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Sharing the Cost of a Capacity Network

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We consider a communication network where each pair of users requests a connection guaranteeing a certain capacity. The cost of building capacity is identical across pairs. Efficiency is achieved by any maximal cost spanning tree. We construct cost sharing methods ensuring standalone core stability, monotonicity of one's cost share in one's capacity requests, and continuity in everyone's requests. We define a solution for simple problems where each pairwise request is zero or one, and extend it piecewise linearly to all problems. The uniform solution obtains if we require one's cost share to be weakly increasing in everyone's capacity request. In the solution, we propose, on the contrary, one's cost share is weakly decreasing in other agents' requests. The computational complexity of both solutions is polynomial in the number of users. The uniform solution admits a closed form expression, and is the analog of a popular solution for the minimal cost spanning tree problem.

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1. Introduction. Game theory applies fairness and incentive compatibility concepts to the analysis of combinatorial cost sharing problems. The earliest example is the minimal cost spanning tree problem (thereafter MCST), that remains to this day the combinatorial game most studied by game theorists: see the literature discussion below. We have n agents and a source, living on $n+1$ distinct nodes; each agent needs to be connected to the source, either directly or indirectly through a path over any number of nodes. Given the cost of connecting each pair of nodes, the cheapest graph meeting this requirement is a tree rooted at the source; this tree is constructed by the following algorithm due to Kruskal [16]: build a spanning tree starting from the cheapest link, always choosing one of the cheapest links that can be added to the ones already selected without creating a cycle.[1]

The related *capacity synthesis* problem that is the object of this paper is a simple version of the well-known *network synthesis* problem (Ahuja et al. [1], Tamir [27]). It has not been discussed with the tools of cooperative game theory before, yet appears to have more potential for economic applications.

The n agents share a network for bilateral exchange of information, transportation of commodities along roads or shipping channels, distributing utilities along a grid, etc. The traffic between agents i and j requires a certain capacity t_{ij} (bandwidth, width of a road, depth of a channel, length of a runway, etc.). The problem is to choose a minimal cost graph such that any pair i, j is connected by a path of which every edge has capacity at least t_{ij}. Computing a feasible network of minimal cost is greatly simplified by the following assumptions: capacity is not divisible across multiple paths; an edge of capacity t can be used jointly by any pair of agents as long as each of their demands is less than t; and the cost function is the same on every edge. The first two assumptions are realistic in transportation problems where there is no congestion (a runway shared by planes of different sizes—see Littlechild and Owen [17], a ship channel or a road where usage is sequential and delay is not a serious issue). The third one clearly limits the applicability of our model.

The MCST and capacity synthesis problems are closely related: in the latter an optimal network is any spanning tree of *maximal* cost with respect to the capacity matrix $t = [t_{ij}]$, hence it is computed efficiently by the same algorithm, applied from the largest capacity demand downward instead of the cheapest link upward.

The cost sharing problem is to divide the efficient cost among the various parties benefiting from the externalities in production. In the literature, the central incentive and fairness property is (i) *standalone core stability*. For instance in the capacity problem the standalone cost of a subset (coalition) S of agents is $v^+(S, t)$, the

[1] This can be implemented with near quadratic running time $O(n^2 \log n)$. Another variant of the algorithm, due to Prim [23], may be implemented to run in quadratic time.

minimal cost of serving all traffic demands of S, including traffic between agents in S and agents outside S. The standalone core property says that S should not be charged more than $v^+(S,t)$. Two other compelling requirements are (ii) *continuity*: cost shares y_i depend continuously upon t, and (iii) *monotonicity*: y_i is weakly increasing in t_{ij} for all j.

To be implementable for problems of any size, centralized mechanisms such as the cost sharing methods discussed here must be of (iv) *polynomial complexity* in the number n of users. In many combinatorial optimization problems, the efficient cost cannot be computed in polynomial time with respect to the number of agents involved. Examples are given in §1.1 below. However, in the few favorable cases where it can, such as our capacity synthesis and the MCST problems, there always exist division methods of polynomial complexity (for instance charging the entire cost to a particular agent), and the interesting question is to find solutions combining computational tractability with other requirements of fairness and incentive compatibility.

The Shapley value of the standalone cooperative game $S \to v^+(S,t)$ illustrates the difficulty that we will resolve. Unlike in the MCST problem, where the Shapley value of the corresponding game is not always core stable (Bird [7]), it is a natural selection of the standalone core in our problem because the game $(N, v^+(\cdot, t))$ is concave (Lemma 3.1), and is clearly continuous. Surprisingly, it is not monotonic (Proposition 4.1); moreover no algorithm of complexity less than exponential is known to compute it.

We construct in §5 a family of solutions meeting the three requirements (i), (ii), and (iii). We define a solution first for capacity matrices t of which every coordinate t_{ij} is zero or one, then extend it linearly to any cone where the ordering of the coordinates remains fixed. We single out two such piecewise linear solutions with very different normative flavors. They are both of polynomial complexity.

The uniform solution (§6), applies the Shapley value not to the original game $(N, v^+(\cdot, t))$ derived from the capacity matrix t, but to the game $(N, v^+(\cdot, t^*))$ derived from the *irreducible matrix* t^*, namely, the largest matrix weakly above t and with the same optimal cost as t. The uniform solution admits a very simple closed form expression (Theorem 6.1). The second solution, dubbed the BHM solution, after the names of the authors, (§7) is more complicated to describe (it is not given in closed form).

In §8 we compare the normative properties of our two solutions. The main observation is that they rely on opposite interpersonal effects of capacity demands.

The *solidarity* axiom requires y_i to be weakly increasing in *all* demands t_{jk}: agent i cannot be charged less when other users' demands increase, she may even pay more. This powerful monotonicity property (combined with a mild dummy axiom) implies *population monotonicity*: cost shares y_i weakly increase when we add a new agent.

The opposite of solidarity is *cross-monotonicity*: y_i is weakly *de*creasing in the demands t_{jk} of other agents. Such an increase is not agent i's doing and may create additional cost savings, thus she should not be charged more. The BHM solution meets cross-monotonicity, whereas the uniform solution meets solidarity.

Any monotonic and solidary solution (in particular the uniform one) must be *reductionist*, that is it only takes into account the irreducible capacity matrix t^* (Lemma 8.1). This simplification erases relevant differences in individual capacity requests. Consider the situation where agent 1 is a communication "hub" in the sense that the only traffic is between agent 1 and other agents: $t_{1i} = 1$ for all $i \neq 1$, and $t_{ij} = 0$ for all $i, j \neq 1$. The corresponding matrix t^* is fully symmetric ($t^*_{ij} = 1$ for all i, j). A reductionist solution gives the same cost shares for all t resulting in the same t^*: therefore if agent 1 bears $1/n$ of the total costs given the symmetric matrix t^*, she must bear the same equal share under t where she is the hub. The difference in the capacity demands of agent 1 versus other agents is ignored. It is more natural to charge more to the hub agent than to others (under BHM she pays half of the total cost).

Interpersonal comparison of demands leads to the *ranking* axiom, generalizing the above discussion: if agent i has strictly higher demands than j, she should be charged more, $\{t_{ik} > t_{jk}$ for all $k \neq i, j\} \Rightarrow y_i > y_j$. Unlike any solidary or reductionist solution, the *BHM* solution meets ranking, and even stricter versions of monotonicity and of ranking defined in §8.1. The pattern of incompatibility between, on the one hand solidarity; population monotonicity; and reductionism; and on the other hand cross-monotonicity; ranking; and a natural decomposition property is quite general, see e.g., Proposition 8.3 in §8.4.

The concluding section discusses some incentive advantages of reductionist (in particular, the uniform) solutions, collects our main results in a table, and presents some open problems.

1.1. Relation to the literature. Many simple combinatorial optimization problems have no known solutions of polynomial complexity, and it is believed that none exists.[2] Examples include the minimal cost Steiner tree

[2] If $P \neq NP$, see Garey and Johnson [12].

problem (Megiddo [20]), and the traveling salesman problem (see Sharkey [25] for a survey). Neither does the literature offer any normatively appealing methods to share the efficient (or even an approximately efficient) cost in these problems.

Thus the fair division literature in combinatorial optimization games focuses almost exclusively on the MCST problem. The earliest solution due to Bird [7] proposes a standalone core selection computed easily once we identify a minimal cost spanning tree. We adapt it to our model in Definition 4.1 and Lemma 4.1. Subsequent authors discussed first some computationally untractable selections of the standalone core, such as the nucleolus (Granot and Huberman [13, 14]) and the Shapley value (Kar [15]).

The construction of our uniform and BHM solutions is inspired by some recent results on the MCST problem. Just as we speak here of the irreducible capacity matrix t^*, in the MCST problem the analog irreducible cost matrix and irreducible core already play the central role in Bird's [7] seminal contribution; see also Tijs et al. [29]. The analog of our uniform solution is a solution that has been (re)discovered many times for the MCST problem: it is the equal remaining obligations rule suggested by Potters and discussed first by Feltkamp et al. [11], then axiomatized by Brânzei et al. [9]; the average of a family of population monotonic solutions in Norde et al. [21]; and the Shapley value of the irreducible standalone game in Bergantiños and Vidal-Puga [3]–[4]. Thus it is dubbed the *folk solution* of the MCST problem in Bogomolnaia and Moulin [8]. The piecewise linear extension technique comes from the MCST literature as well, where its role is similarly crucial; see Bergantiños and Vidal-Puga [5], Norde et al. [21], Brânzei et al. [9], and Tijs et al. [28, 29].

The most recent discussion of the MCST problem focuses on monotonicity properties with respect to costs, corresponding to our axioms of monotonicity and solidarity. Kar [15] axiomatizes the Shapley value of the standalone game, which is not always a core selection. Dutta and Kar [10] construct a monotonic yet not continuous core selection. Bergantiños and Vidal-Puga provide several characterizations of the folk solution: in Bergantiños and Vidal-Puga [3] they rely mostly on solidarity, in Bergantiños and Vidal-Puga [5] on an additivity property implying piecewise linearity.

Finally in their companion paper Bogomolnaia and Moulin [8] develops in the MCST problem the same critique of the folk solution as we apply here to the uniform and all reductionist solutions: by throwing away much relevant, albeit counterfactual, information these solutions violate elementary tests of fairness such as ranking. The paper proposes a plausible polynomial alternative to the folk solution satisfying the analog ranking property. Note that cross-monotonicity makes no sense in the MCST problem,[3] and the BHM solution has no analog in that model. Thus the capacity synthesis model is normatively more versatile than the MCST one.

2. Minimal cost graph, maximal cost spanning tree.

Given is the set N of n agents. An undirected pair ij of N is called an *edge*, their set (of cardinality $n(n-1)/2$) is denoted $N(2)$. We define similarly $S(2)$ for any subset S of N such that $|S| \geq 2$: $ij \in S(2)$ means that i, j are distinct and both in S.

A *capacity synthesis problem* is a pair (N, t) where $t = [t_{ij}]_{ij \in N(2)}$ is the list of nonnegative capacity requests on each edge. We assume that the cost of building x units of capacity is the same between any two agents i, j. This is an important simplifying assumption. Without loss of generality we then set the cost of x units to be \$$x$.

An undirected graph G on N is a subset of $N(2)$. A *capacity network* on N is a pair (G, c) where each edge e in G has a nonnegative capacity c_e. The capacity network (G, c) is *feasible* for (N, t) if it supports the capacity requests t: any two $i, j \in N$ are connected in G by at least one path on all edges of which we have $c_e \geq t_{ij}$. A *minimal cost* capacity network is a feasible network (G, c) minimizing total cost $\sum_{e \in G} c_e$ among all feasible networks. We write $V(N, t)$ or simply $V(t)$ for this minimal cost.

