Games and Economic Behavior **28,** 181–188 (1999)
Article ID game.1998.0694, available online at http://www.idealibrary.com on IDEAL®

On the Least Core and the Mas-Colell Bargaining Set

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Received June 12, 1997

We show that the least core of a TU coalitional game with a finite set of players is contained in the Mas-Colell bargaining set. This result is extended to games with a measurable space of players in which the worth of the grand coalition is at least that of any other coalition in the game. As a consequence, we obtain an existence theorem for the Mas-Colell bargaining set in TU games with a measurable space of players. *Journal of Economic Literature* Classification Number: C71. © 1999 Academic Press

1. INTRODUCTION

Bargaining sets and related solution concepts for coalitional games have been studied intensively (for a comprehensive survey see Maschler (1992)).

The first notions of a bargaining set for a cooperative game were introduced by Aumann and Maschler (1964). Mas-Colell (1989) proposed a new variant of a bargaining set. One of the advantages of the Mas-Colell bargaining set is that it can be defined for games with a continuum of players. Mas-Colell (1989) showed that in atomless pure exchange

* Research supported by the Promotion of Sponsored Research Fund and by the Fund for the Promotion of Research at the Technion.

0899-8256/99 $30.00

economies his bargaining set coincides with the set of competitive equilibria. For such economies, or more generally for games with a non-empty core, the non-emptiness of the bargaining set is automatic, since, by definition, it contains the core. But for games with an empty core, the question of the non-emptiness of the bargaining set arises, and has been studied in the literature. For coalitional games with a finite set of players, the Mas-Colell bargaining set contains the prekernel and is therefore non-empty.

The least core is a core-like non-empty-valued solution concept for coalitional games which was introduced by Maschler *et al.* (1979), who studied its relation to the kernel and the nucleolus. In the present work we show that the least core of a coalitional game with a finite set of players is always contained in the Mas-Colell bargaining set. We give an example which shows that this does not hold for the classical bargaining set. We then extend this result to coalitional games with a measurable set of players in which the worth of the grand coalition is at least that of any other coalition in the game. Since the least core is always non-empty, we thereby obtain an existence theorem for the Mas-Colell bargaining set in games with a measurable space of players. It should be noted that this work is concerned only with TU games, and therefore our results do not bear upon the issue of non-emptiness of the Mas-Colell bargaining set in NTU games.

2. FINITE GAMES

In this section we prove that the Mas-Colell bargaining set contains the least core in any coalitional game with a finite set of players.

A (finite) *coalitional game*, or simply a *game*, is a pair (N, v) where $N = \{1, \dots, n\}$ is the set of players and v: $2^N \to \mathfrak{R}$ is a function which satisfies $v(\emptyset) = 0$. The members of 2^N, i.e., the subsets of N, are called *coalitions*. If $x = (x_1, \dots, x_n) \in \mathfrak{R}^n$ and $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$. An *imputation* for the game (N, v) is a vector $x \in \mathfrak{R}^n$ such that $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. The set of all imputations for (N, v) will be denoted by $I(v)$. A *preimputation* for (N, v) is a vector $x \in \mathfrak{R}^n$ which satisfies $x(N) = v(N)$. The set of all preimputations for (N, v) will be denoted by $I^*(v)$. The *core* of the game (N, v), denoted by $C(v)$, is the set of all preimputations $x \in I^*(v)$ such that $x(S) \geq v(S)$ for all $S \subseteq N$. Clearly, $C(v) \subseteq I(v)$.

Let ε be a real number. The *strong ε-core* of the game (N, v), denoted by $C_\varepsilon(v)$, is the set of all preimputations $x \in I^*(v)$ such that $x(S) \geq v(S) - \varepsilon$ for all $S \subseteq N$, $S \neq \emptyset, N$. The strong ε-core which was introduced by Shapley and Shubik (1966) can be interpreted as the set of efficient payoff vectors that cannot be improved upon by any coalition if forming a coalition entails a cost of ε (or a bonus of $-\varepsilon$, if ε is negative). It is clear that $C_{\varepsilon_1}(v) \supseteq C_{\varepsilon_2}(v)$ for $\varepsilon_1 > \varepsilon_2$. The *least core* of (N, v), denoted by $LC(v)$, is the intersection of all non-empty strong ε-cores of (N, v). Let ε_v be the smallest ε such that $C_\varepsilon(v) \neq \emptyset$ (from here on we assume that $n \geq 2$), i.e.,

$$\varepsilon_v = \min_{x \in I^*(v)} \max_{S \neq \emptyset, N} (v(S) - x(S)).$$

Then $LC(v) = C_{\varepsilon_v}(v)$. The least core was introduced in Maschler *et al.* (1979). Note that if $\varepsilon_v = 0$ then $LC(v) = C(v)$; if $\varepsilon_v > 0$ then $C(v) = \emptyset$, and $\varepsilon_v < 0$ implies $LC(v) \subset C(v)$. Note also that if (N, v) is a zero monotonic game (i.e., $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ for every $S \subseteq N$ and $i \in N \setminus S$) then $LC(v) \subseteq I(v)$ (see Theorem 2.7 in Maschler *et al.* (1979)).

