \text{if } u = v,\ d(u) \text{ is even and } |N(u) \cap B_l| \geqslant \frac{d(u)}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

As the initial 2-colouring x will be fixed throughout the proof, we omit it from our notation and write E_l^t for $E_l^t(x)$.

Due to the truncation, we do not expect E_l^t to be nonincreasing in t. However, we shall estimate from above the difference $\Delta E_l^t = E_l^t - E_l^{t-1}$. First, we rewrite it as

$$\Delta E_l^t = \sum_{u \in B_l} \sum_{v \in B_l} a_l(u,v)(|x^{t+1}(u) - x^t(v)| - |x^{t-1}(u) - x^t(v)|). \tag{2.3}$$

Note that u contributes to the sum only if $x^{t+1}(u) \neq x^{t-1}(u)$. Denote

$$C_i^t = \{u \in S_i \mid x^{t+1}(u) \neq x^{t-1}(u)\}, \quad c_i^t = |C_i^t|,$$

$$\Sigma^+(u) = \sum_{v \in B_l} a_l(u,v)|x^{t+1}(u) - x^t(v)|,$$

$$\Sigma^-(u) = \sum_{v \in B_l} a_l(u,v)|x^{t-1}(u) - x^t(v)|.$$

With these notations (2.3) can be rewritten as follows:

$$\Delta E_l^t = \sum_{0 \leqslant i \leqslant l} \sum_{u \in C_i^t} (\Sigma^+(u) - \Sigma^-(u))$$
$$= \sum_{0 \leqslant i \leqslant l-1} \sum_{u \in C_i^t} (\Sigma^+(u) - \Sigma^-(u)) + \sum_{u \in C_l^t} (\Sigma^+(u) - \Sigma^-(u)).$$

The sum above was separated to vertices u in C_i^t, $i < l$, the neighbours of which are all contained in B_l, and to vertices u in C_l^t which may have neighbours outside B_l. Let us handle the two summands separately.

If $u \in C_i^t$, $i < l$, then the argument in the proof of Claim 2.1 can be carried out to show that $\Sigma^+(u) < \Sigma^-(u)$. As these are integers, we have

$$\Sigma^+(u) - \Sigma^-(u) \leqslant -1 \quad (u \in C_i^t,\ i < l). \tag{2.4}$$

On S_l, this difference may be positive, but we prove that

$$\Sigma^+(u) - \Sigma^-(u) \leqslant \frac{d(u)^-}{2} \quad (u \in C_l^t). \tag{2.5}$$

In the proof of (2.5) we distinguish between three cases:

(i) $d(u)$ is odd. In this case $\Sigma^+(u) \leqslant d(u)^-/2$ due to the majority rule, and we are done.

(ii) $d(u)$ is even and $|N(u) \cap B_l| < d(u)/2$. In this case the number of nonzero summands in $\Sigma^+(u)$ is less than $d(u)/2$, and hence $\Sigma^+(u) \leqslant d(u)^-/2$.

(iii) $d(u)$ is even and $|N(u) \cap B_l| \geqslant d(u)/2$. By the majority rule $\Sigma^+(u) \leqslant d(u)/2$. Thus, if $\Sigma^-(u) > 0$ we are done. Otherwise, we have $x^t(v) = x^{t-1}(u)$, and so $x^t(v) \neq x^{t+1}(u)$, for every $v \in B_1(u) \cap B_l$ (recall that in this case $a_l(u,u) = 1$). The number of these vertices is at least $d(u)/2 + 1$ (by the condition of case (iii) and taking into account the vertex u itself), which is impossible.

By (2.4) and (2.5) we have

$$\Delta E_l^t \leqslant -\sum_{i=0}^{l-1} c_i^t + \sum_{u\in C_l^t} \frac{d(u)^-}{2}. \tag{2.6}$$

From (2.6) we obtain

$$\Delta E_l^t \leqslant -\sum_{i=0}^{l-1} c_i^t + \frac{d_l^- c_l^t}{2}. \tag{2.7}$$