A *spanning tree* is a tree Γ connecting all agents of N. The following facts are well known (Ahuja et al. [1, Chapter 13]). A minimal cost capacity network (G, c) is a tree (more precisely, G is connected and has no cycle with positive edge capacities). A minimal cost capacity network may specify edge capacities that are not inherited from the corresponding capacity requests. For example, in the problem with $N = \{1, 2, 3\}$ and $t_{12} = t_{13} = 1$, $t_{23} = 0$, letting $G = \{12, 13\}$ and $c_{12} = c_{13} = 1$ gives a natural minimal cost capacity network, but letting $G = \{12, 23\}$ and $c_{12} = c_{23} = 1$ gives another one. Yet, among the minimal cost capacity networks (G, c) for a given problem (N, t) we can always find one or more in which $c_{ij} = t_{ij}$ for all $ij \in G$. In fact, for a spanning tree Γ the three following statements are equivalent:

- The capacity network (Γ, t) (where the capacity of any edge ij is exactly t_{ij}) is feasible for (N, t).
- The cost $t(\Gamma) = \sum_{e \in \Gamma} t_e$ is maximal among all spanning trees.
- The capacity network (Γ, t) is a minimal cost capacity network.

[3] It must be violated by any efficient solution charging nonnegative cost shares. We omit the easy two agent one source examples.

We call such a tree a *maximal spanning tree*. It always exists, but may not be unique.

From the computational point of view, Kruskal's algorithm, of near quadratic complexity, computes a maximal cost spanning tree and $V(N, t)$. It constructs recursively the $(n-1)$ edges e_k, $k = 1, \ldots, (n-1)$ of such a tree: pick for e_1 one of the most expensive edges, that is, an edge maximizing t_{e_1}; once $e_1, \ldots, e_{k-1}$ are chosen, pick e_k as one of the most expensive edges among those e for which the graph $\{e_1, \ldots, e_{k-1}, e\}$ contains no cycle.

Next we define the irreducible capacity matrix introduced by Bird [7]. It is the key to the definition of the folk solution by Bergantiños and Vidal-Puga [3].

Lemma 2.1. *Given a problem (N, t), the irreducible capacity matrix $t^* = [t^*_{ij}]$ can be defined in three equivalent ways. It is the largest matrix weakly above $t = [t_{ij}]$ and with the same optimal cost as t:*

$$t^* \geq t; \quad V(t^*) = V(t); \qquad \text{and} \qquad \text{for all } t'\colon t' \gneqq t^* \;\Rightarrow\; V(t') > V(t).$$

Equivalently, t^*_{ij} is the capacity supported between i and j on any maximal spanning tree Γ of (N, t):

$$t^*_{ij} = \min_{e \in \gamma(i,j)} t_e, \quad \text{where the path } \gamma(i,j) \text{ connects } i \text{ and } j \text{ on } \Gamma. \tag{1}$$

Finally, t^*_{ij} is the maximal capacity we can support on any path $\gamma(i,j)$ between i and j, not necessarily following a maximal spanning tree:

$$t^*_{ij} = \max_{\gamma(i,j) = \{il_1, l_1 l_2, \ldots, l_s j\}} \; \min_{e \in \gamma(i,j)} t_e. \tag{2}$$

We omit the easy proof. Note that $t^*_{ij} = t_{ij}$ if, and only if, ij belongs to some maximal spanning tree, and that $(t^*)^* = t^*$. We call the matrix t *irreducible* if $t = t^*$.

3. Standalone core and two Shapley values. A *solution* φ associates to every problem (N, t) a vector of cost shares $y = \varphi(N, t)$ such that

$$y_i \geq 0 \text{ for all } i \qquad \text{and} \qquad \sum_{i \in N} y_i = V(N, t). \tag{3}$$

(Note that we can dispense with the assumption of nonnegative cost shares, a consequence of the standalone core property below.)

The standalone core is the set of cost shares where no coalition S of agents is charged more than the cost of a capacity network feasible for the communication needs of S.

Notations: given (N, t) and a coalition $S \subset N$ with $|S| \geq 1$, the capacity matrix t^S restricted to the needs of coalition S is

$$(t^S)_{ij} = 0 \quad \text{if } ij \in (N \backslash S)(2); \qquad (t^S)_{ij} = t_{ij} \quad \text{otherwise.} \tag{4}$$

Define the standalone cost of coalition S as $v^+(S, t) = V(N, t^S)$. A vector of cost shares y, $y \in \mathbb{R}^N$, is in the standalone core if

$$\sum_{i \in N} y_i = V(N, t); \qquad \sum_{i \in S} y_i \leq v^+(S, t) \quad \text{for all } S \subseteq N. \tag{5}$$

Note that these inequalities imply $y_i \geq 0$, and often not much more. For instance if $\mathbf{1}$ is the fully symmetric matrix $\mathbf{1}_{ij} = 1$ for all $ij \in N(2)$, we have $v^+(S, \mathbf{1}) = n - 1$ for all nonempty $S \subseteq N$. In fact the standalone core is "large" (Shapley [24]) because of the following observation:

Lemma 3.1. *The cooperative game $(N, v^+(\cdot, t))$ is concave, therefore its Shapley value Sh^+ defines a solution in the standalone core.*

Proof. *Step* 1. Fix N and an edge e and consider two capacity matrices t and t' such that $t_e < t'_e$ and $t_{e'} = t'_{e'}$ if $e' \neq e$. Check that (writing for simplicity $V(t')$ instead of $V(N, t')$)

$$V(t') - V(t) = [t'_e - t^*_e]_+, \tag{6}$$

where $[z]_+ = \max\{z, 0\}$. Indeed, pick a maximal spanning tree Γ for t. If $t'_e \leq t^*_e$, Γ is feasible for (N, t') by property (1), therefore it is maximal as well and $V(t') = V(t)$. If $t'_e > t^*_e$, set $e = ij$ and note that there is a link on the path $\gamma(i,j)$ of Γ with capacity t^*_e; replacing it with link e of capacity t'_e creates a spanning tree Γ' feasible for (N, t'), therefore $V(t') = V(t) + (t'_e - t^*_e)$.

Step 2. Fix a problem (N, t), $S \subset N$ and $i, j \notin S$. We will now prove

$$V(t^{S\cup\{i,j\}}) - V(t^{S\cup\{j\}}) \leq V(t^{S\cup\{i\}}) - V(t^S),$$

implying the concavity of $(N, v^+(\cdot, t))$. To see this, set $N\backslash(S \cup \{i, j\}) = \{1, \ldots, p\}$; by definition of t^S, we obtain $t^{S\cup\{i,j\}}$ from $t^{S\cup\{j\}}$ by changing the capacity on the edges $i1, \ldots, ip$, respectively, from zero to $t_{i1}, \ldots, t_{ip}$. Writing $(t^{S\cup\{j\}} \mid t_{i1}, \ldots, t_{ik})$ for the matrix, where only the k first terms have been replaced, and applying (6) repeatedly we get

$$V(t^{S\cup\{i,j\}}) - V(t^{S\cup\{j\}}) = \sum_{k=1}^{p} [t_{ik} - (t^{S\cup\{j\}} \mid t_{i1}, \ldots, t_{i(k-1)})^*_{ik}]_+,$$

(where the first term is simply $[t_{i1} - (t^{S\cup\{j\}})^*_{i1}]_+$). Perform the same computation from t^S to $t^{S\cup\{i\}}$:

$$V(t^{S\cup\{i\}}) - V(t^S) = \sum_{k=1}^{p} [t_{ik} - (t^S \mid t_{i1}, \ldots, t_{i(k-1)})^*_{ik}]_+ + [t_{ij} - (t^S \mid t_{i1}, \ldots, t_{ip})^*_{ij}]_+.$$

For all edges e the mapping $t \to t^*_e$ is weakly increasing, and so, as

$$t^S|t_{i1}, \ldots, t_{i(k-1)} \leq t^{S\cup\{j\}}|t_{i1}, \ldots, t_{i(k-1)},$$

each term in the former sum is no larger than the corresponding term in the latter sum. □

Recall that in the MCST problem, the Shapley value of the standalone game may fall outside the standalone core (see Bird [7]).

In the capacity problem there is another notion of standalone cost, associating to a coalition the minimal cost of a graph supporting the capacity requirements t_{ij} inside S. Define the restricted capacity matrix t_S as follows

$$(t_S)_{ij} = t_{ij} \quad \text{if } ij \in S(2); \qquad (t_S)_{ij} = 0 \quad \text{otherwise}, \tag{7}$$

and the *lower standalone cost* $v_-(S, t)$ of a coalition S as $v_-(S, t) = V(N, t_S)$, with the convention $v_-(\{i\}, t) = 0$. It is a natural lower bound for the total charge of coalition S: if agents outside S were absent, coalition S would pay $v_-(S, t)$;[4] as agents in $N\backslash S$ join, the additional links enjoyed by the agents in S justify that their total share weakly increases. Then we say that the cost shares y pass the lower standalone test if

$$\sum_{i\in S} y_i \geq v_-(S, t) \quad \text{for all } S \subseteq N. \tag{8}$$

Recall that the requests t_{ij}, $j \in N\backslash\{i\}$, are interpreted as an inelastic demand for user i. The upper standalone test is the usual core stability property: should it fail some coalition S would be better off by building its own network. By contrast the lower standalone core is not a stability property, it is only a normative requirement of fairness.

It is interesting that the Shapley value of the lower standalone cooperative game is another selection of the standalone core.

LEMMA 3.2. *The game $(N, v_-(\cdot, t))$ is superadditive but generally not convex; its Shapley value Sh_- defines a solution in the standalone core. Both Sh^+ and Sh_- may fail the lower standalone test.*

PROOF. *Step* 1. Superadditivity of $(N, v_-(\cdot, t))$ is clear. The four agents problem with $t_{12} = 1$; $t_{ij} = 10$ for $ij \neq 12$ gives

$$v_-(\{1,2\}, t) + v_-(\{1,2,3,4\}, t) = 31 < v_-(\{1,2,3\}, t) + v_-(\{1,2,4\}, t) = 40,$$

so that $(N, v_-(\cdot, t))$ is not convex.

Step 2. We fix t and a nonempty coalition S. We mimic the notation and argument in Step 2 in the proof of Lemma 3.1 to show that for any $T \subseteq N$, possibly empty, and any $i \in S\backslash T$, we have

$$V(t_{T\cup\{i\}}) - V(t_T) \leq V(t^S_{T\cup\{i\}}) - V(t^S_T), \tag{9}$$

where the notation $t^S_T \stackrel{\text{def}}{=} (t^S)_T = (t_T)^S$ is justified by comparing (4) and (7). This implies $(\text{Sh}_-)_i(N, t) \leq (\text{Sh}_-)_i(N, t^S)$ for all $i \in S$, hence $\sum_S (\text{Sh}_-)_i(N, t) \leq \sum_N (\text{Sh}_-)_i(N, t^S) = V(t^S)$, as desired.

[4] Unlike in the MCST model, in the capacity model it is not useful for S to build a graph using some of the "nodes" in $N\backslash S$.

The claim is obviously true if T is empty as both sides of (9) are null, so we assume it is not and set $T = \{1, \ldots, p\}$. Repeated applications of (6) give

$$V(t_{T\cup\{i\}}) - V(t_T) = \sum_{k=1}^{p}[t_{ik} - (t_T \mid t_{i1}, \ldots, t_{i(k-1)})^*_{ik}]_+ \qquad \text{and}$$

$$V(t^S_{T\cup\{i\}}) - V(t^S_T) = \sum_{k=1}^{p}[t_{ik} - (t^S_T \mid t_{i1}, \ldots, t_{i(k-1)})^*_{ik}]_+.$$

As in the previous proof, $(t^S_T \mid t_{i1}, \ldots, t_{i(k-1)})^*_{ik} \le (t_T \mid t_{i1}, \ldots, t_{i(k-1)})^*_{ik}$ follows from the monotonicity of $t \to t^*_e$.

Step 3. Consider the six person problem $t_{ij} = 1$ if $i = 1, 2$ and $j = 3, 4, 5, 6$, $t_{ij} = 0$ otherwise. The lower standalone bound implies $\sum_{i=2}^{6} y_i \ge 4 \Rightarrow y_1 \le 1$. However one computes easily

$$(\mathrm{Sh}_-)_1(t) = \tfrac{17}{15}; \qquad \mathrm{Sh}^+_1(t) = \tfrac{41}{30}.$$

Both Shapley values are very natural solutions, however only the Sh_- solution meets the basic requirement of monotonicity (Proposition 4.1 below). Moreoever, no algorithm to compute them that has polynomial complexity in the number of agents is known, and we suspect that none exists (although we do not have a formal proof of that). This disqualifies them for practical applications involving more than about twenty agents. In the rest of the paper we are looking for computationally tractable selections of the standalone core.