We now proceed to the definition of the Mas-Colell bargaining set, introduced in Mas-Colell (1989).

Let x be a preimputation for the game (N, v) and let $A \subseteq N$. A pair (A, y) is an *objection* to x if $y \in \Re^n$, $y(A) \leq v(A)$, and $y_i \geq x_i$ for all $i \in A$, with at least one strict inequality. A *counter-objection* to (A, y) is a pair (B, z) such that $B \subseteq N$, $z \in \Re^n$, $z(B) \leq v(B)$, and the following two conditions are satisfied:

$$z_i \geq y_i \text{ for all } i \in A \cap B, \tag{2.1}$$

$$z_i \geq x_i \text{ for all } i \in B \setminus A, \tag{2.2}$$

with at least one strict inequality in (2.1) or (2.2).

An objection (A, y) to x is *justified* if there is no counter-objection to it. The *Mas-Colell bargaining set* of the game (N, v), denoted by $MB(v)$, is the set of all preimputations for (N, v) to which there is no justified objection. Note that $C(v) \subseteq MB(v)$.

Two games (N, v) and (N, w) are *strategically equivalent* if there exist $\alpha > 0$ and $b \in \Re^n$ such that $w(S) = \alpha v(S) + b(S)$ for every $S \subseteq N$. A set-valued mapping $F(v)$ from games on N to $\Re^n$ is *covariant under strategic equivalence* if the previous relation implies that $F(w) = \{\alpha x + b | x \in F(v)\}$. It is clear that $LC(v)$ and $MB(v)$ are covariant under strategic equivalence.

THEOREM A. *Let (N, v) be a finite coalitional game. Then*

$$LC(v) \subseteq MB(v).$$

Proof. Let $x \in LC(v)$. Assume, on the contrary, that $x \notin MB(v)$. Since $LC(v)$ and $MB(v)$ are covariant under strategic equivalence, by a suitable translation of v and x we may assume that

$$x_i > 0 \text{ for all } i \in N. \tag{2.3}$$

Let (A, y) be a justified objection to x, and assume, w.l.o.g., that $y(A) = v(A)$. As $x(N) = v(N)$, $A \neq N$ and hence, by (2.3), $x(N \setminus A) > 0$. As $v(A) > x(A)$, we may choose $\alpha < 1$ such that $\alpha v(A) > x(A)$. Define

$$\beta = \frac{v(N) - \alpha v(A)}{x(N \setminus A)}$$

and

$$z_i = \begin{cases} \alpha y_i & \text{if } i \in A, \\ \beta x_i & \text{if } i \in N \setminus A. \end{cases}$$

Note that $z(N) = v(N)$. Let B be any coalition, $B \neq \emptyset, N$. Since (A, y) is a justified objection to x,

$$v(B) \leq y(B \cap A) + x(B \cap (N \setminus A)).$$

Therefore

$$\begin{aligned} v(B) - z(B) &\leq (1-\alpha) y(B \cap A) + (1-\beta) x(B \cap (N \setminus A)) \\ &\leq (1-\alpha) y(A) + \frac{\alpha v(A) - x(A)}{x(N \setminus A)} x(B \cap (N \setminus A)) \\ &\leq (1-\alpha) v(A) + \alpha v(A) - x(A) = v(A) - x(A) \leq \varepsilon_v. \end{aligned}$$

Moreover, $v(B) - z(B) = \varepsilon_v$ is impossible. Indeed, this would require that $y(B \cap A) = y(A)$ and $x(B \cap (N \setminus A)) = x(N \setminus A)$, which means that $B = N$. It follows that

$$\max_{S \neq \emptyset, N} (v(S) - z(S)) < \varepsilon_v,$$

which contradicts the definition of ε_v. Q.E.D.

