TWO AND A HALF REMARKS ON THE MARICA–SCHÖNHEIM INEQUALITY

RON AHARONI AND RON HOLZMAN

ABSTRACT

The Marica–Schönheim inequality states that the number of distinct differences of the form $A \backslash B$, with A, B taken from a given finite family $\mathscr{A}$ of sets is at least $|\mathscr{A}|$. We prove that equality occurs essentially if and only if $\mathscr{A}$ is the product of an ideal and a filter. We also prove an infinite version of the theorem, conjectured (in weaker form) by Daykin and Lovász. Finally, we note that a generalization (due to Ahlswede and Daykin) of the inequality which considers two families $\mathscr{A}$ and $\mathscr{B}$ holds under a weaker assumption on the relation between $\mathscr{A}$ and $\mathscr{B}$.

0. *The Marica–Schönheim inequality*

Given two families of sets $\mathscr{A}$ and $\mathscr{B}$, we write

$$\mathscr{A} \sim \mathscr{B} = \{A \backslash B : A \in \mathscr{A}, B \in \mathscr{B}\}.$$

(We should remark that by a 'family of sets' we really mean a set of sets, that is, repetitions are not allowed.) Motivated by a problem of Graham in combinatorial number theory, Marica and Schönheim proved the following inequality.

THEOREM 0.1 [5]. *For every finite family $\mathscr{A}$ we have that $|\mathscr{A} \sim \mathscr{A}| \geqslant |\mathscr{A}|$.*

This simple inequality turned out to be closely related to several more sophisticated correlation inequalities in combinatorics, all subsumed by the Four Functions Theorem of Ahlswede and Daykin [1]. In this paper we make a few separate observations connected to the Marica–Schönheim inequality. The background for each of them will be presented in the respective section of the paper. However, all our remarks are based on a proof of the inequality which was given by Daykin and Lovász [4]. For the reader's convenience, and in order to establish notation, we produce that proof here.

Let V be the ground set of the family $\mathscr{A}$, that is, $A \subseteq V$ for all $A \in \mathscr{A}$. For $x \in V$ we write

$$\mathscr{A}_x = \{A \in \mathscr{A} : x \in A\}.$$

Given $x \in V$, we consider two families of subsets of $V \backslash \{x\}$ derived from $\mathscr{A}$ as follows:

$$\mathscr{A}_x^+ = \{A \backslash \{x\} : A \in \mathscr{A}_x\},$$
$$\mathscr{A}_x^- = \mathscr{A} \backslash \mathscr{A}_x.$$

Once x is fixed, we shall omit the subscript and write simply $\mathscr{A}^+$, $\mathscr{A}^-$. We also form the following two families:

$$\overline{\mathscr{A}} = \mathscr{A}^+ \cup \mathscr{A}^-, \quad \underline{\mathscr{A}} = \mathscr{A}^+ \cap \mathscr{A}^-.$$

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For any $B \subseteq V\backslash\{x\}$, we write $\hat{B} = B \cup \{x\}$. Thus, $\overline{\mathscr{A}}$ consists of those $B \subseteq V\backslash\{x\}$ such that either B or $\hat{B}$ is a member of $\mathscr{A}$, while $\underline{\mathscr{A}}$ consists of those $B \subseteq V\backslash\{x\}$ such that both B and $\hat{B}$ are in $\mathscr{A}$.

The proof of Theorem 0.1 proceeds by induction on $n = |V|$. (We may assume that V is finite, because $\mathscr{A}$ is finite and we may identify points in the ground set which are undistinguished by $\mathscr{A}$.) If $n \geqslant 1$, fix some $x \in V$ and apply the induction hypothesis to each of the families $\overline{\mathscr{A}}$ and $\underline{\mathscr{A}}$, to obtain:

$$|\mathscr{A}| = |\overline{\mathscr{A}}| + |\underline{\mathscr{A}}| \leqslant |\overline{\mathscr{A}} \sim \overline{\mathscr{A}}| + |\underline{\mathscr{A}} \sim \underline{\mathscr{A}}|. \tag{0.1}$$

Note that $\underline{\mathscr{A}} \sim \underline{\mathscr{A}} \subseteq \overline{\mathscr{A}} \sim \overline{\mathscr{A}}$, and so the right hand side of (0.1) counts the number of sets in $\overline{\mathscr{A}} \sim \overline{\mathscr{A}}$, with each set in $\underline{\mathscr{A}} \sim \underline{\mathscr{A}}$ counted twice. Suppose that $D = B_1\backslash B_2 \in \overline{\mathscr{A}} \sim \overline{\mathscr{A}}$. Then either B_1 or $\hat{B}_1$ is a member of $\mathscr{A}$ and either B_2 or $\hat{B}_2$ is a member of $\mathscr{A}$. Therefore either D or $\hat{D}$ is a member of $\mathscr{A} \sim \mathscr{A}$. Moreover, if $D \in \underline{\mathscr{A}} \sim \underline{\mathscr{A}}$ then B_1, $\hat{B}_1$ and B_2 are all members of $\mathscr{A}$ and so both of D and $\hat{D}$ are members of $\mathscr{A} \sim \mathscr{A}$. It follows that

$$|\overline{\mathscr{A}} \sim \overline{\mathscr{A}}| + |\underline{\mathscr{A}} \sim \underline{\mathscr{A}}| \leqslant |\mathscr{A} \sim \mathscr{A}|. \tag{0.2}$$

Combining (0.1) and (0.2) yields the proof.