4. The Bird and Bird$^{1/2}$ solutions. The simplest way to construct a tractable core selection (that also passes the lower standalone test) is to adapt the celebrated Bird [7] solution for the MCST problem. Select a maximal spanning tree and an arbitrary agent to be the nonpaying "source," then charge to every other agent the cost of its adjacent upstream edge.

Notation: given a spanning tree Γ of N, the path from user i to user j along Γ is written $\gamma(i, j) = \{ik_1, k_1k_2, \ldots, k_mj\}$.

Definition 4.1. Given (N, t), choose a maximal spanning tree Γ, and an arbitrary agent $i \in N$. Define the (Γ, i)-solution[5] as follows:

$$y_i^{\Gamma,i} = 0; \quad \text{for } j \ne i\text{: } y_j^{\Gamma,i} = t_{jk}, \text{ where } jk \text{ is the first edge of } \gamma(j, i).$$

Lemma 4.1. *The* (Γ, i)*-solution is in the standalone core* (5) *and passes the lower standalone test* (8).

Proof. *Standalone core.* Fix a coalition $S \subset N$ and let Δ be the graph of all edges paid by members of S. Any such edge e is adjacent to at least an agent in S, therefore $t_e = t^S_e$. As Δ is contained in Γ, a spanning tree, we have $\sum_S y_j^{\Gamma,i} = t(\Delta) = t^S(\Delta) \le t^S(\Gamma) \le V(N, t^S)$.

Lower standalone test. As agent i pays zero and $v_-(\cdot, t)$ is weakly increasing with respect to inclusion, it suffices to check the lower standalone test for coalitions S that include i. Fix such a coalition S. Take away the $|S| - 1$ edges paid by members of S, and call $\Gamma(-S)$ the resulting graph. Choose a maximal spanning tree Δ of (S, t_S); its cost is $v_-(S)$. The graph $\Gamma(-S) \cup \Delta$ has exactly $n - 1$ edges, moreover it spans N, because for any $j \in N\backslash S$, the path from j to i in Γ stays in $\Gamma(-S)$ until we reach a node in S. Thus $\Gamma(-S) \cup \Delta$ is a spanning tree of N with cost $(V(N, t) - \sum_S y_j^{\Gamma,i}) + v_-(S)$. This cost is not larger than $V(N, t)$, which gives the desired inequality. □

Kruskal's algorithm computes a maximal spanning tree in near quadratic time, thus a (Γ, i)-solution is just as easy to compute. On the other hand, such a solution violates the basic requirement of horizontal equity because it selects an arbitrary agent who pays nothing. This problem disappears if we take an appropriate average.

Definition 4.2. The Bird solution, denoted B, is the uniform average of the (Γ, i)-solutions over all maximal spanning trees and all agents.

By Lemma 4.1, and the fact that (5) and (8) are stable by convex combination, the Bird solution passes the standalone core and the lower standalone test.

Just like in the MCST problem, the Bird solution fails some elementary requirements of continuity and fairness. Consider the four-person problem

$$t_{12} = a; \quad t_{13} = t_{24} = 2; \quad t_{34} = 1; \quad t_{14} = t_{23} = 0.$$

[5] With an abuse of language: strictly speaking Γ is a mapping selecting for each problem (N, t) a maximal spanning tree Γ.

Compute $B(t)$ successively for $a<1$ and $a>1$:

$$a<1 \;\Rightarrow\; B(t)=\left(\frac{3}{2},\frac{3}{2},1,1\right); \qquad a>1 \;\Rightarrow\; B(t)=\left(\frac{a+1}{2},\frac{a+1}{2},\frac{3}{2},\frac{3}{2}\right).$$

Thus, as t_{12} crosses the threshold of one upwards, the cost shares of agents one and two drop down. Even in the absence of strategic maneuvers, the fact that cost shares vary discontinuously in t means that small measurement errors can have a dramatic impact on cost shares.

We turn to another natural selection of the standalone core. It divides equally between its two adjacent nodes the cost of each edge in the maximal spanning tree.

Definition 4.3. Given (N,t) choose a maximal spanning tree Γ. Define the $\Gamma^{1/2}$-solution[6] as follows:

$$\text{for all } i\colon\; y_i^{\Gamma,1/2}=\tfrac{1}{2}\sum_{j:\,ij\in\Gamma} t_{ij}. \tag{10}$$

The Bird$^{1/2}$ solution, denoted $B^{1/2}$, is the uniform average of the $\Gamma^{1/2}$-solutions over all maximal spanning trees of (N,t).

Lemma 4.2. *All $\Gamma^{1/2}$-solutions and the Bird$^{1/2}$ solution are in the standalone core* (5).

Proof. Fix (N,t) and Γ as in the premises of Definition 4.3, and a coalition S. Compute from (10):

$$2\sum_{i\in S} y_i^{\Gamma,1/2}=\sum_{ij\in\Gamma\cap S(2)} t_{ij}+\sum_{ij\in\Gamma\cap S(2)} t_{ij}+\sum_{ij\in\Gamma,\,i\in S,\,j\notin S} t_{ij}.$$

The first term in the sum is not larger than $V(t_S)$, as $\Gamma\cap S(2)$ is a subforest (graph without cycles) of $S(2)$. The sum of the next two terms is bounded by $V(t^S)$ because the edges in $\Gamma\cap S(2)$ together with those in Γ joining an agent in S and one outside S, form a subforest of the nonnull edges of t^S. Note that we proved a stronger inequality, namely

$$\sum_{i\in S} y_i \le \tfrac{1}{2}(v^+(S,t)+v_-(S,t)) \quad \text{for all } S\subseteq N. \quad \square \tag{11}$$

In the four-person example above, we have

$$a<1 \;\Rightarrow\; B^{1/2}(t)=\left(1,1,\frac{3}{2},\frac{3}{2}\right); \qquad a>1 \;\Rightarrow\; B^{1/2}(t)=\left(\frac{a+2}{2},\frac{a+2}{2},1,1\right),$$

so that as t_{12} increases, both y_1 and y_2 weakly increase (while y_3 and y_4 weakly decrease), as one would expect. Proposition 4.1 below states the general monotonicity property. On the other hand in the example the shares $B^{1/2}(t)$ are not continuous in t.

We shall impose the following minimal requirements of a solution φ. Recall our notation $y_i=\varphi_i(N,t)$.

- *Standalone Core*: (5)
- *Monotonicity*: y_i is weakly increasing in t_{ij} for all i,j
- *Continuity*: y_i is continuous in t
- *Equal treatment of equals*: for all N,t,i,j $\{t_{ik}=t_{jk}$ for all $k\neq i,j\}\Rightarrow y_i=y_j$
- *Polynomial complexity* with respect to $n=|N|$

Monotonicity prevents the wasteful inflation of capacity, and is at the same time a compelling fairness property. Note that the first four properties above are stable by convex combinations of solutions with fixed coefficients.

When there is a unique maximal cost spanning tree (e.g., when all demands t_{ij} are different), the Bird and Bird$^{1/2}$ solutions are both easy to compute. But in general there may be as many as n^{n-2} maximal spanning trees (this is achieved when all demands t_{ij} are equal). Therefore, carrying out the computation for each of them and averaging is not efficient. Yet, these two solutions can be computed in polynomial time, as shown below.

Proposition 4.1.
(i) *The Sh^+ solution meets all properties except monotonicity and possibly polynomial complexity.*
(ii) *The Sh_- solution meets all properties except possibly polynomial complexity.*
(iii) *The Bird solution meets all properties except monotonicity and continuity.*
(iv) *The Bird$^{1/2}$ solution meets all properties except continuity.*

[6] With the same abuse of language as in Definition 4.1.

Proof. *Step* 1. *Two Shapley values.* Continuity is clear because $v_-(S,t)$ and $v^+(S,t)$ are continuous in t. Equal treatment is obvious and Lemmas 2 and 3 prove standalone core stability. Monotonicity of Sh_- is clear: given a problem (N,t), an agent i, an edge ij and a coalition S not containing i, the term $v_-(S\cup\{i\},t)-v_-(S,t)$ is monotonic in t_{ij} because $v_-(S,t)$ does not depend on t_{ij} and $t_{ij}\to v_-(S\cup\{i\},t)$ is monotonic.

To see a violation of monotonicity for Sh^+, consider the eight-agent problem (N,t) in which $t_{12}=1$ and $t_{ij}=1$ for all $i\in\{2,3\}$ and all $j\in\{4,5,6,7,8\}$, and $t_e=0$ otherwise. It is easy to see that $\text{Sh}_1^+(t)=\frac{1}{2}$. When we move from this problem to (N,t'), which differs from it only in $t'_{13}=1$, a little calculation shows that $\text{Sh}_1^+(t')=83/168<\frac{1}{2}$.

Step 2. *Bird solution.* Lemma 4.1 gives standalone core stability and the example after Definition 4.2 provides a violation of continuity and monotonicity. Equal treatment is obvious.

Before presenting a polynomial algorithm that computes the Bird solution, we describe an iterative random process that generates the uniform maximal spanning tree[7] in a graph G with edge capacities t_e. Let $t^1>t^2>\cdots>t^k$ be all the different edge capacities appearing in G. We start with the graph G^1 consisting only of those edges e with $t_e=t^1$, and choose uniformly at random a spanning tree in each connected component. Next, we contract each connected component of G^1 to a point and consider the multigraph[8] $\tilde{G}^2$ with these points as nodes and with edges e such that $t_e=t^2$, and e was not contracted to a point. We choose uniformly at random a spanning tree in each connected component of $\tilde{G}^2$, then contract these components and consider the multigraph $\tilde{G}^3$ with edges of capacity t^3 not yet contracted, and so on. The random spanning tree formed by this sequence of independent uniform choices is the uniform maximal spanning tree in G.

Step 2.1. We show first that the following auxiliary problem can be solved in polynomial time: Given a multigraph $\tilde{G}$, and edge $e=xy$ in $\tilde{G}$, and a node z in the same connected component of $\tilde{G}$, find the probability that e appears in the uniform spanning tree of that component so that y is between x and z.

To this end, we recall that for a directed multigraph $\tilde{D}$ and a given node z, an arborescence of $\tilde{D}$ rooted at z is a subdirected graph of $\tilde{D}$ that is a spanning tree such that all its arcs are in the outward direction from z. A formula of Tutte (see Berge [6, Theorem 21]) gives the number of arborescences of $\tilde{D}$ rooted at z as the determinant of a suitable $(n-1)\times(n-1)$ matrix, where n is the number of nodes and the entries in the matrix can be filled in by counting arcs between pairs of nodes. Hence this number of arborescences is polynomially computable.

Back to our auxiliary problem, we construct from the relevant connected component of $\tilde{G}$ a directed multigraph $\tilde{D}$ by putting an arc from x to y, and for every other edge $e'=x'y'$ we put two arcs: one from x' to y' and one from y' to x'. There is a bijection between arborescences of $\tilde{D}$ rooted at z, and spanning trees in the undirected multigraph that either avoid e or use it so that x is between y and z. Thus we can compute the number of such spanning trees, and subtracting it from the total number of spanning trees (also computable in a similar way, but doubling e as well), we solve our problem.

Step 2.2. *A polynomial algorithm to compute Bird.* We show how to solve in polynomial time the following problem: Given a graph G with edge capacities t_e, a node x and another node z in the same connected component, calculate $f(G,x,z)$, defined as the expected capacity of the edge adjacent to x toward z, in the uniform maximal spanning tree of that component. This is enough because the Bird solution charges $(1/n)\sum_{j\in N\setminus\{i\}} f(N(2),i,j)$ to agent i.

We calculate the function $f(G,\cdot,\cdot)$ by an inductive process, in which we first calculate the function $f(G^1,\cdot,\cdot)$ where G^1 consists of the edges of maximal capacity t^1, then we calculate $f(G^{12},\cdot,\cdot)$ where G^{12} consists of the edges of the two largest capacities $t^1>t^2$, and so on, until we get $f(G,\cdot,\cdot)$. The basis of the induction is clear: $f(G^1,x,z)=t^1$ for all distinct x and z in the same connected component. For the next step, consider the multigraph $\tilde{G}^2$ described above, obtained by contracting connected components of G^1 and putting in the edges of capacity t^2. Let x and z be two distinct nodes in the same connected component of G^{12}. If x and z are contracted to the same point in $\tilde{G}^2$ then we know the relevant value by induction: $f(G^{12},x,z)=f(G^1,x,z)$. If they are not, let X and Z be the contracted blocks they belong to. For each node $x'\in X$ and each edge $e'=x'y'$ of capacity t^2 with y' outside X, we use the procedure of the auxiliary problem (Step 2.1) to find the probability that e' appears in the uniform spanning tree of the relevant component of $\tilde{G}^2$ so that y' is between x' and Z.