For $n > k \geqslant 0$ we construct the functional $E_{n,k}^t$ which is a weighted combination of E_l^t, $k+1 \leqslant l \leqslant n$, as follows:

$$\begin{aligned} E_{n,k}^t = {} & E_n^t + \frac{2}{d_{n-1}^-}E_{n-1}^t + \left(1+\frac{2}{d_{n-1}^-}\right)\frac{2}{d_{n-2}^-}E_{n-2}^t \\ & + \left(1+\frac{2}{d_{n-1}^-}\right)\left(1+\frac{2}{d_{n-2}^-}\right)\frac{2}{d_{n-3}^-}E_{n-3}^t + \cdots + \prod_{j=k+2}^{n-1}\left(1+\frac{2}{d_j^-}\right)\frac{2}{d_{k+1}^-}E_{k+1}^t. \end{aligned} \tag{2.8}$$

These weights were chosen so that, when we apply (2.6) to E_n^t and (2.7) to the other terms in (2.8), the quantities c_i^t for $k+1 \leqslant i \leqslant n-1$ will be cancelled out. Indeed, using the identity

$$\begin{aligned} & 1 + a_1 + (1+a_1)a_2 + (1+a_1)(1+a_2)a_3 + \cdots + (1+a_1)\cdots(1+a_{m-1})a_m \\ & \quad = (1+a_1)(1+a_2)\cdots(1+a_m), \end{aligned} \tag{2.9}$$

we obtain

$$\begin{aligned} \Delta E_{n,k}^t = {} & \Delta E_n^t + \frac{2}{d_{n-1}^-}\Delta E_{n-1}^t + \left(1+\frac{2}{d_{n-1}^-}\right)\frac{2}{d_{n-2}^-}\Delta E_{n-2}^t + \cdots \\ & + \prod_{j=k+2}^{n-1}\left(1+\frac{2}{d_j^-}\right)\frac{2}{d_{k+1}^-}\Delta E_{k+1}^t \\ \leqslant {} & -\sum_{i=0}^{n-1} c_i^t + \sum_{u\in C_n^t}\frac{d(u)^-}{2} + \frac{2}{d_{n-1}^-}\left(-\sum_{i=0}^{n-2} c_i^t + \frac{d_{n-1}^- c_{n-1}^t}{2}\right) \\ & + \left(1+\frac{2}{d_{n-1}^-}\right)\frac{2}{d_{n-2}^-}\left(-\sum_{i=0}^{n-3} c_i^t + \frac{d_{n-2}^- c_{n-2}^t}{2}\right) + \cdots \\ & + \prod_{j=k+2}^{n-1}\left(1+\frac{2}{d_j^-}\right)\frac{2}{d_{k+1}^-}\left(-\sum_{i=0}^{k} c_i^t + \frac{d_{k+1}^- c_{k+1}^t}{2}\right) \\ = {} & -\prod_{j=k+1}^{n-1}\left(1+\frac{2}{d_j^-}\right)\sum_{i=0}^{k} c_i^t + \sum_{u\in C_n^t}\frac{d(u)^-}{2}. \end{aligned} \tag{2.10}$$

(Here and in the sequel we adopt the convention that a product of the form $\prod_{j=n}^{n-1}$ equals one; a sum of the form $\sum_{j=n}^{n-1}$ equals zero.) If we follow the dynamic process from time t to time $t+T$ and use (2.10), we obtain an upper bound on the increment of $E_{n,k}$:

$$E_{n,k}^{t+T} - E_{n,k}^{t} = \sum_{\tau=t+1}^{t+T} \Delta E_{n,k}^{\tau} \leqslant - \prod_{j=k+1}^{n-1} \left(1 + \frac{2}{d_j^-}\right) \sum_{\tau=t+1}^{t+T} \sum_{i=0}^{k} c_i^{\tau} + T \sum_{u \in S_n} \frac{d(u)^-}{2}. \tag{2.11}$$

In particular, setting $t=0$ and recalling that $D_n = \sum_{u \in S_n} d(u)$, we have

$$E_{n,k}^{T} - E_{n,k}^{0} \leqslant - \prod_{j=k+1}^{n-1} \left(1 + \frac{2}{d_j^-}\right) \sum_{\tau=1}^{T} \sum_{i=0}^{k} c_i^{\tau} + TD_n. \tag{2.12}$$