1. *Characterizing equality*

Which families $\mathscr{A}$ satisfy $|\mathscr{A} \sim \mathscr{A}| = |\mathscr{A}|$? One class of examples is when $\mathscr{A}$ is an ideal (that is, $\mathscr{A}$ is closed downwards, meaning that $A \in \mathscr{A}$, $B \subset A$ imply that $B \in \mathscr{A}$). In this case $\mathscr{A} \sim \mathscr{A} = \mathscr{A}$.

Denoting $\{V\backslash A : A \in \mathscr{A}\}$ by $\mathscr{A}^c$, we observe that $\mathscr{A}^c \sim \mathscr{A}^c = \mathscr{A} \sim \mathscr{A}$. Therefore, another class of examples is when $\mathscr{A}$ is a filter (that is, closed upwards). In this case $\mathscr{A} \sim \mathscr{A} = \mathscr{A}^c$.

Given two families $\mathscr{B}$ and $\mathscr{C}$ on disjoint ground sets, we define their product as

$$\mathscr{B} \times \mathscr{C} = \{B \cup C : B \in \mathscr{B}, C \in \mathscr{C}\}.$$

We observe that if $\mathscr{A} = \mathscr{B} \times \mathscr{C}$ then $\mathscr{A} \sim \mathscr{A} = (\mathscr{B} \sim \mathscr{B}) \times (\mathscr{C} \sim \mathscr{C})$. This gives rise to a new class of examples of equality, namely when $\mathscr{A}$ is a product of an ideal and a filter on disjoint ground sets. The previous examples are special cases, where one of the ground sets is empty. Note that the product of two ideals (filters) is an ideal (filter), so we do not obtain new examples by iterating this operation. Also, this class is closed under complementation $\mathscr{A} \mapsto \mathscr{A}^c$.

We shall prove that this class essentially exhausts all examples of equality. By 'essentially' we mean that any other example (such as $\mathscr{A} = \{\varnothing, V\}$) can be reduced to one in this class by identifying points in the ground set which are undistinguished by $\mathscr{A}$. The formal definition and statement of our result follow.

DEFINITION 1.1. We say that $\mathscr{A}$ is *weakly separating* if $\mathscr{A}_x = \mathscr{A}_y$ (where $x, y \in V$, $x \neq y$) implies that either $\mathscr{A}_x = \mathscr{A}$ or $\mathscr{A}_x = \varnothing$.

THEOREM 1.2. *Let $\mathscr{A}$ be a weakly separating family of subsets of a finite set V. Then $|\mathscr{A} \sim \mathscr{A}| = |\mathscr{A}|$ if and only if there exist a (possibly trivial) partition of V into sets S and T, an ideal $\mathscr{I}$ on S and a filter $\mathscr{F}$ on T, so that $\mathscr{A} = \mathscr{I} \times \mathscr{F}$.*

Proof. The 'if' direction has been shown above. We prove the 'only if' part by induction on $n = |V|$. For definiteness, assume that $V = [n] = \{1, \ldots, n\}$ and that the arguments in the inductive step of the proof of Theorem 0.1 are performed with respect to the element n of the ground set (that is, $\mathscr{A}^+ = \mathscr{A}^+_n$, $\mathscr{A}^- = \mathscr{A}^-_n$). The equality $|\mathscr{A} \sim \mathscr{A}| = |\mathscr{A}|$ requires that equality hold in both (0.1) and (0.2). The first of these requires that $|\overline{\mathscr{A}} \sim \overline{\mathscr{A}}| = |\overline{\mathscr{A}}|$ and $|\underline{\mathscr{A}} \sim \underline{\mathscr{A}}| = |\underline{\mathscr{A}}|$; the second requires that for no $D \in (\overline{\mathscr{A}} \sim \overline{\mathscr{A}}) \setminus (\underline{\mathscr{A}} \sim \underline{\mathscr{A}})$ do both D and $\hat{D}$ belong to $\mathscr{A} \sim \mathscr{A}$. The latter entails in particular that:

$$\mathscr{A}^+ \sim \underline{\mathscr{A}} \subseteq \underline{\mathscr{A}} \sim \underline{\mathscr{A}}, \tag{1.1}$$

$$\underline{\mathscr{A}} \sim \mathscr{A}^- \subseteq \underline{\mathscr{A}} \sim \underline{\mathscr{A}}. \tag{1.2}$$

It follows, since $\mathscr{A}$ is weakly separating, that $\overline{\mathscr{A}}$ is also weakly separating.

Assertion 1.2a. *We also have that $\underline{\mathscr{A}}$ is weakly separating.*

Proof. Suppose that x, y witness the contrary. Then $\underline{\mathscr{A}}$ has a set containing x and y, and a set omitting them. This and (1.1) imply that if $\mathscr{A}^+$ has a set containing one of x, y and not the other, then so has $\underline{\mathscr{A}} \sim \underline{\mathscr{A}}$, contradicting the choice of x, y. Similarly, $\mathscr{A}^-$ does not separate x and y. Thus $x \in B$ if and only if $y \in B$ for all $B \in \overline{\mathscr{A}}$. Since $\overline{\mathscr{A}}$ is weakly separating, this means that either $\overline{\mathscr{A}}_x = \overline{\mathscr{A}}$ or $\overline{\mathscr{A}}_x = \varnothing$. Hence, as $\underline{\mathscr{A}} \subseteq \overline{\mathscr{A}}$, we have that $\underline{\mathscr{A}}_x = \underline{\mathscr{A}}$ or $\underline{\mathscr{A}}_x = \varnothing$, a contradiction.