[7] By the uniform maximal spanning tree we mean a probability space whose elements are all the maximal spanning trees, each with the same probability. We use the same terminology without the adjective "maximal" when the graph in question is unweighted (i.e., no edge capacities are specified).

[8] In a multigraph we have a finite number of edges (possibly zero) between any two nodes.

Next, by summing these probabilities for a given x' over all y', we get the probability $p(x')$ that X is connected to Z via a path leaving X at x'. This allows us to compute inductively:

$$f(G^{12}, x, z) = p(x)\cdot t^2 + \sum_{x'\in X\setminus\{x\}} p(x')f(G^1, x, x').$$

In the next step, $f(G^{123}, \cdot, \cdot)$ is computed similarly using $f(G^{12}, \cdot, \cdot)$, and so on, until we get, after at most $n(n-1)/2$ iterations, the desired $f(G, \cdot, \cdot)$.

Step 3. *Bird*$^{1/2}$ *solution.* Lemma 4.2 gives standalone core stability, and equal treatment is obvious. The example introduced after Definition 4.2 shows that $B^{1/2}$ fails Continuity.

Step 3.1. *A polynomial algorithm to compute Bird*$^{1/2}$. Given a problem (N, t) and an agent 1, we compute his cost share $B_1^{1/2}(N, t) = y_1$ by summing up, for each edge e adjacent to one, the probability p_e of e in the uniform maximal spanning tree of t, multiplied by $t_e/2$. Fix such an edge e, set $t_e = a$ and let b be the next value of some capacities $t_{e'}$ (i.e., $a < t_{e''} < b$ is impossible); if a is the largest capacity, let $b = \infty$. We compute p_e in two steps. First we find the connected components of the graph H formed by edges with capacity at least b and contract each of them to a point. If the edge e is contracted, then $p_e = 0$. Otherwise, denoting by $\tilde{G}$ the multigraph of all edges of capacity a remaining after the contraction, p_e is the probability $p_e(\tilde{G})$ that e is in the uniform spanning tree of its connected component in $\tilde{G}$. This probability can be calculated in polynomial time by computing suitable determinants (see Step 2.1 above).

Step 3.2. To prove monotonicity we need a preliminary result concerning the probabilities $p_e(\tilde{G})$ defined above. If $\tilde{G}$ is a multigraph and e' is an edge, we use the notation $\tilde{G} - e'$ for $\tilde{G}\setminus\{e'\}$ and $\tilde{G} \mid e'$ for the multigraph obtained from $\tilde{G}$ by contracting the two nodes of e' to a point. Then we have for any $e' \neq e$

$$p_e(\tilde{G} \mid e') \le p_e(\tilde{G}) \le p_e(\tilde{G} - e').$$

The right-hand side inequality is proven for instance in Lyons and Peres [19]. The left-hand side (LHS) inequality then follows from the identity

$$p_e(\tilde{G}) = p_{e'}(\tilde{G})p_e(\tilde{G} \mid e') + (1 - p_{e'}(\tilde{G}))p_e(\tilde{G} - e'),$$

and the fact that $p_{e'}(\tilde{G}) > 0$.

Step 3.3. *Monotonicity.* Using the notations of the previous steps, we fix an edge e adjacent to agent one, and prove monotonicity in two steps: first when t_e increases from a to x, $a < x < b$, next when it moves from x, $a < x < b$, to b.

Step 3.3.1. Here t_e goes from a to x, $a < x < b$. This move can only affect p_e and $p_{e'}$ for $e' \in \tilde{G}$. If e is contracted ($e \notin \tilde{G}$) then $\tilde{G}$ is unchanged and so are $p_{e'}$ for $e' \in \tilde{G}$. Moreover $p_e = 0$ before and after so y_1 is unchanged. Suppose next $e \in \tilde{G}$, i.e., it links two connected components of H. Then p_e jumps to one and for $e' \in \tilde{G} - e$, $p_{e'}$ decreases (at least weakly): $p_{e'}(\tilde{G}) \ge p_{e'}(\tilde{G} \mid e)$ (Step 3.2). Moreover the spanning trees of the connected components of $\tilde{G}$ now have one less edge, hence

$$\sum_{e'\in\tilde{G}} p_{e'}(\tilde{G}) = 1 + \sum_{e'\in\tilde{G}-e} p_{e'}(\tilde{G} \mid e).$$

Combining these inequalities

$$\pi \overset{\text{def}}{=} \sum_{e'\in\tilde{G};\,1\in e'} p_{e'}(\tilde{G}) \le 1 + \sum_{e'\in\tilde{G}-e;\,1\in e'} p_{e'}(\tilde{G} \mid e) \overset{\text{def}}{=} 1 + \pi'.$$

Now we have $\pi a \le x + \pi' a$ so the relevant change of y_1 is upward.

Step 3.3.2. Now t_e goes from x, $a < x < b$, to b. The probabilities $p_{e'}(\tilde{G} \mid e)$, $e' \in \tilde{G} - e$, are unchanged, so we only worry about the impact of this move on $p_{e''}(\tilde{K})$, $e'' \in \tilde{K}$, where $\tilde{K}$ is the multigraph of all edges of capacity b remaining after the contraction of the connected components in the graph of capacities higher than b. Assume first $e \notin \tilde{G}$, i.e., e is in a connected component of $\tilde{K}$, or already contracted. Then $p_e = 0$ before the move. If e is already contracted in $\tilde{K}$, then all $p_{e''}(\tilde{K})$ are unchanged and so is y_1. If e is in a connected component of $\tilde{K}$, after it is added to $\tilde{K}$ obtaining the multigraph that we denote $\tilde{K} + e$, the spanning trees of its connected components have the same number of edges, hence

$$\sum_{e''\in\tilde{K}} p_{e''}(\tilde{K}) = p_e(\tilde{K} + e) + \sum_{e''\in\tilde{K}} p_{e''}(\tilde{K} + e).$$

Moreover by Step 3.2 $p_{e''}(\tilde{K}+e) \le p_{e''}(\tilde{K})$ for all $e'' \ne e$, so that

$$\pi \stackrel{\text{def}}{=} \sum_{e''\in\tilde{K};\, 1\in e''} p_{e''}(\tilde{K}) \le p_e(\tilde{K}+e) + \sum_{e''\in\tilde{K};\, 1\in e''} p_{e''}(\tilde{K}+e) \stackrel{\text{def}}{=} \pi',$$

and the relevant change in y_1 is from πb to $\pi' b$.

Next we suppose $e \in \tilde{G}$, i.e., it links two connected components of $\tilde{K}$. Then $p_e = 1$ before and after the move, and $p_{e''}(\tilde{K}) = p_{e''}(\tilde{K}+e)$ for all $e'' \in \tilde{K}$ so y_1 does not change. □

5. Piecewise linear solutions. We now construct a rich, convex family of solutions meeting all five properties listed before Proposition 4.1.

We write the set of capacity matrices as $\mathcal{T}(N) = \{t = (t_{ij})\} = \mathbb{R}_+^{n(n-1)/2}$, and we speak of elementary matrices if all numbers t_{ij} are either zero or one. The set of elementary matrices is

$$\mathcal{G}(N) = \{t \in \mathcal{T}(N) \mid t_{ij} \in \{0,1\} \text{ for all } ij\}.$$

The graph of a matrix $t \in \mathcal{G}(N)$ is the graph $G(t) = \{ij \in N(2) \mid t_{ij} = 1\}$. When this causes no confusion, we identify the matrix t and its graph $G(t)$, and thus refer to $t \in \mathcal{G}(N)$ either as an elementary matrix or as a capacity graph.

In the construction of piecewise linear solutions, we focus on the cones of $\mathcal{T}(N)$ in which the relative ordering of the coordinates t_{ij} is fixed. Let σ: $\{1, 2, \dots, n(n-1)/2\} \to N(2)$ be a labeling (one-to-one mapping) of $N(2)$. The cone $\mathcal{T}^\sigma(N)$ is defined by the system:

$$t_{\sigma(1)} \le t_{\sigma(2)} \le \cdots \le t_{\sigma(n(n-1)/2)}. \tag{12}$$

The key observation is that, for any σ, inside the cone $\mathcal{T}^\sigma(N)$ the values $V(N,t)$, $v_-(S,t)$, $v^+(S,t)$ are positively linear in t. Indeed Kruskal's algorithm described at the end of §2 picks a unique maximal spanning tree Γ whenever $t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(n(n-1)/2)}$, and this tree is still maximal for any $t \in \mathcal{T}^\sigma(N)$. Therefore the equation $V(N,t) = \sum_{e\in\Gamma} t_e$ proves the claim. The argument is identical for $v_-(S,t)$ and $v^+(S,t)$.

Now a positive linear basis of $\mathcal{T}^\sigma(N)$ consists of the following elementary matrices $b^{\sigma,k} \in \mathcal{G}(N) \cap \mathcal{T}^\sigma(N)$, $k = 1, \dots, n(n-1)/2$:

$$b_{ij}^{\sigma,1} = 1 \quad \text{for all } e; \qquad \text{for } k \ge 2\text{:} \quad b_{\sigma(l)}^{\sigma,k} = 0 \quad \text{for } l = 1, \dots, k-1; \qquad b_{ij}^{\sigma,k} = 1 \quad \text{otherwise.}$$

For any $t \in \mathcal{T}^\sigma(N)$, the following identity is easy to check

$$t = \sum_{k=1}^{n(n-1)/2} (t_{\sigma(k)} - t_{\sigma(k-1)}) b^{\sigma,k}, \tag{13}$$

with the convention $t_{\sigma(0)} = 0$.

For any elementary matrix $t \in \mathcal{G}(N)$, write $\lambda(t)$ for the number of connected components of its graph $G(t)$. Clearly $V(N,t) = n - \lambda(t)$. Choose now a solution ψ on $\mathcal{G}(N)$, namely a mapping from $\mathcal{G}(N)$ into $\mathbb{R}_+^N$ such that

$$\sum_{i\in N} \psi_i(t) = n - \lambda(t) \quad \text{for all } t \in \mathcal{G}(N). \tag{14}$$

For any labeling σ, ψ has a unique linear extension to the cone $\mathcal{T}^\sigma(N)$:

$$\varphi(N,t) = \sum_{k=1}^{n(n-1)/2} (t_{\sigma(k)} - t_{\sigma(k-1)}) \psi(N, b^{\sigma,k}) \quad \text{for all } t \in \mathcal{T}^\sigma(N). \tag{15}$$

This formula defines a full fledged solution φ on $\mathcal{T}(N)$. Indeed, if for two labelings σ, σ' of $N(2)$ we have $t \in \mathcal{T}^\sigma(N) \cap \mathcal{T}^{\sigma'}(N)$, then $t_{\sigma(k)} = t_{\sigma'(k)}$ for all k as this number is the k-th lowest among the $n(n-1)/2$ numbers t_e. Thus φ is well defined. Moreover, φ satisfies (3): nonnegativity is clear, and budget balance holds on $\mathcal{G}(N)$ by (14), and, for any σ, in $\mathcal{T}^\sigma(N)$ because $V(N,t)$ is linear in that cone.

We will call piecewise linear the solutions decomposable as in (15).

Theorem 5.1. *Let ψ: $\mathcal{G}(N) \to \mathbb{R}_+^N$ satisfy (14). Then Equation (15) defines a continuous solution φ on $\mathcal{T}(N)$, that we call the piecewise linear extension of ψ. The following five properties extend from ψ to φ:*

(i) *Standalone core stability (5).*

(ii) *Lower standalone test (8).*

(iii) *Monotonicity: y_i is weakly increasing when we add $e = ij$ to the graph of $t \in \mathcal{G}(N)$.*

(iv) *Equal treatment of equals.*
(v) *Polynomial complexity.*

PROOF. *Step* 1. *φ is continuous.* Continuity of φ is clear inside each cone $\mathcal{T}^\sigma(N)$, and extends at once to $\mathcal{T}(N)$ of which these cones are a closed covering.