The next step is to give an upper bound on $E_{n,k}^t$, which will imply a lower bound on $E_{n,k}^T - E_{n,k}^0$. First, by the majority rule, we have

$$E_l^t \leqslant \sum_{u \in B_l} \frac{d(u)}{2}. \tag{2.13}$$

If we assume that $l > k \geqslant 0$ and denote

$$D_{\leqslant k+1} = \sum_{i=0}^{k+1} D_i,$$

we can deduce from (2.13), being generous about division by 2, that

$$E_l^t \leqslant D_{\leqslant k+1} + \sum_{i=k+2}^{l} D_i \quad (l > k \geqslant 0). \tag{2.14}$$

Using (2.14) to estimate the terms of (2.8), and applying identity (2.9), we get

$$\begin{aligned} E_{n,k}^t &\leqslant D_{\leqslant k+1} + \sum_{i=k+2}^{n} D_i + \frac{2}{d_{n-1}^-}\left(D_{\leqslant k+1} + \sum_{i=k+2}^{n-1} D_i\right) \\ &\quad + \left(1 + \frac{2}{d_{n-1}^-}\right) \frac{2}{d_{n-2}^-}\left(D_{\leqslant k+1} + \sum_{i=k+2}^{n-2} D_i\right) + \cdots \\ &\quad + \prod_{j=k+2}^{n-1}\left(1 + \frac{2}{d_j^-}\right) \frac{2}{d_{k+1}^-} D_{\leqslant k+1} \\ &= \prod_{j=k+1}^{n-1}\left(1 + \frac{2}{d_j^-}\right) D_{\leqslant k+1} + \sum_{i=k+2}^{n} \prod_{j=i}^{n-1}\left(1 + \frac{2}{d_j^-}\right) D_i. \end{aligned}$$

Thus,

$$E_{n,k}^{T} - E_{n,k}^{0} \geqslant -E_{n,k}^{0} \geqslant - \prod_{j=k+1}^{n-1}\left(1 + \frac{2}{d_j^-}\right) D_{\leqslant k+1} - \sum_{i=k+2}^{n} \prod_{j=i}^{n-1}\left(1 + \frac{2}{d_j^-}\right) D_i. \tag{2.15}$$

From (2.12) and (2.15) we get

$$-\prod_{j=k+1}^{n-1}\left(1+\frac{2}{d_j^-}\right)D_{\leqslant k+1}-\sum_{i=k+2}^{n}\prod_{j=i}^{n-1}\left(1+\frac{2}{d_j^-}\right)D_i$$
$$\leqslant-\prod_{j=k+1}^{n-1}\left(1+\frac{2}{d_j^-}\right)\sum_{\tau=1}^{T}\sum_{i=0}^{k}c_i^\tau+TD_n. \tag{2.16}$$

Dividing by $\prod_{j=k+1}^{n-1}(1+2/d_j^-)$ in (2.16) and rearranging the terms we obtain

$$\sum_{\tau=1}^{T}\sum_{i=0}^{k}c_i^\tau\leqslant D_{\leqslant k+1}+\sum_{i=k+2}^{n}\frac{D_i}{\prod_{j=k+1}^{i-1}(1+2/d_j^-)}+\frac{TD_n}{\prod_{j=k+1}^{n-1}(1+2/d_j^-)}.$$

With the notation $R_k=\prod_{j=1}^{k}(1+2/d_j^-)$ we get

$$\sum_{\tau=1}^{T}\sum_{i=0}^{k}c_i^\tau\leqslant D_{\leqslant k+1}+R_k\sum_{i=k+2}^{n}\frac{D_i}{\prod_{j=1}^{i-1}(1+2/d_j^-)}+\frac{R_kTD_n}{\prod_{j=1}^{n-1}(1+2/d_j^-)}. \tag{2.17}$$

We are ready now to make use of the theorem's assumption (1.2). Let us denote by S the sum of the series in (1.2). For fixed k and T and $n\to\infty$ (2.17) yields

$$\sum_{\tau=1}^{T}\sum_{i=0}^{k}c_i^\tau\leqslant D_{\leqslant k+1}+R_kS, \tag{2.18}$$

since the last term of (2.17) is a constant multiple of the general term of the convergent series in (1.2).