Since $\overline{\mathscr{A}}$ and $\underline{\mathscr{A}}$ are weakly separating and using the induction hypothesis, we have that $\overline{\mathscr{A}}$ and $\underline{\mathscr{A}}$ are products of ideals and filters. Thus $\overline{\mathscr{A}} = \overline{\mathscr{I}} \times \overline{\mathscr{F}}$ where $\overline{\mathscr{I}}$ is an ideal on $\overline{S}$, $\overline{\mathscr{F}}$ a filter on $\overline{T}$, and $(\overline{S}, \overline{T})$ is a partition of $[n-1]$; also $\underline{\mathscr{A}} = \underline{\mathscr{I}} \times \underline{\mathscr{F}}$ where $\underline{\mathscr{I}}$ is an ideal on $\underline{S}$, $\underline{\mathscr{F}}$ a filter on $\underline{T}$, and $(\underline{S}, \underline{T})$ is a partition of $[n-1]$.

Assume first that $\underline{\mathscr{A}} \neq \varnothing$. Then $\underline{T} \in \underline{\mathscr{A}}$. By (1.1) and the fact that $\underline{\mathscr{I}} \sim \underline{\mathscr{I}} = \underline{\mathscr{I}}$, this implies that

$$A \cap \underline{S} \in \underline{\mathscr{I}} \quad \text{for all } A \in \mathscr{A}^+. \tag{1.3}$$

Similarly, by (1.2) and the fact that $\underline{\mathscr{F}} \sim \underline{\mathscr{F}} = \{\underline{T} \setminus B : B \in \underline{\mathscr{F}}\}$, it follows that

$$A \cap \underline{T} \in \underline{\mathscr{F}} \quad \text{for all } A \in \mathscr{A}^-. \tag{1.4}$$

Assertion 1.2b. *Either $\mathscr{A}^+ \subseteq \mathscr{A}^-$ or $\mathscr{A}^- \subseteq \mathscr{A}^+$.*

Proof. Suppose for contradiction that there exist $A_1 \in \mathscr{A}^+ \setminus \mathscr{A}^-$ and $A_2 \in \mathscr{A}^- \setminus \mathscr{A}^+$. Let

$$A = (A_1 \cap \overline{T} \cap \underline{T}) \cup (A_2 \cap \overline{S} \cap \underline{S}) \cup (\underline{S} \cap \overline{T}).$$

Then $A \cap \overline{S} = A_2 \cap \overline{S} \cap \underline{S} \subseteq A_2 \cap \overline{S}$. But $A_2 \in \mathscr{A}^- \subseteq \overline{\mathscr{A}}$, hence $A_2 \cap \overline{S} \in \overline{\mathscr{I}}$, and thus $A \cap \overline{S} \in \overline{\mathscr{I}}$. On the other hand, $A \cap \overline{T} = (A_1 \cap \overline{T} \cap \underline{T}) \cup (\underline{S} \cap \overline{T}) \supseteq A_1 \cap \overline{T}$. Since $A_1 \in \mathscr{A}^+ \subseteq \overline{\mathscr{A}}$ we have that $A_1 \cap \overline{T} \in \overline{\mathscr{F}}$, and hence $A \cap \overline{T} \in \overline{\mathscr{F}}$. We have shown, in fact, that $A \in \overline{\mathscr{A}}$. By (1.3) and (1.4) this implies that either $A \cap \underline{S} \in \underline{\mathscr{I}}$ (if $A \in \mathscr{A}^+$) or $A \cap \underline{T} \in \underline{\mathscr{F}}$ (if $A \in \mathscr{A}^-$).

Assume the first possibility, that is, assume that $A \cap \underline{S} \in \underline{\mathscr{I}}$. Since

$$A \cap \underline{S} = (A_2 \cap \overline{S} \cap \underline{S}) \cup (\underline{S} \cap \overline{T}) \supseteq A_2 \cap \underline{S},$$

this would imply that $A_2 \cap \underline{S} \in \underline{\mathscr{I}}$. But by (1.4) $A_2 \cap \underline{T} \in \underline{\mathscr{F}}$, which, together with the above implies that $A_2 \in \underline{\mathscr{A}}$, a contradiction.

Similarly, if $A \cap \underline{T} \in \underline{\mathscr{F}}$ then, since $A \cap \underline{T} = A_1 \cap \overline{T} \cap \underline{T} \subseteq A_1 \cap \underline{T}$, it would follow that $A_1 \cap \underline{T} \in \underline{\mathscr{F}}$. Together with (1.3) this would yield $A_1 \in \underline{\mathscr{A}}$, a contradiction again.

By passing, if necessary, to $\mathscr{A}^c$, we may choose which of the two cases, $\mathscr{A}^+ \subseteq \mathscr{A}^-$ or $\mathscr{A}^- \subseteq \mathscr{A}^+$, we wish to consider. So, assume that $\underline{\mathscr{A}} = \mathscr{A}^- \subseteq \mathscr{A}^+ = \overline{\mathscr{A}}$. Let

$$\begin{aligned} S^* &= \overline{S} \cap \underline{S}, \\ T^* &= \overline{T} \cup \underline{T} \cup \{n\}, \\ \mathscr{I}^* &= \{B \subseteq S^* : B \in \underline{\mathscr{I}}\}, \\ \mathscr{F}^* &= \{C \subseteq T^* : C \cap \underline{T} \in \underline{\mathscr{F}} \text{ or } (n \in C \text{ and } C \cap \overline{T} \in \overline{\mathscr{F}})\}, \\ \mathscr{A}^* &= \mathscr{I}^* \times \mathscr{F}^*. \end{aligned}$$