Mas-Colell (1989) pointed out that in finite coalitional games the prekernel is always contained in his bargaining set (for a proof see Proposition 3.2 in Vohra (1991)). Since the prekernel is non-empty, so is the Mas-Colell bargaining set. The proof of Theorem A provides an alternative elementary proof for the existence of the Mas-Colell bargaining set in finite games, and as we shall see in Section 3, it can be extended under additional mild conditions to infinite games, where no appropriate definition is known for the prekernel (except for games with a countable set of players).

We now give an example which shows that the analog of Theorem A does not hold for the classical bargaining set $M_1^{(i)}$ (for a definition see Davis and Maschler (1963)).

EXAMPLE 2.1. Let $N = \{1, 2, 3, 4\}$ and $w_1 = w_2 = 2$, $w_3 = w_4 = 1$. Define a game (N, v) by

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq 4, \\ 0 & \text{otherwise}. \end{cases}$$

A direct computation gives:

$$M_1^{(i)}(v) = \left\{\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{6}\right)\right\} \quad \text{and} \quad LC(v) = \left\{\left(\tfrac{1}{3}, \tfrac{1}{3}, \alpha, \tfrac{1}{3} - \alpha\right) \middle| 0 \leq \alpha \leq \tfrac{1}{3}\right\}.$$

Thus $LC(v)$ is not contained in $M_1^{(i)}(v)$.

3. INFINITE GAMES

The purpose of this section is to extend Theorem A to games with an infinite set of players.

Let (T, Σ) be a measurable space, i.e., T is a set and Σ is a σ-field of subsets of T. The members of T are the *players* and those of Σ are the *coalitions*. A *coalitional game* on (T, Σ) is a bounded function v: $\Sigma \to \Re$ with $v(\emptyset) = 0$. Let *ba* be the space of all bounded finitely additive measures on (T, Σ) with the variation norm. It is well known that *ba* is the norm dual of the space of all bounded Borel-measurable functions on (T, Σ). A *preimputation* for the game v is a member x of *ba* which satisfies $x(T) = v(T)$. Like in the finite case we denote the set of all preimputations for v by $I^*(v)$.

The *core* and the *strong ε-core* of v (for any real number ε) are defined as in the finite case. Since the family $\{C_\varepsilon(v) | \varepsilon \in \Re\}$ is an increasing family of weak*-compact sets, by the finite intersection property the intersection of all non-empty strong ε-cores is non-empty. This intersection is the *least core* of v and as in the finite case is denoted by $LC(v)$. Let

$$\varepsilon_v = \inf_{x \in I^*(v)} \sup_{S \in \Sigma \setminus \{\emptyset, T\}} (v(S) - x(S)).$$

Then it is easy to see that ε_v is finite, the infimum is attained, and $LC(v) = C_{\varepsilon_v}(v)$.

The next definition is taken from Einy *et al.* (1996); it extends the definition of the Mas-Colell bargaining set to infinite games.

Let v be a game on (T, Σ) and x be a preimputation for v. An *objection* to x is a pair (A, y) such that $A \in \Sigma$, and $y \in ba$ satisfies $y(A) \leq v(A)$, $y(A) > x(A)$, and $y(S) \geq x(S)$ for every coalition $S \subset A$. A *counter-objection* to (A, y) is a pair (B, z) such that $B \in \Sigma$, $z \in ba$ satisfies $z(B) \leq v(B)$, and the following conditions are satisfied:

$$z(S) \geq y(S) \quad \text{for every coalition} \quad S \subseteq A \cap B, \tag{3.1}$$

$$z(S) \geq x(S) \quad \text{for every coalition} \quad S \subseteq B \setminus A, \tag{3.2}$$

$$z(B) > y(A \cap B) + x(B \setminus A). \tag{3.3}$$

A *justified objection* is an objection to which there is no counter-objection. The *Mas-Colell bargaining set* of v is the set $MB(v)$ of all preimputations to which there is no justified objection.

We are now ready to extend Theorem A to games with a measurable set of players.

THEOREM B. *Let v be a game on (T, Σ). Assume that $v(T) \geq v(S)$ for all non-empty $S \in \Sigma$. Then*

$$LC(v) \subseteq MB(v).$$

In particular, $MB(v) \neq \emptyset$.