The left-hand side of (2.18) represents the total number of occurrences of $x^{\tau+1}(u)\neq x^{\tau-1}(u)$ inside $B_k(v_0)$ during the first T iterations of the dynamic process. Thus, (2.18) shows that for fixed k this number is bounded uniformly in T. Since any given vertex v belongs to $B_k(v_0)$ for some k, the number of occurrences of $x^{\tau+1}(v)\neq x^{\tau-1}(v)$ is finite. We conclude that there exists $t_0=t_0(v,x)$ such that $x^{\tau+1}(v)=x^{\tau-1}(v)$ for all $\tau>t_0$, and the local period-two-property at the vertex v holds. □

3. Remarks and extensions

3.1. Spheres with maximal degree two

In order to avoid dividing by zero in (1.2), we had to assume $d_j(v_0)\geqslant 3$ for all j. Here we show how to adapt the condition to the case when $d_j(v_0)=2$ for some values of j (obviously $d_j(v_0)<2$ is impossible in an infinite, locally finite connected graph). We distinguish between the two cases.

(i) $d_j(v_0)=2$ for finitely many values of j. In this case, let j_0 be such that $d_j(v_0)\geqslant 3$ for all $j\geqslant j_0$. Then if we replace the product $\prod_{j=1}^{i-1}(1+2/d_j(v_0)^-)$ in the denominator of the general term of the series in (1.2) by $\prod_{j=j_0}^{i-1}(1+2/d_j(v_0)^-)$ and ignore the first j_0 terms of the series, the theorem remains true with essentially the same proof.

(ii) $d_j(v_0)=2$ for infinitely many values of j. In this case the graph is a puppet without any further assumptions. To see this, follow the argument in the proof up to (2.7) which gives, for l such that $d_l(v_0)=2$,

$$\Delta E_l^t \leqslant -\sum_{i=0}^{l-1} c_i^t. \tag{3.1}$$

Thus, given a vertex v, we can choose l such that v belongs to $B_{l-1}(v_0)$ and (3.1) holds. This permits to conclude that the lp2p holds at the vertex v.

3.2. A weaker condition for the p2p

Here we point out that the first part of the proof of the theorem suffices for establishing the period-two-property, in fact, under a somewhat weaker condition than (1.2). Suppose that the graph G and the vertex v_0 satisfy

$$\liminf_{n\to\infty} \frac{\sum_{u\in S_n(v_0)} d(u)^-}{\prod_{j=1}^{n-1}(1+2/d_j(v_0)^-)} = 0. \tag{3.2}$$

Then we claim that G has the p2p. Indeed, assume that $x^{t+T}=x^t$. Then, by (2.11) we have for every $n>k\geqslant 0$,

$$0 = E_{n,k}^{t+T}(x) - E_{n,k}^t(x) \leqslant -\prod_{j=k+1}^{n-1}\left(1+\frac{2}{d_j(v_0)^-}\right)\sum_{\tau=t+1}^{t+T}\sum_{i=0}^{k} c_i^\tau + T\sum_{u\in S_n(v_0)}\frac{d(u)^-}{2}.$$

It follows that

$$\sum_{\tau=t+1}^{t+T}\sum_{i=0}^{k} c_i^\tau \leqslant \frac{T}{2}\cdot\frac{\sum_{u\in S_n(v_0)} d(u)^-}{\prod_{j=k+1}^{n-1}(1+2/d_j(v_0)^-)}. \tag{3.3}$$

Hence, given a vertex v, we can choose k such that $v\in B_k(v_0)$ and then let n tend to infinity along a sequence realizing the lim inf in (3.2), thereby concluding that the left-hand side of (3.3) is zero. This implies that $x^{\tau+1}(v)=x^{\tau-1}(v)$ for $t+1\leqslant\tau\leqslant t+T$. As this holds for every v, we have $x^{t+2}=x^t$.

3.3. General threshold dynamics

The majority action (1.1) is a special case of the *symmetric threshold dynamics* $\tilde{M}: 2^V \to 2^V$ defined by

$$(\tilde{M}x)(v) = \begin{cases} 1 & \text{if } \sum_{u\in B_1(v)} \alpha(u,v)x(u) \geqslant \beta(v), \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

where $\alpha(u,v)$ and $\beta(v)$ are real-valued coefficients of the dynamics, satisfying the symmetry condition $\alpha(u,v)=\alpha(v,u)$; recall that our graphs are locally finite, and hence the sum that appears in (3.4) is always a finite one. This framework is rather flexible and permits to express, inter alia, majority rule with different tie breaking provisions, as well as minority rule (note that negative weights are allowed).