We shall show that $\mathscr{A} = \mathscr{A}^*$. Suppose that $A \in \mathscr{A}$. Then $A \setminus \{n\} \in \overline{\mathscr{A}} = \mathscr{A}^+$, and so by (1.3) $A \cap \underline{S} \in \underline{\mathscr{I}}$. Now, $A \cap S^* \subseteq A \cap \underline{S}$, and therefore $A \cap S^*$ is also in $\underline{\mathscr{I}}$, and hence in $\mathscr{I}^*$. If $n \in A$ then $n \in A \cap T^*$, and since $A \setminus \{n\} \in \overline{\mathscr{A}}$, we have $A \cap \overline{T} \in \overline{\mathscr{F}}$. If $n \notin A$ then, since $A \in \mathscr{A}^-$, we have $A \cap \underline{T} \in \underline{\mathscr{F}}$ (by (1.4) or since $\mathscr{A}^- = \underline{\mathscr{A}}$). In both cases, the definition of $\mathscr{F}^*$ implies that $A \cap T^* \in \mathscr{F}^*$. Thus $A \in \mathscr{A}^*$.

Assume next that $A \in \mathscr{A}^*$. By the definition of $\mathscr{F}^*$ one of the two following cases holds.

Case 1: $A \cap \underline{T} \in \underline{\mathscr{F}}$. To prove $A \in \mathscr{A}$ we need to show in this case that $A \cap \underline{S} \in \underline{\mathscr{I}}$. Since $A \in \mathscr{A}^*$, we have $A \cap \overline{S} \cap \underline{S} \in \underline{\mathscr{I}}$. Hence $(A \cap \overline{S} \cap \underline{S}) \cup \underline{T} \in \underline{\mathscr{A}} \subseteq \overline{\mathscr{A}}$. Adding elements from $\overline{T}$ to a member of $\overline{\mathscr{A}}$ keeps it in $\overline{\mathscr{A}}$. Hence also $(A \cap \underline{S}) \cup \underline{T} \in \overline{\mathscr{A}} = \mathscr{A}^+$. By (1.3) this implies that $A \cap \underline{S} \in \underline{\mathscr{I}}$.

Case 2: *$n \in A$ and $A \cap \overline{T} \in \overline{\mathscr{F}}$.* In this case we have to show that $A \cap \overline{S} \in \overline{\mathscr{I}}$. As noted above, $(A \cap \overline{S} \cap \underline{S}) \cup \underline{T} \in \overline{\mathscr{A}}$, and hence $(A \cap \overline{S} \cap \underline{S}) \cup (\underline{T} \cap \overline{S}) \in \overline{\mathscr{I}}$. But the last set contains $A \cap \overline{S}$, and hence $A \cap \overline{S} \in \overline{\mathscr{I}}$.

We turn now our attention to the remaining case when $\underline{\mathscr{A}} = \varnothing$. Recalling that $\overline{\mathscr{A}} = \overline{\mathscr{I}} \times \overline{\mathscr{F}}$ and $\overline{\mathscr{A}} = \mathscr{A}^+ \cup \mathscr{A}^-$, we see that if $\mathscr{A}^+$ or $\mathscr{A}^-$ is empty then $\mathscr{A}$ is a product of an ideal and a filter and we are done. We shall assume that $\mathscr{A}^+ \neq \varnothing$, $\mathscr{A}^- \neq \varnothing$ and derive a contradiction.

Since $\underline{\mathscr{A}} \sim \underline{\mathscr{A}} = \varnothing$, equality in (0.2) requires that for no $D \in \overline{\mathscr{A}} \sim \overline{\mathscr{A}}$ do both D and $\hat{D}$ belong to $\mathscr{A} \sim \mathscr{A}$. Thus,

$$[(\mathscr{A}^+ \sim \mathscr{A}^+) \cup (\mathscr{A}^- \sim \mathscr{A}^-)] \cap (\mathscr{A}^+ \sim \mathscr{A}^-) = \varnothing. \tag{1.5}$$

ASSERTION 1.2c. *We have that $\mathscr{A}^-$ is closed downwards relative to $\overline{\mathscr{A}}$, that is, if $A \in \mathscr{A}^-$, $B \subset A$ and $B \in \overline{\mathscr{A}}$ then $B \in \mathscr{A}^-$.*

Proof. Otherwise, we would have $\varnothing = B \setminus A \in \mathscr{A}^+ \sim \mathscr{A}^-$, but also

$$\varnothing = A \setminus A \in \mathscr{A}^- \sim \mathscr{A}^-,$$

contradicting (1.5).

Clearly $\overline{T} \in \overline{\mathscr{A}}$. Either $\overline{T} \in \mathscr{A}^+$ or $\overline{T} \in \mathscr{A}^-$, and by considering $\mathscr{A}^c$ if necessary, we may choose which of the two cases we wish to handle. So, assume $\overline{T} \in \mathscr{A}^-$.

ASSERTION 1.2d. *For $B \in \overline{\mathscr{A}}$, membership of B in $\mathscr{A}^+$ or $\mathscr{A}^-$ depends only on $B \cap \overline{S}$, that is, $B \in \mathscr{A}^-$ if and only if $B \cup \overline{T} \in \mathscr{A}^-$.*

Proof. By Assertion 1.2c, we need only rule out the possibility that $B \in \mathscr{A}^-$ but $B \cup \overline{T} \in \mathscr{A}^+$. In this case $\overline{T} \backslash B = (B \cup \overline{T}) \backslash B \in \mathscr{A}^+ \sim \mathscr{A}^-$, but since $\overline{T} \in \mathscr{A}^-$ we also have $\overline{T} \backslash B \in \mathscr{A}^- \sim \mathscr{A}^-$, contradicting (1.5).