Step 2. *Standalone core and inequality* (8). Both sides of the inequality $\sum_{i\in S}\varphi_i(N,t)\le v^+(S,t)=V(N,t^S)$ are linear inside any cone $\mathcal{T}^\sigma(N)$, so if it holds for all $t\in\mathcal{G}(N)$, it holds everywhere. The argument for (8) is identical.

Step 3. *Monotonicity.* Fix a problem (N,t), an agent i, and an edge e containing i. Because φ is continuous, it is enough to check monotonicity when t_e varies in an interval $]a,b[$ and no other $t_{e'}$ is in this interval. Then t stays in the same cone $\mathcal{T}^\sigma(N)$ and in the i-th coordinate of the sum (15) t_e only appears with the coefficients $\psi_i(N,b)$ and $-\psi_i(N,b')$, where $b_{e'}=0$ if $t_{e'}\le a$, $b_{e'}=1$ otherwise, and the only difference between b' and b is $b'_e=0$. Monotonicity of ψ gives $\psi_i(N,b')\le\psi_i(N,b)$ and we are done.

Step 4. *Polynomial complexity.* A sorting algorithm identifies a cone $\mathcal{T}^\sigma(N)$ containing a given matrix t in $O(n^2\log n)$ time, then Equation (15) delivers the vector of cost shares $\varphi(N,t)$ as the sum of $O(n^2)$ such vectors for capacity graphs. □

Note a consequence of standalone core stability for a solution ψ on $\mathcal{G}(N)$. For any capacity graph t, if A is a connected component of the associated graph G, then $\sum_{i\in A}\psi_i(t)=|A|-1$. But these conditions are not sufficient to ensure core stability of ψ.

The two Shapley values Sh^+, Sh_- (§3) are two examples of piecewise linear solutions: the cost share $\varphi_i(t)$ is a linear combination of the numbers $v^+(S,t)$ or $v_-(S,t)$, each of which is linear inside any cone $\mathcal{T}^\sigma(N)$.

6. The uniform solution. For a capacity graph $t\in\mathcal{G}(N)$, the Bird solution $B(t)$ is especially easy to compute. Fix a connected component A of the graph t, a spanning tree Γ and an agent $i\in A$. All edges of Γ cost one, so the (Γ,i)-solution charges one to all j, $j\neq i$. Therefore

$$B_i(t)=\frac{|A|-1}{|A|},\quad \text{where } A \text{ is the connected component of } i \text{ in graph } t. \tag{16}$$

DEFINITION 6.1. The uniform solution, denoted *Un*, is the piecewise linear solution derived from the restriction of the Bird solution to $\mathcal{G}(N)$.

It is already clear that the uniform solution is of low complexity because we only need to order the capacities t_{ij}, then identify the connected components of the $n(n-1)/2$ graphs $b^{\sigma,k}$. We give below a closed form expression for this solution, based on the irreducible capacity matrix, that makes it especially easy to compute in many examples. Theorem 6.1 also shows that this solution is the counterpart of the folk solution for the MCST problem (Bergantiños and Vidal-Puga [3]).

THEOREM 6.1. *The uniform solution meets our five minimal requirements*: *standalone core stability, monotonicity, continuity, equal treatment of equals, and polynomial complexity. It also satisfies the lower standalone test. Moreover*

(i) *It is the Shapley value of both games $(N,v^+(\cdot,t^*))$ and $(N,v_-(\cdot,t^*))$.*
(ii) *It is computed by arranging the numbers $(t^*_{ij})_{j\in N\setminus\{i\}}$ decreasingly*

$$\begin{gathered}\max_{j\in N\setminus\{i\}}t^*_{ij}=t_i^{*1}\ge t_i^{*2}\ge\cdots\ge t_i^{*(n-1)}=\min_{j\in N\setminus\{i\}}t^*_{ij},\\ Un_i(t)=\frac{1}{2}t_i^{*1}+\frac{1}{6}t_i^{*2}+\cdots+\frac{1}{k(k+1)}t_i^{*k}+\cdots+\frac{1}{(n-1)n}t_i^{*(n-1)}.\end{gathered}\tag{17}$$

PROOF. Proposition 4.1 and Theorem 5.1 imply continuity, standalone core stability and the lower standalone test (8). Check now that B is monotonic for capacity graphs[9] $b\in\mathcal{G}(N)$. Adding an edge $e=ij$ to a capacity graph G can only enlarge the connected component A of i, in which case the ratio $(|A|-1)/|A|$ goes up.

Next we prove Equation (17). Fix $t\in\mathcal{G}(N)$ with corresponding graph G. By Lemma 2.1 the irreducible matrix t^* is the following capacity graph

$$t^*_{ij}=1\quad\text{if } i,j \text{ are connected in } G;\qquad t^*_{ij}=0\quad\text{otherwise.}$$

[9] Recall that B is not monotonic on $\mathcal{T}(N)$.

Therefore if the connected component of $i \in N$ in G is A, the numbers t_i^{*k} defined in the statement of Theorem 6.1 are $t_i^{*k} = 1$ for $1 \le k \le |A| - 1$, $t_i^{*k} = 0$ for $k \ge |A|$, so the right-hand side in (17) is $\frac{1}{2} + \frac{1}{6} + \cdots + 1/(|A|(|A|-1)) = (|A|-1)/|A|$. This proves Equation (17) if $t \in \mathcal{G}(N)$. The mapping $t \to t_i^{*k}$ is linear in the cone $\mathcal{T}^{\sigma}(N)$, for any σ, i, and k, so the same is true of the right-hand side in (17). Therefore the equality holds everywhere.

As noticed at the end of §5, the Shapley value of $(N, v^+(\cdot, t^*))$ and that of $(N, v_-(\cdot, t^*))$ are piecewise linear solutions. Thus they both coincide with Un if they do so for any $t \in \mathcal{G}(N)$. The latter follows at once from the fact that t^* is the complete graph inside every connected component of t. □

For the irreducible matrix t^* the game $(N, v^+(\cdot, t^*))$ is concave (this holds true for all t) and one can also check that the game $(N, v_-(\cdot, t^*))$ is convex; the proof, omitted for brevity, consists of showing that for all S and i, $i \notin S$, we have $v_-(S \cup \{i\}, t^*) - v_-(S, t^*) = \max_{j \in S} t^*_{ij}$.

Example 6.1. It is interesting to compare the five solutions identified so far in the "generic" three agents problem with $t_{23} < t_{13} < t_{12}$. There is a single maximal spanning tree $2 \leftrightarrow 1 \leftrightarrow 3$, and one computes easily

$$B(t) = \left(\tfrac{1}{3}(t_{12}+t_{13}), \tfrac{2}{3}t_{12}, \tfrac{2}{3}t_{13}\right),$$
$$B^{1/2}(t) = \left(\tfrac{1}{2}(t_{12}+t_{13}), \tfrac{1}{2}t_{12}, \tfrac{1}{2}t_{13}\right),$$
$$\mathrm{Sh}^+(t) = \mathrm{Sh}_-(t) = \left(\tfrac{1}{2}(t_{12}+t_{13}) - \tfrac{1}{3}t_{23}, \tfrac{1}{2}t_{12} + \tfrac{1}{6}t_{23}, \tfrac{1}{2}t_{13} + \tfrac{1}{6}t_{23}\right).$$

Both Bird and Bird$^{1/2}$ ignore t_{23} altogether, whereas the Shapley values charge strictly more to agents 2, 3 when t_{23} increases. This is related to the discontinuity of the Bird and Bird$^{1/2}$ shares as t_{23} increases above t_{13}, at which point agent 1's share jumps up in $B(t)$ and down in $B^{1/2}(t)$. One computes further

$$Un(t) = \left(\tfrac{1}{2}t_{12} + \tfrac{1}{6}t_{13}, \tfrac{1}{2}t_{12} + \tfrac{1}{6}t_{13}, \tfrac{2}{3}t_{13}\right).$$

The uniform solution treats agents 1 and 2 equally, ignoring the fact that agent 2 demands less than 1: $t_{23} < t_{13}$. Bird is even worse: it charges more to agent 2 than to 1.

Example 6.2. Fix n numbers a_i such that

$$a_1 \ge a_2 \ge \cdots \ge a_n,$$

and set $t_{ij} = \max\{a_i, a_j\}$ for all i, j. Thus agent i requires a capacity a_i while communicating with any other user. The efficient tree (maximal spanning tree) is the star centered on agent 1. Much like in the classic airport landing game (Littlechild and Owen [17], Littlechild and Thompson [18]), we expect agent 1 to be charged more. However the uniform (and Bird) solution ignores all differences between the users because the irreducible capacity matrix is $t^*_{ij} = a_1$ for all i, j. Therefore

$$Un_i(t) = B_i(t) = \frac{n-1}{n}a_1 \quad \text{for all } i,$$

a paradoxical outcome when $a_2, \ldots, a_n$ are much smaller than a_1.

On the other hand, the Bird$^{1/2}$ solution charges more to agent 1; in particular if $a_1 > a_2$ there is a unique maximal spanning tree and

$$B_1^{1/2} = \frac{n-1}{2}a_1; \qquad \text{for } i \ge 2: \quad B_i^{1/2} = \frac{1}{2}a_1.$$

Only the Shapley values and the BHM solution (defined in the next section) differentiate finely between the n agents. Their computation is a little more involved. See the discussion after Theorem 7.1.

Examples 6.1, 6.2 illustrate the major weakness of the uniform solution. It only takes into account the irreducible traffic matrix t^*, in which at most $(n-1)$ numbers are different, as opposed to the $n(n-1)/2$ coordinates of the initial capacity matrix t. In doing so it throws away much information relevant to fair division, about demands t_{ij} such that $t_{ij} < t^*_{ij}$, i.e., demands that will be more than met in any maximal spanning tree. In §8 we discuss further this property that we call *reductionism*.

7. The BHM solution. For a connected capacity graph $t \in \mathcal{G}(N)$ a maximal spanning tree is simply a spanning tree Γ. The $\Gamma^{1/2}$-solution charges to an agent i one half of his degree (the number of edges adjacent to i) in Γ (10), and the Bird$^{1/2}$ solution charges one half of his expected degree in the uniform maximal spanning tree.

Definition 7.1. The BHM solution is the piecewise linear solution derived from the restriction of the Bird$^{1/2}$ solution to $\mathcal{G}(N)$.

Theorem 7.1. *The BHM solution meets our five minimal requirements, standalone core stability, monotonicity, continuity, equal treatment of equals, and polynomial complexity. It fails the lower standalone test.*

Proof. Follows directly from Lemma 4.2, Proposition 4.1, Theorem 5.1, and Proposition 8.4 below. □

In Example 6.1 with three agents, we have $\text{BHM}(t) = \text{Sh}^+(t) = \text{Sh}_-(t)$.

Unlike the uniform one, the BHM solution does not have a closed form expression, and for that reason it is harder to compute in simple examples such as Example 6.2. Assume four agents and $a_1 \geq a_2 \geq a_3 \geq a_4$. Then

$$\text{BHM}_1(t) = \tfrac{3}{2}a_1 - \tfrac{5}{8}a_2 - \tfrac{1}{8}a_3, \qquad \text{BHM}_2(t) = \tfrac{1}{2}a_1 + \tfrac{3}{8}a_2 - \tfrac{1}{8}a_3,$$
$$\text{BHM}_3(t) = \text{BHM}_4(t) = \tfrac{1}{2}a_1 + \tfrac{1}{8}a_2 + \tfrac{1}{8}a_3.$$

However, the BHM solution is very easy to compute in the special case of irreducible capacity matrices:

$$t = t^* \implies \text{BHM}(t) = Un(t) = B(t).$$

At the other extreme, consider a matrix t such that $t_{ij} > 0$ if ij is an edge in the unique spanning tree Γ, and $t_{ij} = 0$ otherwise. Then $\text{BHM}(t) = B^{1/2}(t)$, much different than the uniform and Bird solutions. For instance in Example 6.2 assume $a_1 = 1$ and $a_i = 0$ for $i \geq 2$; then $BHM_1(t) = (n-1)/2$ whereas $Un_1(t) = B_1(t) = (n-1)/n$.