For any fixed system of coefficients $\alpha(u,v), \beta(v)$ the action $\tilde{M}$ defined in (3.4) gives rise, for every initial 2-colouring x, to a string of colourings $\{\tilde{M}^t(x)\}$ generated by repeated applications of $\tilde{M}$ to x. We say that G is an *$\tilde{M}$-puppet* if it satisfies the analogous condition to Definition 1.3.

By suitably modifying the proof of Theorem 1.4 we can obtain a sufficient condition for G to be an $\tilde{M}$-puppet, expressed as the convergence of an appropriate series, the terms of which depend on the graph G and on the coefficients $\alpha(u,v), \beta(v)$. Instead of giving this rather complicated condition, we state here a simpler condition, expressed in terms of the graph G alone. A graph obeying this condition is an $\tilde{M}$-puppet for a whole class of 'nicely behaved' threshold dynamics. Denote by Λ the class of symmetric threshold dynamics $\tilde{M}$ satisfying the following two conditions:

Weight boundedness: There exists $W = W(\tilde{M}) < \infty$ such that $|\alpha(u,v)| \leqslant W$ for all $u,v \in V$ with $u \in B_1(v)$.

Uniform separation: There exists $\varepsilon = \varepsilon(\tilde{M}) > 0$ such that for all $v \in V$ and all $A, A' \subseteq B_1(v)$,

$$\sum_{u\in A} \alpha(u,v) < \beta(v) \leqslant \sum_{u\in A'} \alpha(u,v) \Rightarrow \sum_{u\in A'} \alpha(u,v) - \sum_{u\in A} \alpha(u,v) \geqslant \varepsilon.$$

The proof of the following theorem goes along the general lines of the proof of Theorem 1.4, and is therefore omitted.

Theorem 3.1. *Let G be a connected graph of bounded degree with growth $g(G)=1$. Then G is an $\tilde{M}$-puppet for every $\tilde{M} \in \Lambda$.*

3.4. Multiple colours

Here we consider the case when $\{0,1\}$ is replaced by a finite set of r colours. An *r-colouring* of a graph G is a function $x : V \to \{0,1,\dots,r-1\}$, where V denotes, as usual, the set of vertices of G. For updating r-colourings the majority rule is naturally replaced by the plurality rule, but we need a different tie breaking rule. The precise definition of the *plurality action* $P_r : \{0,1,\dots,r-1\}^V \to \{0,1,\dots,r-1\}^V$ is

$$(P_r x)(v) = i \Leftrightarrow \begin{array}{l} |\{u \in N(v) \,|\, x(u) = i\}| \geqslant |\{u \in N(v) \,|\, x(u) = j\}| \\ \text{for all } j, \text{ with strict inequality for all } j > i. \end{array}$$

For every initial r-colouring of x this generates a string $\{P_r^t(x)\}$ of r-colourings. We say that G is an *r-puppet* if the analogous condition to Definition 1.3 holds. Note that we still require the period to be two, even though the number of colours may be higher. In order to state a sufficient condition for G to be an r-puppet, we need the following notation. For an integer m let

$$m^{(r)} = \begin{cases} \frac{m-1}{2} & \text{if } m \text{ is odd,} \\ \frac{m}{2} - \frac{1}{r} & \text{if } m \text{ is even.} \end{cases}$$

Theorem 3.2. *Let G be an infinite, locally finite connected graph and let $r \geqslant 2$. Suppose there exists a vertex $v_0 \in V$ satisfying*

$$\sum_i \frac{D_i(v_0)}{\prod_{j=1}^{i-1}(1+1/(rd_j(v_0)^{(r)}))} < \infty.$$

Then G is an r-puppet.

The proof of this theorem is in the spirit of the proof of Theorem 1.4 and is therefore omitted. Similar to Corollary 1.5, we have here the following.

Corollary 3.3. *Let G be a connected graph and let $r, d \geqslant 2$. Suppose that all the vertices of G have degree at most d, and that*

$$g(G) < 1 + \frac{1}{rd^{(r)}}. \tag{3.5}$$

Then G is an r-puppet.

Moran [3, Theorem 4] proved that if G is a connected graph with degrees bounded by $d \geqslant 2$, and

$$g(G) < \left(1 + \frac{2}{d-1}\right)^{1/r}, \tag{3.6}$$

then G has the period-two-property under the plurality action P_r. Note that (3.5) is a weaker condition than (3.6) if d is odd, whereas for even values of d the relative strength of the two conditions depends on d and r.

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