By Assertion 1.2d, there is a (non-trivial) partition of $\overline{\mathscr{I}}$ into $\mathscr{I}^+$ and $\mathscr{I}^-$ so that $\mathscr{A}^+ = \mathscr{I}^+ \times \overline{\mathscr{F}}$ and $\mathscr{A}^- = \mathscr{I}^- \times \overline{\mathscr{F}}$. Then (1.5) induces

$$[(\mathscr{I}^+ \sim \mathscr{I}^+) \cup (\mathscr{I}^- \sim \mathscr{I}^-)] \cap (\mathscr{I}^+ \sim \mathscr{I}^-) = \varnothing. \tag{1.6}$$

By Assertion 1.2c, $\mathscr{I}^-$ is an ideal.

ASSERTION 1.2e. *The ideal $\mathscr{I}^-$ is closed under unions relative to $\overline{\mathscr{I}}$, that is, if $I_1, I_2 \in \mathscr{I}^-$ and $I_1 \cup I_2 \in \mathscr{I}$ then $I_1 \cup I_2 \in \mathscr{I}^-$.*

Proof. Suppose that $I_1 \cup I_2 \in \mathscr{I}^+$. Since $I_1 = (I_1 \cup I_2) \backslash (I_2 \backslash I_1)$ and $I_2 \backslash I_1 \subseteq I_2 \in \mathscr{I}^-$, we have $I_1 \in \mathscr{I}^+ \sim \mathscr{I}^-$. On the other hand $I_1 = I_1 \backslash \varnothing \in \mathscr{I}^- \sim \mathscr{I}^-$, contradicting (1.6).

ASSERTION 1.2f. *We have that $\mathscr{I}^+$ is intersecting, that is, if I_1, $I_2 \in \mathscr{I}^+$ then $I_1 \cap I_2 \neq \varnothing$.*

Proof. Suppose that $I_1 \cap I_2 = \varnothing$. Then $I_1 = I_1 \backslash I_2 \in \mathscr{I}^+ \sim \mathscr{I}^+$, but also $I_1 = I_1 \backslash \varnothing \in \mathscr{I}^+ \sim \mathscr{I}^-$, contradicting (1.6).

How many singletons belong to $\mathscr{I}^+$? By Assertion 1.2e, at least one (otherwise $\mathscr{I}^- = \overline{\mathscr{I}}$). By Assertion 1.2f, at most one. Let $\{m\}$ be the unique singleton in $\mathscr{I}^+$. By the above, a set $I \in \overline{\mathscr{I}}$ belongs to $\mathscr{I}^+$ or $\mathscr{I}^-$ according as $m \in I$ or $m \notin I$. It follows that a set in $\mathscr{A}$ contains m if and only if it contains n. This fact and our assumptions $\mathscr{A}^+ \neq \varnothing$, $\mathscr{A}^- \neq \varnothing$ yield a contradiction to $\mathscr{A}$ being weakly separating.

2. *An infinite version*

Let us call a function $\phi : \mathscr{A} \to \mathscr{A}$ *good* if $A \backslash \phi(A) \neq B \backslash \phi(B)$ for all $A \neq B$ in $\mathscr{A}$. We say that a family $\mathscr{A}$ is *good* if there exists a good bijection $\phi : \mathscr{A} \to \mathscr{A}$. By a sharper analysis of their proof for the Marica–Schönheim theorem, Daykin and Lovász proved the following strengthening of the theorem.

THEOREM 2.1 [4]. *Every finite family $\mathscr{A}$ is good. Moreover, if $|\mathscr{A}| \geqslant 2$ then there exists a good bijection such that $\phi(A) \neq A$ for all $A \in \mathscr{A}$.*

This bijective version of the Marica–Schönheim inequality is meaningful in the case of infinite $\mathscr{A}$ as well. Unfortunately, as noted by Daykin and Lovász, it fails in the infinite case. Indeed, if $\mathscr{A} = \{A_i : i < \omega\}$ is an ascending chain (that is, $A_i \subset A_{i+1}$), then there is not even a good injection $\phi : \mathscr{A} \to \mathscr{A}$. Daykin and Lovász conjectured however that every countable family of sets of finitely bounded size (that is, $|A| \leqslant k$ for all $A \in \mathscr{A}$, with $k < \infty$) has a good injection without fixed points (that is, $\phi(A) \neq A$ for all $A \in \mathscr{A}$). They remarked that they had proved this for $k = 2$. Here we confirm the conjecture for all $k < \infty$, as we prove the following.

THEOREM 2.2. *Let $\mathscr{A}$ be a family of sets so that, for some $k < \infty$, $|A| \leqslant k$ for all $A \in \mathscr{A}$. Then $\mathscr{A}$ is good.*

Our result generalizes the Daykin–Lovász conjecture in allowing uncountable $\mathscr{A}$ as well, and strengthens it by insisting on a bijection rather than an injection. The requirement that the bijection have no fixed points can be added to our result, but we shall not bother with it.

For the proof of Theorem 2.2 we shall need some notation. Let $\mathscr{B}$ be a family of subsets of V. A *matching* in $\mathscr{B}$ is a subfamily of $\mathscr{B}$ consisting of non-empty mutually disjoint sets. We set

$$\nu(\mathscr{B}) = \sup\{|\mathscr{M}| : \mathscr{M} \text{ is a matching in } \mathscr{B}\}.$$

A *cover* of $\mathscr{B}$ is a subset of V meeting all non-empty members of $\mathscr{B}$. We set

$$\tau(\mathscr{B}) = \min\{|C| : C \text{ is a cover of } \mathscr{B}\}.$$

Clearly $\nu(\mathscr{B}) \leqslant \tau(\mathscr{B}) \leqslant |\mathscr{B}|$.