8. Comparing the uniform and BHM solutions.

8.1. Ranking. We formalize the critique of the uniform solution suggested by Examples 6.1, 6.2. Whenever the capacity demands of one agent dominate those of another agent, cost shares should reflect this hierarchy. In the three definitions below, we assume $|N| \geq 3$.

Weak ranking: for all $i, j \in N$, $\{t_{ih} \leq t_{jh}$ for all $h \neq i, j\} \Rightarrow y_i \leq y_j$.

Ranking: for all $i, j \in N$, $\{t_{ih} < t_{jh}$ for all $h \neq i, j\} \Rightarrow y_i < y_j$.

Strict ranking: for all $i, j \in N$, $\{t_{ih} \leq t_{jh}$ for all $h \neq i, j$; $t_{ih} < t_{jh}$ for at least one $h\} \Rightarrow y_i < y_j$.

Note that strict ranking implies ranking. For a solution that treats equals equally, strict ranking implies weak ranking. For a continuous solution, ranking implies weak ranking.

In the same spirit, we can talk about

Strict monotonicity: y_i is strictly increasing in t_{ij} for all $i, j \in N$.

Theorem 8.1.

(i) *A piecewise linear solution φ meets strict ranking (respectively, weak ranking) if its restriction ψ to $\mathcal{G}(N)$ does.*

(ii) *The BHM solution meets strict ranking (hence also the other two variants).*

(iii) *The Bird$^{1/2}$ solution meets ranking and weak ranking, but not strict ranking.*

(iv) *The uniform solution meets weak ranking but not ranking.*

(v) *The Bird solution fails all three variants.*

(vi) *The two Shapley values Sh^+, Sh_-, meet strict ranking (hence also the other two variants).*

Proof. *Step* 1. Fix a solution ψ on $\mathcal{G}(N)$ as in Theorem 5.1 and its piecewise linear extension φ. Suppose ψ meets weak ranking. Fix $t \in \mathcal{T}(N)$ and two agents i, j as in the premises of weak ranking, and consider a matrix $b^{\sigma,k}$ with positive coefficient in the decomposition (13) of t. If for some third agent h we have $b_{ih}^{\sigma,k} = 1$ then $t_{ih} \geq t_{\sigma(k)} \Rightarrow t_{jh} \geq t_{\sigma(k)} \Rightarrow b_{jh}^{\sigma,k} = 1$. By our assumption on ψ this implies $\psi_i(b^{\sigma,k}) \leq \psi_j(b^{\sigma,k})$ hence after summing in (15) $\varphi_i(t) \leq \varphi_j(t)$ obtains.

Repeat the argument with strict ranking in lieu of weak ranking, calling $\bar{h}$ an agent such that $t_{i\bar{h}} < t_{j\bar{h}}$. We still have $\psi_i(b^{\sigma,k}) \leq \psi_j(b^{\sigma,k})$ for all k. Moreover we can find an index k such that $t_{j\bar{h}} \geq t_{\sigma(k)} > t_{\sigma(k-1)} \geq t_{i\bar{h}}$. Then $b_{i\bar{h}}^{\sigma,k} = 0$ and $b_{j\bar{h}}^{\sigma,k} = 1$, so by strict ranking for ψ we get $\psi_i(b^{\sigma,k}) < \psi_j(b^{\sigma,k})$. Summing up in (15) gives $\varphi_i(t) < \varphi_j(t)$.

Step 2. *BHM meets Strict Ranking.* By Step 1 it is enough to show that $\text{Bird}^{1/2}$ meets strict ranking on elementary matrices. Fix $t \in \mathscr{G}(N)$ and $1, 2 \in N$ such that $t_{1h} \le t_{2h}$ for all $h \neq 1, 2$, with at least one strict inequality. In the corresponding graph G this means that the set $B(1, G)$ of 1's neighbors in G other than 2 (agents such that $1i \in G, i \neq 2$) is a strict subset of $B(2, G)$. Observe first that if 1 and 2 are in different connected components of G, then 1 is isolated in G and 2 is not, implying $y_1 = 0 < y_2$. From now on we assume that 1 and 2 are in the same connected component of G. Without loss of generality we restrict attention to this component, also denoted N, and we assume that G is connected. Using the notation of Step 3.2 in the proof of Proposition 4.1, we must prove

$$\sum_{i \in B(1, G)} p_{1i}(G) < \sum_{j \in B(2, G)} p_{2j}(G).$$

Pick agent 3 in $B(2, G) \backslash B(1, G)$. We claim

$$p_{2j}(G + 13) \le p_{2j}(G) \quad \text{for all } j \in B(2, G) \qquad \text{and} \qquad p_{23}(G + 13) < p_{23}(G). \tag{18}$$

The first inequality was already used in Step 3.2 and is proven in Lyons and Peres [19]. To prove the second one, let $m(H)$ denote the number of spanning trees of a graph H. Setting $H = G - 23$, our inequality reduces to

$$\frac{m(H+13)}{m(H+13+23)} > \frac{m(H)}{m(H+23)},$$

which results from the Stong [26] identity

$$m(H+13)m(H+23) - m(H)m(H+13+23) = (K(H, 13, 23))^2,$$

where $K(H, 13, 23)$ is the number of near trees L (graphs $L \subseteq H$ such that $L + 13$ and $L + 23$ are spanning trees of N). To show this number is positive, we prove that a near tree in H must exist. Choose a spanning tree Γ of N in $H + 23 = G$, and compare the position of 1, 2, 3 in Γ. If 2 is between 1 and 3, we take away one edge between 2 and 3 to get a near tree; similarly if 1 is between 2 and 3; if 3 is between 1 and 2, the first agent from 1 toward 3 is $i \neq 3$ (because $13 \notin G$) and $1i \in G \Rightarrow 2i \in G$, so by removing an edge of Γ between i and 3 and adding $2i$ we create a spanning tree where 2 is between 1 and 3.

Claim (18) just proven implies that y_2 strictly decreases as we add the edge 13 to G (noting that if $12 \in G$ then the first part of (18) applies to this edge as well). On the other hand y_1 weakly increases by monotonicity of $\text{Bird}^{1/2}$ (Proposition 4.1). Repeating this argument for all agents in $B(2, G) \backslash B(1, G)$, we obtain a graph where 1 and 2 are symmetric hence pay the same cost.

Step 3. *$\text{Bird}^{1/2}$ solution.* The following four-person example shows that it fails strict ranking: $t_{14} = t_{24} = t_{34} = 2$; $t_{23} = 1$; $t_{12} = t_{13} = 0$. The $\text{Bird}^{1/2}$ solution charges the same to agents 1 and 2.

To show next that it meets ranking, fix a problem (N, t) with $|N| \ge 3$ and two agents $1, 2$ such that $t_{1h} < t_{2h}$ for all $h \ge 3$. In any maximal spanning tree Γ, 1 must be a leaf, i.e., he has exactly one neighbor: if i, j are two neighbors of 1 in Γ, at least one of i or j, say i, is not on the path from 1 to 2 in Γ, and substituting $2i$ to $1i$ contradicts the maximality of Γ. If the unique neighbor of 1 in Γ is 2, then $y_1^{\Gamma, 1/2} = t_{12}/2 < y_2^{\Gamma, 1/2}$. So assume that there are other agents on the path from 1 to 2 in Γ: let 3 be 1's neighbor and $3'$ be 2's (3 and $3'$ could coincide). We have $t_{13} < t_{23} \le t_{23'}$, the latter by maximality of Γ. Therefore the $\Gamma^{1/2}$ solution charges less to 1:

$$y_1^{\Gamma, 1/2} = \frac{t_{13}}{2} < \frac{t_{23'}}{2} \le y_2^{\Gamma, 1/2}.$$

Turning to weak ranking, we fix a problem (N, t) and two agents $1, 2$ such that $t_{1h} \le t_{2h}$ for all $h \ge 3$. Let us now move from (N, t) to (N, t') in which $t'_{1h} = t_{2h}$ for all $h \ge 3$, and otherwise t and t' are identical. By monotonicity (Proposition 4.1) and cross-monotonicity of $\text{Bird}^{1/2}$ (Proposition 8.1 below), the charge y_1 weakly increases and y_2 weakly decreases. By symmetry, the two charges are equal at (N, t').

Step 4. *Uniform solution.* An example where this solution fails ranking is the three-person problem $t_{12} = 2$, $t_{13} = 0$, $t_{23} = 1$, for which the shares are $y_1 = y_2 = \frac{7}{6}$, $y_3 = \frac{2}{3}$, despite $y_{13} < y_{23}$.

To prove weak ranking, fix (N, t) and two agents $1, 2$ such that $t_{1h} \le t_{2h}$ for all $h \ge 3$. Using the expression of t^*_{ij} as (2) in Lemma 2.1, we have $t^*_{1h} \le t^*_{2h}$ for all $h \ge 3$ as well (compare the paths $\{1l_1, l_1 l_2, \ldots, l_s h\}$ and $\{2l_1, l_1 l_2, \ldots, l_s h\}$). In turn this gives $t_1^{*k} \le t_2^{*k}$ for $k = 1, \ldots, n-1$ and $y_1 \le y_2$ follows from (17).

Step 5. *Bird solution.* In the three-person example of the previous step, the Bird solution gives $y = (\frac{4}{3}, 1, \frac{2}{3})$, violating ranking and weak ranking.

Step 6. *Two Shapley values.* We fix a problem (N, t) and two agents 1, 2 such that $t_{1h} \le t_{2h}$ for all $h \ge 3$, with at least one strict inequality. For any $S \subset N$ containing neither 1 nor 2, inequalities $v_-(S \cup \{1\}, t) \le v_-(S \cup \{2\}, t)$ and $v^+(S \cup \{1\}, t) \le v^+(S \cup \{2\}, t)$ are straightforward. Next $v^+(\{1\}, t) = \sum_{h \ne 1} t_{1h} < \sum_{h \ne 2} t_{2h} = v^+(\{2\}, t)$ implies $\mathrm{Sh}_1^+(N, t) < \mathrm{Sh}_2^+(N, t)$. And if $t_{13} < t_{23}$, $v_-(\{1,3\}, t) = t_{13} < t_{23} = v_-(\{2,3\}, t)$ gives the same inequality for Sh_-. □

THEOREM 8.2.
(i) *A piecewise linear solution φ meets strict monotonicity if its restriction ψ to $\mathcal{G}(N)$ does.*
(ii) *The BHM solution, and the Shapley value Sh_- meet strict monotonicity.*
(iii) *The uniform solution, the Bird$^{1/2}$ solution, the Bird solution, and the Shapley value Sh^+, fail strict monotonicity.*

PROOF. *Step* 1. The first statement obtains by the same argument as in the corresponding part of Theorem 5.1.

Step 2. By Step 1, to check that the *BHM* solution is strictly monotonic, we only need to check that the Bird$^{1/2}$ solution is strictly monotonic on $\mathcal{G}(N)$, or that if $t, t' \in \mathcal{G}(N)$ and $t_{ij} = 0 < t'_{ij} = 1$ while $t_e = t'_e$ for all $e \ne ij$, then $B_i^{1/2}(t) < B_i^{1/2}(t')$. But $B_i^{1/2}(t)$ is one half of i's expected degree in the uniform maximal spanning tree. Thus, we need to show that when we add an edge ij to the graph $G(t)$ with $t_{ij} = 0$, this expected degree strictly increases.

Case 1. Edge ij unites two connected components of the graph $G(t)$ into one connected component of the graph $G(t')$. Here we have a bijection between MSTS of $G(t)$ and $G(t')$: to any MST Γ of $G(t)$ corresponds the MST Γ' of $G(t')$ with $\Gamma' = \Gamma + ij$. Because the degree of agent i in each Γ is one less than in Γ', we obtain strict monotonicity.