Proof. If $C(v) \neq \emptyset$ then we have $LC(v) \subseteq C(v) \subseteq MB(v)$. So assume that $C(v) = \emptyset$. Therefore $\varepsilon_v > 0$, and thus, by the theorem's assumption,

$$\sup_{S \in \Sigma \setminus \{\emptyset, T\}} v(S) < v(T) + \varepsilon_v. \tag{3.4}$$

Let $x \in LC(v)$. Assume, on the contrary, that $x \notin MB(v)$. Let (A, y) be a justified objection to x, satisfying, w.l.o.g., $y(A) = v(A)$. Clearly $A \neq T$. Now, by a suitable translation of v, x, and y, we can make sure that

$$x(S) \geq 0 \quad \text{for all} \quad S \in \Sigma, \quad \text{and} \quad x(T \setminus A) > 0. \tag{3.5}$$

This is a weak version of (2.3), but it will allow us to adapt the proof of Theorem A to the current context. Let α, β be chosen as in the proof of Theorem A. For each $S \in \Sigma$ let

$$z(S) = \alpha y(S \cap A) + \beta x(S \cap (T \setminus A)).$$

Then $z(T) = v(T)$, and the same chain of inequalities as in the proof of Theorem A shows that $v(B) - z(B) \leq \varepsilon_v$ for every $B \in \Sigma \setminus \{\emptyset, T\}$. We show that $\sup_{S \in \Sigma \setminus \{\emptyset, T\}}(v(S) - z(S)) < \varepsilon_v$ and this will contradict the definition of ε_v. Assume, on the contrary, that $\sup_{S \in \Sigma \setminus \{\emptyset, T\}}(v(S) - z(S)) = \varepsilon_v$.

Then there is a sequence $\{S_n\}_{n=1}^{\infty}$ of coalitions $S_n \neq \emptyset, T$ such that $\lim_{n\to\infty}(v(S_n) - z(S_n)) = \varepsilon_v$. Therefore, as in the proof of Theorem A, $\lim_{n\to\infty} y(S_n \cap A) = y(A)$ and $\lim_{n\to\infty} x(S_n \cap (T \setminus A)) = x(T \setminus A)$, which implies that $\lim_{n\to\infty} z(S_n) = v(T)$. But then, by (3.4), $\varepsilon_v = \lim_{n\to\infty}(v(S_n) - z(S_n)) \leq \sup_{S \in \Sigma \setminus \{\emptyset, T\}} v(S) - v(T) < \varepsilon_v$, which is a contradiction. Q.E.D.

Remark 3.1. It is clear from the proof of Theorem A that the condition "$v(T) \geq v(S)$ for every non-empty $S \in \Sigma$" may be replaced by condition (3.4). Moreover, even if v does not satisfy (3.4), it suffices that v is strategically equivalent to a game which satisfies this condition. We note that any finite game is strategically equivalent to a game which satisfies this condition, and this explains why no condition was needed in the finite case.

Remark 3.2. Once we have an existence result for the Mas-Colell bargaining set, it is natural to ask whether the measure whose existence is asserted may be required to satisfy some additional desirable properties. We mention two facts that we have proved in this respect (omitting the precise definitions and the proofs). If we replace the assumption of Theorem B by the stronger, but quite standard, assumption that v is non-negative-valued and superadditive, then we can show that $MB(v)$ contains an imputation (rather than just a preimputation). If, in addition to the above, we assume that v is absolutely continuous with respect to some non-negative countably additive measure on (T, Σ), then $MB(v)$ contains a countably additive imputation.

Note. Shimomura (1997), in a paper that appeared after this work was submitted for publication, proved a result—his Theorem 1—that is related to our Theorem A. Using the terminology defined there, his theorem asserts that the individually rational quasicore is contained in the Mas-Colell bargaining set for every finite TU game satisfying grand coalition zero monotonicity. The individually rational quasicore differs only slightly from the least core: it is defined in essentially the same way, but with respect to imputations rather than preimputations. There is, however, an important difference in the definitions of the Mas-Colell bargaining set. Contrary to the standard definition that we use here, Shimomura requires that an objection be strictly beneficial to all the members of the objecting coalition (that is, $y_i > x_i$ for all $i \in A$). This may result in the bargaining set being larger under his definition than under ours. For example, let (N, v) be defined by $N = \{1, 2, 3\}$ and

$$v(S) = \begin{cases} 1 & \text{if } |S| = 3, \\ \frac{2}{3} & \text{if } |S| = 2, \\ 0 & \text{otherwise}. \end{cases}$$

Then the Mas-Colell bargaining set according to the usual definition consists of the single point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, whereas according to Shimomura's definition it is the union of three line segments joining this point to the points $(\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$, and $(0, \frac{1}{2}, \frac{1}{2})$, respectively. This difference renders Shimomura's result weaker than ours.

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