LEMMA 2.3. *Suppose that $\mathscr{B}$ is a family of finite sets and $\tau(\mathscr{B})$ is infinite. Then there exists a matching $\mathscr{M}$ in $\mathscr{B}$ such that $|\mathscr{M}| = \tau(\mathscr{B})$ and $M = \bigcup \mathscr{M}$ is a cover of $\mathscr{B}$. In particular, $\nu(\mathscr{B}) = \tau(\mathscr{B})$ and the supremum defining $\nu(\mathscr{B})$ is actually a maximum.*

Proof. Let $\mathscr{M}$ be a matching in $\mathscr{B}$ which is not contained in any other matching (it exists, for example, by Zorn's lemma). Then $M = \bigcup \mathscr{M}$ is a cover of $\mathscr{B}$. If $|\mathscr{M}| < \tau(\mathscr{B})$ then, since $\tau(\mathscr{B})$ is infinite and the sets in $\mathscr{M}$ are finite, also $|M| < \tau(\mathscr{B})$, contradicting the definition of $\tau(\mathscr{B})$. On the other hand, $|\mathscr{M}| > \tau(\mathscr{B})$ would contradict $\nu(\mathscr{B}) \leqslant \tau(\mathscr{B})$. Hence $|\mathscr{M}| = \tau(\mathscr{B})$ and the other statements follow.

LEMMA 2.4. *Let κ be an infinite cardinal, and let $\mathscr{B}$ and $\mathscr{C}$ be families of sets of size less than κ satisfying $|\mathscr{B}| = |\mathscr{C}| = \tau(\mathscr{B}) = \tau(\mathscr{C}) = \kappa$. Then there exists a bijection $\mu : \mathscr{B} \to \mathscr{C}$ such that $B \cap \mu(B) = \varnothing$ for all $B \in \mathscr{B}$.*

Proof. Let $\mathscr{B} = \{B_i : i < \kappa\}$, $\mathscr{C} = \{C_j : j < \kappa\}$. We define μ inductively. At each even stage $\alpha < \kappa$ we find the first B_i which is still unmatched, then find the first unmatched C_j disjoint from B_i (such a C_j exists because $\tau(\mathscr{C}) = \kappa$) and match them: $\mu(B_i) = C_j$. At each odd stage $\alpha < \kappa$ we find the first unmatched C_j, then the first unmatched B_i disjoint from C_j and match them: $\mu(B_i) = C_j$. After κ stages we have the required bijection.

The proof of Theorem 2.2 is by induction on θ, where $|\mathscr{A}| = \aleph_\theta$. We remark that our arguments in the countable case ($\theta = 0$) are substantially different from those required in the uncountable case.

Case I: $|\mathscr{A}| = \aleph_0$. We distinguish two possible subcases.

Case I.1: $\tau(\mathscr{A}) = \aleph_0$. In this case we apply Lemma 2.4 with $\mathscr{B} = \mathscr{C} = \mathscr{A}$ to obtain a good bijection.

Case I.2: $\tau(\mathscr{A}) < \aleph_0$. We proceed in this case by induction, with respect to the lexicographic order, on the pair of positive integers (k,t) associated with $\mathscr{A}$ as follows:

$$k = \rho(\mathscr{A}) = \max\{|A| : A \in \mathscr{A}\},$$
$$t = \tau(\mathscr{A}_{(k)}), \quad \text{where } \mathscr{A}_{(k)} = \{A \in \mathscr{A} : |A| = k\}.$$

Thus, we assume that every countable family $\mathscr{B}$ such that $\rho(\mathscr{B}) < k$ or $\rho(\mathscr{B}) = k$ and $\tau(\mathscr{B}_{(k)}) < t$ is good. The induction step is based on the following assertion, using the notation introduced in the proof of Theorem 0.1.

ASSERTION 2.2a. *If $\overline{\mathscr{A}}$ and $\underline{\mathscr{A}}$ are good then $\mathscr{A}$ is good.*

Before proving the assertion, we indicate how we apply it to obtain the inductive proof that $\mathscr{A}$ is good. Let T be a cover of $\mathscr{A}_{(k)}$ with $|T| = t$. We choose some $x \in T$ and form the families $\mathscr{A}^+ = \mathscr{A}^+_x$, $\mathscr{A}^- = \mathscr{A}^-_x$, $\overline{\mathscr{A}}$ and $\underline{\mathscr{A}}$. We have $\rho(\mathscr{A}^+) < k$. Since $\underline{\mathscr{A}} \subseteq \mathscr{A}^+$, this implies that $\underline{\mathscr{A}}$ is good. Also, $\overline{\mathscr{A}}_{(k)} = \mathscr{A}^-_{(k)}$, implying that $T\backslash\{x\}$ is a cover of $\overline{\mathscr{A}}_{(k)}$ and therefore $\tau(\overline{\mathscr{A}}_{(k)}) < t$. So the induction hypothesis tells us that $\overline{\mathscr{A}}$ is also good, and we can invoke Assertion 2.2a to deduce that $\mathscr{A}$ is good.