Case 2. Edge ij is inside some connected component of the graph $G(t)$ (which is then also a connected component of the graph $G(t')$). We assume without loss of generality that this connected component is N. Let $i = i_1, i_2, \ldots, i_{p-1}, i_p = j$ be a path from i to j in $G(t)$, and let $k = i_{p-1}$. We claim that $p_e(G(t')) \le p_e(G(t))$ for every $e \in G(t)$, and the inequality is strict for $e = jk$. The proof is similar to that of (18); the existence of a near tree follows from the existence of a spanning tree of $G(t)$ in which j does not lie between i and k (such a spanning tree may be found by augmenting the path $i = i_1, i_2, \ldots, i_{p-1} = k$). Now, the expected degree of i in the uniform spanning tree of $G(t)$ is given by

$$\sum_{e \in G(t);\, i \in e} p_e(G(t)) = n - 1 - \sum_{e \in G(t);\, i \notin e} p_e(G(t)),$$

whereas the expected degree of i for $G(t')$ is

$$\sum_{e \in G(t');\, i \in e} p_e(G(t')) = n - 1 - \sum_{e \in G(t);\, i \notin e} p_e(G(t')),$$

and the latter expression is strictly larger than the above.

Step 3. All the remaining statements are easily derived by essentially the same reasoning as in the proof of the previous theorem (and from Proposition 4.1 in the case of the Shapley value Sh^+). □

8.2. Solidarity vs. cross-monotonicity. Going beyond the monotonicity property, we examine the impact of an increase in the capacity demand t_{jh} on agent i's cost share, where $i \ne j, h$. We find that the uniform and Bird solutions stand out against the other four solutions: BHM, Bird$^{1/2}$, Sh^+, and Sh_-.

Solidarity: for all distinct i, j, h, y_i is weakly *in*creasing in t_{jh}.

Cross-monotonicity: for all distinct i, j, h, y_i is weakly *de*creasing in t_{jh}.

PROPOSITION 8.1.
(i) *A piecewise linear solution φ is cross-monotonic* (*respectively, solidary*) *if its restriction ψ to $\mathcal{G}(N)$ is.*
(ii) *The BHM, Bird$^{1/2}$, Sh^+, and Sh_- solutions are all cross-monotonic.*
(iii) *The uniform solution is solidary.*
(iv) *The Bird solution is neither solidary nor cross-monotonic.*

PROOF. *Step* 1. The proof of the first statement follows exactly that of Step 3 in Theorem 5.1, where we showed that if ψ is monotonic on $\mathcal{G}(N)$, so is its piecewise linear extension φ.

Step 2. *BHM and Bird$^{1/2}$*. First we show that Bird$^{1/2}$ is cross-monotonic on $\mathcal{G}(N)$. By Step 1 this implies that BHM is cross-monotonic everywhere. Fix $t \in \mathcal{G}(N)$ with corresponding graph G. With the notation of Step 3.2 in the proof of Proposition 4.1, the cost share $B_1^{1/2}(t)$ is, up to a factor $\frac{1}{2}$, the sum of $p_e(G)$ over all edges in G adjacent to agent 1. When an edge 23 is added to G we have $p_e(G + 23) \le p_e(G)$ and the desired conclusion. Finally we can mimic the proof that Bird$^{1/2}$ is monotonic in Steps 3.3.1 and 3.3.2 of the proof of Proposition 4.1, to show that it is cross-monotonic on $\mathcal{T}(N)$. We omit the details.

Step 3. *Two Shapley values.* Fix (N, t), $S \subset N$, $1 \notin S$, and two other agents 2, 3. If one of them is at least outside S, the term $v_-(S \cup \{1\}, t) - v_-(S, t)$ does not change as we increase t_{23}. If both 2 and 3 are in S and t_{23} increases to x, by Equation (6) in the proof of Lemma 3.1 the net change is

$$[x - (t_{S\cup\{1\}})^*_{23}]_+ - [x - (t_S)^*_{23}]_+,$$

a nonpositive term because $t \to t^*_{23}$ is weakly increasing. Similarly as we increase t_{23} to x, the term $v^+(S \cup \{1\}, t) - v^+(S, t)$ can only change if at least one of 2, 3 is in S, and then the net change is

$$[x - (t^{S\cup\{1\}})^*_{23}]_+ - [x - (t^S)^*_{23}]_+.$$

Step 4. *Uniform solution.* Monotonicity of $t \to t^*$ ensures that for any $i \in N$ and $k = 1, \dots, n-1$, t_i^{*k} is weakly increasing in all t_e. By Equation (17) the same is true for the *uniform* solution.

Step 5. *Bird solution.* In the three-person example $t_{12} = 2$, $t_{13} = 0$, $t_{23} = 1$ (Step 4 in the proof of Theorem 8.1), it chooses $y = (\frac{4}{3}, 1, \frac{2}{3})$; after t_{13} increases to $t'_{13} = 2$, agent 2's share goes up to $\frac{4}{3}$. In the four-person example

$$t_{12} = 2, \quad t_{14} = 3, \quad t_{34} = 1, \quad t_{23} = a; \quad t_{13} = t_{24} = 0,$$

the Bird solution charges 2 to agent 1 if $a < 1$, and $\frac{7}{4}$ if $a > 1$. □

8.3. Reductionism and population monotonicity. We turn to two consequences of solidarity. The first one says that the solution φ is entirely determined by its action on irreducible capacity matrices.[10]

Reductionism: $\varphi(N, t) = \varphi(N, t^*)$ for all t.

As noted above, reductionism considerably simplifies the computation of our solution, from $n(n-1)/2$ free variables in t to only $n-1$ in t^*. However, it also throws away much relevant counterfactual information (Example 6.2).

The second property requires the entry of one more agent to be always weakly disadvantageous to all previous participants.

Population Monotonicity: for all N, j, t, $\varphi_i(N\backslash\{j\}, t) \le \varphi_i(N, t)$ for all $i \ne j$.

(Here the restriction of t to $N\backslash\{j\}$ is also written t.)

Among the six solutions we discuss, only the *uniform* solution is reductionist, as well as population monotonic. All others—Shapley values Sh^+ and Sh_-, Bird and $\text{Bird}^{1/2}$, and *BHM*—fail both properties. Checking the latter claim for Bird is straightforward; for the other four it follows from Proposition 8.3 below and the fact that these solutions satisfy cross-monotonicity.

In our next result we use a mild dummy-type property to connect problems of different sizes.

Dummy: for all $N, i, j, i \ne j$, and t: $\{t_{jh} = 0$ for all $h \ne j\} \Rightarrow \varphi_i(N\backslash\{j\}, t) = \varphi_i(N, t)$.

All solutions discussed here, including the asymmetric ones in Definitions 6.1, 6.2, satisfy dummy. Note that population monotonicity implies dummy.

Lemma 8.1.
(i) *If a monotonic solution is solidary, it is reductionist.*
(ii) *If a monotonic and solidary solution meets dummy, it is population monotonic.*

Proof. Under monotonicity and solidarity, each coordinate $\varphi_i(N, t)$ is weakly increasing in t. As t increases to t^*, the sum $\sum_N \varphi_i(N, t)$ remains constant, hence so does each $\varphi_i(N, t)$. This proves reductionism.

For any N, j, t define t^{-j} by $t_e^{-j} = t_e$ if e does not contain j, $t_e^{-j} = 0$ if it does. By solidarity and monotonicity, $\varphi_i(N, t^{-j}) \le \varphi_i(N, t)$ for all $i \ne j$ and by dummy $\varphi_i(N, t^{-j}) = \varphi_i(N\backslash\{j\}, t)$. This proves population monotonicity. □

The piecewise linear property provides a simple characterization of the uniform solution.

Proposition 8.2. *The uniform solution is the only piecewise linear solution that is a standalone core selection, treats equals equally, and is reductionist.*

Proof. It is enough to show that the uniform solution on $\mathcal{G}(N)$ (16) is characterized by standalone core selection, equal treatment and reductionism. Let ψ be a solution on $\mathcal{G}(N)$ with those properties. Fix $t \in \mathcal{G}(N)$ with associated graph G, and let $A_1, \dots, A_r$ be its connected components. Then t^* is represented by the graph G^* with the same connected components and the complete graph inside each component A. Standalone core selection implies $\sum_{i\in A} \psi_i(t^*) = |A| - 1$ (as noted after the proof of Theorem 5.1), equal treatment of the agents in A gives $\psi_i(t^*) = (|A|-1)/|A|$ for $i \in A$, and reductionism implies $\psi_i(t) = \psi_i(t^*)$. □

[10] It is equivalent to the property called *Independence of Irrelevant Trees* in Bergantiños and Vidal-Puga [3] and Ozsoy [22] for the mcst problem. In our model that property reads: if Γ is a maximal spanning tree for (N, t) and (N, t'), and if t and t' coincide on all edges of Γ, then $\varphi(N, t) = \varphi(N, t')$.

Compare this proposition to the characterization result in Bergantiños and Vidal-Puga [5]. The folk solution of the MCST problem is characterized there by equal treatment of equals, a separability property reminiscent of our decomposition axiom below, and restricted additivity, a stronger requirement than piecewise linearity.

8.4. A sharp pattern of incompatibilities. Partition the five axioms introduced in this section into solidarity, reductionism and population monotonicity on one side, and on the other side cross-monotonicity and ranking. Proposition 8.3 below says that, essentially, any axiom in one group is incompatible with any axiom in the other. Before stating the precise result, we introduce one last property drawing a wedge between the uniform and BHM solutions.

Suppose that two disjoint subsets of users, M and P, do not communicate directly with each other, and that $N\backslash\{M\cup P\}$ contains a single agent h. Then the capacity built inside $\{h\}\cup P$ cannot help the needs of agents in M, and similarly the network inside $\{h\}\cup M$ is of no use to P. This suggests to uncouple the cost sharing problems in $\{h\}\cup P$ and $\{h\}\cup M$ as follows.

Decomposition: for any partition $N = M\cup P\cup\{h\}$

$$\{t_{ij}=0 \text{ for all } i\in M,\ j\in P\} \implies \varphi(N,t)=\varphi(M\cup\{h\}, t_{M\cup\{h\}})+\varphi(P\cup\{h\}, t_{P\cup\{h\}}). \tag{19}$$

(Here we abuse notation in the obvious way as these three vectors are in different euclidian spaces.)[11]

Under the premises of decomposition, a maximal spanning tree of (N,t) connects at h a maximal spanning tree in $(M\cup\{h\}, t_{M\cup\{h\}})$ and one in $(P\cup\{h\}, t_{P\cup\{h\}})$. The uniform MST chooses these two trees independently, therefore the expected degree of i in M is the same in (N,t) and in $(M\cup\{h\}, t_{M\cup\{h\}})$. Thus Bird$^{1/2}$ meets decomposition, and so does BHM, because all terms in (19) are piecewise linear.

A similar argument shows $v_-(S,t)=v_-(S\cap(M\cup\{h\}), t_{M\cup\{h\}})+v_-(S\cap(P\cup\{h\}), t_{P\cup\{h\}})$ (ditto for $v^+(S,t)$), implying that both Sh_- and Sh^+ meet decomposition.

On the other hand, the Bird and uniform solutions fail decomposition, for instance in the three-agent capacity graph $t_{12}=t_{13}=1$, $t_{23}=0$, where both solutions give $y_1=\frac{2}{3}$, whereas $y_1(\{1,2\}, t_{\{1,2\}})=y_1(\{1,3\}, t_{\{1,3\}})=\frac{1}{2}$.

Proposition 8.3. *Each of the axioms solidarity, reductionism, population monotonicity is incompatible with any of the axioms cross-monotonicity, decomposition, ranking, with one exception*: *population monotonicity is compatible with ranking, even with strict ranking.*

Proof. *Step* 1. *Solidarity and cross-monotonicity.* Set $N=\{1,2,3\}$ and $t_{ij}=d_k$. Solidarity and cross-monotonicity together imply that y_1 depends only upon d_2 and d_3. So we get for all $d\in\mathbb{R}^3_+$

$$y_1(d_2,d_3)+y_2(d_3,d_1)+y_3(d_1,d_2)=d_1+d_2+d_3-\min\{d_1,d_2,d_3\},$$

which is clearly impossible.

Step 2. *Solidarity* (*or reductionism*) *and decomposition.* Fix three agents 1, 2, 3; write $\varphi_i(\{i,j\}, t_{ij}=1)=y_i^{ij}$ and a_i for i's share in the symmetric three-agent problem $t_{12}=t_{13}=t_{23}=1$. In the three-agent problem $t_{12}=t_{13}=1$, $t_{23}=0$, Decomposition gives $y_1=y_1^{12}+y_1^{13}$ and Solidarity implies $y_1\le a_1$ (reductionism gives $y_1=a_1$). Permuting the role of the agents gives $\sum_i y_i^{ij}+y_i^{ik}\le\sum_i a_i=2$, contradicting budget balance in the two-agent problems $(\{i,j\}, t_{ij}=1)$.

Step 3. *Solidarity and ranking.* Consider the three-agent problem $t_{12}=t_{13}=t_{23}=1$ with shares a_1,a_2,a_3. Assume without loss of generality that $a_1=\min_i a_i$. For the new problem obtained by setting $t'_{23}=0$ and $t'_e=t_e$ otherwise. Solidarity gives $y'_1\le a_1$, and ranking gives $y'_2<y'_1$ and $y'_3<y'_1$, implying $\sum_i y'_i<3a_1\le\sum_i a_i$, a contradiction.

Step 4. *Reductionism and cross-monotonicity.* We use the notation of Step 2, with the slight difference that now y_i^{ij} is i's share in the three-agent problem with $t_{ij}=1$ and $t_{ik}=t_{jk}=0$. For the three-agent problem $t_{12}=t_{13}=1$, $t_{23}=0$, cross-monotonicity gives $y_2\le y_2^{12}$ and reductionism gives $y_2=a_2$. Hence $a_2\le y_2^{12}$ and similarly $a_1\le y_1^{12}$, implying $a_1+a_2\le 1$. In the same way $a_2+a_3\le 1$ and $a_3+a_1\le 1$, yielding the contradiction $\sum_i a_i\le\frac{3}{2}$.

Step 5. *Reductionism and ranking.* In the three-agent problem $t_{12}=3$, $t_{13}=t_{23}=2$, assume, without loss of generality, $y_2\le y_1$. Reductionism preserves this inequality in the three-agent problem $t_{12}=3$, $t_{13}=1$, $t_{23}=2$, where it contradicts ranking.

[11] Our definition has no counterpart in the MCST model. A Decomposition axiom used in Kar [15] and Bergantiños and Vidal-Puga [3] corresponds in our model to the property $\{t_{ij}=0$ for all $i\in M, j\in P\}\Rightarrow\varphi(N,t)=\varphi(M,t_M)+\varphi(P,t_P)$ when M,P partition N. All our solutions meet this much weaker requirement.

Step 6. *Population monotonicity and decomposition.* Fix four agents $1, 2, 3, 4$ and the benchmark problem $t_{12} = t_{23} = t_{34} = t_{41} = 1$, $t_{13} = t_{24} = 0$. In the three-agent problem obtained by removing 4, Decomposition gives the cost shares $(y_1^{12}, y_2^{12} + y_2^{23}, y_3^{23})$, where we use the notation of Step 2. By population monotonicity these are lower bounds on the shares of agents $1, 2, 3$ in the benchmark. By removing 2 in the benchmark, we get similarly the lower bounds $(y_1^{14}, y_3^{34}, y_4^{14} + y_4^{34})$ for the shares of $1, 3, 4$ in the benchmark. Budget balance implies

$$\max\{y_1^{12}, y_1^{14}\} + (y_2^{12} + y_2^{23}) + \max\{y_3^{23}, y_3^{34}\} + (y_4^{14} + y_4^{34}) \leq 3. \tag{20}$$

Exchanging the role of $1, 3$ versus $2, 4$ gives

$$(y_1^{12} + y_1^{14}) + \max\{y_2^{12}, y_2^{23}\} + (y_3^{23} + y_3^{34}) + \max\{y_4^{14}, y_4^{34}\} \leq 3.$$

Summing up and using $y_i^{ij} + y_j^{ij} = 1$

$$\max\{y_1^{12}, y_1^{14}\} + \max\{y_2^{12}, y_2^{23}\} + \max\{y_3^{23}, y_3^{34}\} + \max\{y_4^{14}, y_4^{34}\} \leq 2,$$

then using $y_i^{ij} + y_j^{ij} = 1$ several times we get

$$y_1^{12} = y_1^{14} = y_3^{23} = y_3^{34}; \qquad y_2^{12} = y_2^{23} = y_4^{14} = y_4^{34},$$

and finally as (20) must be an equality, all eight numbers are $\frac{1}{2}$, and the share of each agent in the benchmark is at least 1, contradiction.

Step 7. *Population monotonicity and cross-monotonicity.* We prove their incompatibility by showing that these two axioms imply decomposition, and then invoking Step 6. Suppose $N = M \cup P \cup \{h\}$ and t satisfy the premises of decomposition. Population monotonicity implies that $\varphi_i(M \cup \{h\}, t_{M\cup\{h\}}) \leq \varphi_i(N, t)$ for all $i \in M$. On the other hand, dummy (a consequence of population monotonicity) and cross-monotonicity imply $\varphi_i(M \cup \{h\}, t_{M\cup\{h\}}) = \varphi_i(N, t_{M\cup\{h\}}) \geq \varphi_i(N, t)$ for all $i \in M$. Thus the two sides of (19) agree for all $i \in M$, and similarly for all $i \in P$, hence also for h.

Step 8: *Population monotonicity and strict ranking.* We define a solution ψ on $\mathcal{G}(N)$, for all N, by

$$\psi_i(t) = \left(1 - \frac{1}{n_i^3}\right) B_i(t) + \frac{1}{n_i^3} B_i^{1/2}(t),$$

where n_i is the number of nodes in the connected component of i in t. We extend it piecewise linearly to a solution φ on $\mathcal{T}(N)$. It suffices to prove that ψ satisfies the two properties on capacity graphs (for strict ranking this was shown in Theorem 8.1, we omit the straightforward argument for population monotonicity).

To check population monotonicity of ψ, let t be a graph, and let t' be obtained from it by adding a new node and linking it to some of the old nodes. Let i be an old node. If the new node is in a different connected component than i in t', then $\psi_i(t') = \psi_i(t)$. If it is in the same connected component as i, then $B_i(t') \geq n_i/(n_i + 1)$ whereas $B_i(t) = (n_i - 1)/n_i$ and $B_i^{1/2}(t) \leq (n_i - 1)/2$; a little computation shows that these imply $\psi_i(t') > \psi_i(t)$.

Next we check strict ranking of ψ. If i and j satisfy the premises of this property then either i is isolated and j is not (in which case $y_i = 0 < y_j$), or they are in the same connected component. In the latter case, the conclusion follows because the Bird solution satisfies weak ranking on $\mathcal{G}(N)$ and the Bird$^{1/2}$ solution satisfies strict ranking there (see Theorem 8.1). □

In Lemma 4.2, we saw that all $\Gamma^{1/2}$-solutions not only are in the standalone core, but meet the tighter upper bound (11). This bound is also met by the BHM solution.[12] It is easy to check that the Bird$^{1/2}$ solution fails the lower standalone test (8), whereas the Bird solution misses the tight upper bound (11). We can say more.

Proposition 8.4.
(i) *The lower standalone test* (8) *and the tight upper bound* (11) *are incompatible.*
(ii) *Solidarity is incompatible with the tight upper bound* (11).
(iii) *Cross-monotonicity is incompatible with the lower bound* (8).

Proof. *Step* 1. (8) *and* (11). Consider the six-person problem $N = \{1, \ldots, 6\}$; $t_{ij} = 1$ if $i \in \{1, 2\}$, $j \in \{3, 4, 5, 6\}$; $t_{ij} = 0$ otherwise. Here $V(N, t) = 5$ and $v_-(\{2, 3, 4, 5, 6\}, t) = 4$, so the lower standalone bound implies $y_1 \leq 1$; by symmetry $y_2 \leq 1$. Next we have $v_-(\{3, 4, 5, 6\}, t) = 0$ and $v^+(\{3, 4, 5, 6\}, t) = 5$, so the tight upper bound implies $y_3 + y_4 + y_5 + y_6 \leq \frac{5}{2}$. Together these inequalities bring a contradiction.

[12] Indeed under the premises of Theorem 5.1, the inequality (11) extends from ψ to φ.

Step 2. *Solidarity and* (11). In the three-agent problem $t_{12} = t_{13} = t_{23} = 1$, assume without loss of generality that agent 1's share a_1 satisfies $a_1 \le \frac{2}{3}$. Then by solidarity his share must still be at most $\frac{2}{3}$ upon deleting the 23 edge, implying that (11) is violated for $S = \{2, 3\}$.

Step 3. *Cross-monotonicity and* (8). Consider the ten-agent capacity graph (N, t) composed of two connected components, a complete graph on $\{1, \ldots, 5\}$ and a complete graph on $\{6, \ldots, 10\}$. Without loss of generality, assume that $y_1 + y_2 + y_6 + y_7 \ge 3.2$. Now let us add the four crossing edges between 1, 2 and 6, 7 to get a new graph (N, t'). Cross-monotonicity requires that the added cost of 1 be borne entirely by agents 1, 2, 6, 7, and hence $y'_1 + y'_2 + y'_6 + y'_7 \ge 4.2$. Assuming without loss of generality that $y'_1 > 1$, we get a violation of (8) for $S = N \setminus \{1\}$. □

9. Concluding comments.

9.1. A consequence of reductionism. Capacities t_{ij} represent communication needs between i and j. If they are only known to these two agents, they are potentially subject to strategic manipulations. Under a monotonic solution, reporting a capacity larger than one's true need is obviously not advantageous, but the opposite move may well be.

Consider the problem with four agents 1, 2, 3, 4, and

$$t_{ij} = 10 \quad \text{for all } ij \neq 12, \ \text{ and } \ 0 < t_{12} < 10.$$

Monotonicity says that the share of agents 1 and 2 is weakly increasing in t_{12}. This capacity is irrelevant in the sense that an optimal spanning tree uses three links of capacity 10, and routes the traffic between 1 and 2 through those links. Therefore if the shares of these two agents are strictly increasing in t_{12}, they benefit by reporting a lower capacity need t'_{12}, knowing full well that their actual capacity need t_{12} will be covered in any optimal spanning tree. An example is the Sh_ solution, charging $y_1 = y_2 = \frac{1}{12} t_{12} + 6\frac{2}{3}$.

A reductionist solution is not manipulable by underreporting of capacities. Fix a problem (N, t), two nodes 1, 2 and consider t' such that $t'_{12} < t_{12}$ and $t'_e = t_e$ otherwise. In (N, t) the effective capacity sustained between 1 and 2 is t^*_{12}. If $t_{12} < t^*_{12}$, we have $(t')^* = t^*$, so the effective capacity on 12 does not change, and by reductionism neither do the cost shares of 1 and 2. If $t_{12} = t^*_{12}$ and $(t')^*_{12} < t^*_{12}$, the capacity needs of 12 are not covered. Finally if $t_{12} = t^*_{12}$ and $(t')^*_{12} = t^*_{12}$, we have $(t')^* = t^*$ again and we conclude as above that the move to t' is pointless.

9.2. Summary of results.

	Sh$^+$	Sh_	B	$B^{1/2}$	Un	BHM
Standalone core	+	+	+	+	+	+
Monotonicity	−	+	−	+	+	+
Continuity	+	+	−	−	+	+
Equal treatment of equals	+	+	+	+	+	+
Polynomial complexity	?	?	+	+	+	+
Lower standalone test	−	−	+	−	+	−
Weak ranking	+	+	−	+	+	+
Ranking	+	+	−	+	−	+
Strict ranking	+	+	−	−	−	+
Strict monotonicity	−	+	−	−	−	+
Cross-monotonicity	+	+	−	+	−	+
Solidarity	−	−	−	−	+	−
Reductionism	−	−	−	−	+	−
Population monotonicity	−	−	−	−	+	−
Decomposition	+	+	−	+	−	+

9.3. Open questions. 1. Is there a simple characterization of BHM analogous to that of *Un* in Proposition 8.2? Can we find a characterization of either solution without the piecewise linearity axiom?

2. The BHM solution meets the five properties listed in §4, and satisfies in addition cross-monotonicity, strict ranking and decomposition. It does not have a closed form expression like that of *Un* in Theorem 6.1. Can we define a solution with a closed form expression, sharing some of the distinctive normative properties of the BHM solution?

3. How should we adapt our model when capacity requests are *oriented*, namely user i requests t_{ij}, j requests t_{ji}, and we need to sustain the capacity $\max\{t_{ij}, t_{ji}\}$ on some path between i and j?

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