Assertion 2.2a is proved in [4] for the finite case, and the proof carries over to the infinite case with only minor modifications. For the sake of completeness, we sketch the argument. Let $\overline{\phi}$ be a good bijection for $\overline{\mathscr{A}}$ and $\underline{\phi}$ be a good bijection for $\underline{\mathscr{A}}$. We define a mapping $\overline{\pi} : \overline{\mathscr{A}} \to \mathscr{A}$ as follows: if $A \in \underline{\mathscr{A}}$ then $\overline{\pi}(A) = A$, and if $A \in \overline{\mathscr{A}}\backslash\underline{\mathscr{A}}$ then $\overline{\pi}(A)$ is the unique $B \in \mathscr{A}$ such that $B\backslash\{x\} = A$. We also define $\underline{\pi} : \underline{\mathscr{A}} \to \mathscr{A}$ by $\underline{\pi}(A) = \hat{A}$ (recall that $\hat{A} = A \cup \{x\}$). The pair of maps $\overline{\pi}$ and $\underline{\pi}$ serve to present $\mathscr{A}$ as the disjoint union of a copy of $\overline{\mathscr{A}}$ and a copy of $\underline{\mathscr{A}}$. This presentation permits us to define a bijection $\phi : \mathscr{A} \to \mathscr{A}$ as follows: for $A \in \overline{\mathscr{A}}$ let $\phi(\overline{\pi}(A)) = \overline{\pi}(\overline{\phi}(A))$ and for $A \in \underline{\mathscr{A}}$ let $\phi(\underline{\pi}(A)) = \underline{\pi}(\underline{\phi}(A))$. For each subfamily $\mathscr{B} \subseteq \underline{\mathscr{A}}$ let $\sigma_{\mathscr{B}} : \mathscr{A} \to \mathscr{A}$ be defined as follows: if $B \in \mathscr{B}$ then $\sigma_{\mathscr{B}}(B) = \hat{B}$ and $\sigma_{\mathscr{B}}(\hat{B}) = B$, and otherwise $\sigma_{\mathscr{B}}$ is the identity. Then $\phi_{\mathscr{B}} = \sigma_{\mathscr{B}} \circ \phi$ is a bijection for each $\mathscr{B} \subseteq \underline{\mathscr{A}}$. Let us call an ordered pair (A,B) with $A \in \overline{\mathscr{A}}$ and $B \in \underline{\mathscr{A}}$ *dangerous* if $A\backslash\overline{\phi}(A) = B\backslash\underline{\phi}(B)$. It can be seen that, for each $A \in \overline{\mathscr{A}}$, $\overline{\pi}(A)\backslash\phi_{\mathscr{B}}(\overline{\pi}(A))$ is either $A\backslash\overline{\phi}(A)$ or $\hat{A}\backslash\overline{\phi}(A)$. Similarly, for each $B \in \underline{\mathscr{A}}$, $\underline{\pi}(B)\backslash\phi_{\mathscr{B}}(\underline{\pi}(B))$ is either $B\backslash\underline{\phi}(B)$ or $\hat{B}\backslash\underline{\phi}(B)$. Therefore, since $\overline{\phi}$ and $\underline{\phi}$ are good, it follows that $\phi_{\mathscr{B}}$ is a good bijection if and only if for every dangerous pair (A,B) exactly one of the differences $\overline{\pi}(A)\backslash\phi_{\mathscr{B}}(\overline{\pi}(A))$ and $\underline{\pi}(B)\backslash\phi_{\mathscr{B}}(\underline{\pi}(B))$ contains x. Thus it suffices to show that there is a way to choose $\mathscr{B} \subseteq \underline{\mathscr{A}}$ so that this condition is satisfied.

Let us call a sequence $\ldots, C_{-1}, C_0, C_1, C_2, \ldots$ of sets in $\underline{\mathscr{A}}$ (which may terminate or not on either side) a *chain* if for every (C_i, C_{i+1}) in the sequence the pair $(\overline{\phi}^{-1}(C_i), \underline{\phi}^{-1}(C_{i+1}))$ is dangerous, and if the sequence terminates on either side then it cannot be extended on that side so as to maintain the above condition. Clearly, two distinct chains have no member in common. Considering each chain separately we decide which of its members to include in $\mathscr{B}$, as follows:

(i) If the chain does not terminate on the left, we include all its members in $\mathscr{B}$.

(ii) Suppose that the chain terminates on the left, say at C_0. Note that the chain has no repetitions (otherwise it would be periodic and hence infinite on both sides). We make our choice on whether or not $C_i \in \mathscr{B}$ inductively, starting at C_0. If there is a dangerous pair (A,B) such that $\underline{\phi}(B) = C_0$ (necessarily such a pair is unique and $\overline{\phi}(A) \notin \underline{\mathscr{A}}$), make the choice $C_0 \in \mathscr{B}$ or $C_0 \notin \mathscr{B}$ so as to render $\underline{\pi}(B)\backslash\phi_{\mathscr{B}}(\underline{\pi}(B))$ distinct from $\bar{\pi}(A)\backslash\phi_{\mathscr{B}}(\bar{\pi}(A))$; note that $\phi_{\mathscr{B}}(\bar{\pi}(A)) = \bar{\pi}(\bar{\phi}(A))$ is independent of $\mathscr{B}$ and thus known in advance. If no such dangerous pair exists, make the choice for C_0 arbitrarily. Having made the choice for C_i, if C_{i+1} exists and (A, B) is the dangerous pair so that $\bar{\phi}(A) = C_i$ and $\phi(B) = C_{i+1}$, make the choice for C_{i+1} so as to render $\underline{\pi}(B)\backslash\phi_{\mathscr{B}}(\underline{\pi}(B))$ distinct from $\bar{\pi}(A)\backslash\phi_{\mathscr{B}}(\bar{\pi}(A))$.

It is easy to verify that this choice of $\mathscr{B}$ satisfies the required condition.

Case II: $|\mathscr{A}| = \aleph_\theta$, $\theta > 0$. We shall construct a sequence $T_\alpha(\alpha \leqslant \theta)$ of subsets of $V = \bigcup\mathscr{A}$, satisfying the following conditions:

1. $T_\alpha \subseteq T_\beta$ for $\alpha < \beta$;
2. $T_\theta = V$;
3. $T_\zeta = \bigcup\{T_\alpha : \alpha < \zeta\}$ whenever $\zeta \leqslant \theta$ is a limit ordinal;
4. $|T_\alpha| = \aleph_\alpha$ $(\alpha \leqslant \theta)$.

For the fifth (and most important) condition, we need first some notation. Given a family of sets $\mathscr{B}$, a set T and a subset R of T, we set

$$h(\mathscr{B}, T, R) = \{B : B \cap T = \varnothing, B \cup R \in \mathscr{B}\}.$$

For $\alpha < \theta$ we set $\mathscr{A}^\alpha = \{A \in \mathscr{A} : A \subseteq T_{\alpha+1}$ but $A \nsubseteq T_\alpha\}$.

5. For all $\alpha < \theta$ and all $A \in \mathscr{A}^\alpha$, $\tau(h(\mathscr{A}^\alpha, T_\alpha, A \cap T_\alpha)) = \aleph_{\alpha+1}$.

Our construction of the sequence T_α $(\alpha \leqslant \theta)$ relies on the following notion of closure. We say that $T \subseteq V$ is *closed* if $A \in \mathscr{A}$, $A \nsubseteq T$ imply that

$$\tau(h(\mathscr{A}, T, A \cap T)) > |T|.$$

ASSERTION 2.2b. *Let S be a subset of V of cardinality $\kappa \geqslant \aleph_0$. Then there exists a closed set $T = \mathrm{cl}(S)$ such that $S \subseteq T$ and $|T| = \kappa$.*

Proof. We define an ascending chain S_i $(i < \omega)$ of subsets of V of cardinality κ as follows. Let $S_0 = S$. Assuming S_i has been defined, let

$$\mathscr{U}_i = \{R \subseteq S_i : \tau(h(\mathscr{A}, S_i, R)) \leqslant \kappa\},$$

and for each $R \in \mathscr{U}_i$ choose a cover C_R of $h(\mathscr{A}, S_i, R)$ of minimal size. Let $S_{i+1} = S_i \cup \bigcup\{C_R : R \in \mathscr{U}_i\}$. Finally, let $T = \bigcup\{S_i : i < \omega\}$. If $R \in \mathscr{U}_i$ and $|R| > k$ then $C_R = \varnothing$ and so it follows inductively that $|S_{i+1}| = \kappa$ for each $i < \omega$. Thus $|T| = \kappa$, and clearly $S \subseteq T$. To prove that T is closed, assume that $A \in \mathscr{A}$ and $A \nsubseteq T$. Let $R = A \cap T$. Since $|R| \leqslant k$ there exists $i < \omega$ such that $R \subseteq S_i$. Since $A\backslash R \in h(\mathscr{A}, S_i, R)$ and $S_{i+1} \cap (A\backslash R) = \varnothing$, it follows from the definition of S_{i+1} that $R \notin \mathscr{U}_i$ and thus, from the definition of $\mathscr{U}_i$, that $\tau(h(\mathscr{A}, S_i, R)) > \kappa$. Because $|T| = \kappa$ it follows also that $\tau(h(\mathscr{A}, T, R)) > \kappa$.

We define the sequence T_α $(\alpha \leqslant \theta)$ inductively. The first step is to choose an arbitrary subset S_0 of V of size $\aleph_0$, and let $T_0 = \mathrm{cl}(S_0)$.

Once a set T_α is defined, we define an auxiliary system of sets K_α^β $(\alpha \leqslant \beta < \theta)$; the subscript α indicates that the set was defined from T_α, the superscript β indicates that it will be used later in the definition of $T_{\beta+1}$. Let

$$\mathscr{W}_\alpha = \{R \subseteq T_\alpha : \tau(h(\mathscr{A}, T_\alpha, R)) > \aleph_\alpha\}.$$

For each $R \in \mathscr{W}_\alpha$, define $\zeta = \zeta(R) > \alpha$ by $\tau(h(\mathscr{A}, T_\alpha, R)) = \aleph_\zeta$. Applying Lemma 2.3 to the family $h(\mathscr{A}, T_\alpha, R)$, we find a matching $\mathscr{M}_R$ such that $|\mathscr{M}_R| = \aleph_\zeta$ and $M_R = \bigcup \mathscr{M}_R$ is a cover of $h(\mathscr{A}, T_\alpha, R)$. Since $\aleph_\zeta = \sum\{\aleph_{\beta+1} : \alpha \leqslant \beta < \zeta\}$, we can write $\mathscr{M}_R = \bigcup\{\mathscr{M}_R^\beta : \alpha \leqslant \beta < \zeta(R)\}$ with $|\mathscr{M}_R^\beta| = \aleph_{\beta+1}$. Letting $M_R^\beta = \bigcup \mathscr{M}_R^\beta$, we define for $\alpha \leqslant \beta < \theta$,

$$K_\alpha^\beta = \bigcup\{M_R^\beta : R \in \mathscr{W}_\alpha, \zeta(R) > \beta\}.$$

Since $|R| \leqslant k$ for each $R \in \mathscr{W}_\alpha$ it follows that $|\mathscr{W}_\alpha| \leqslant \aleph_\alpha$, and so $|K_\alpha^\beta| \leqslant \aleph_{\beta+1}$.

Assuming that $\alpha \leqslant \theta$ and the sets T_γ as well as the systems K_γ^β $(\gamma \leqslant \beta < \theta)$ have been defined for all $\gamma < \alpha$, we now define T_α. First, if α is a successor ordinal, say $\alpha = \beta + 1$, we choose a subset S_α of V of size $\aleph_\alpha$ which contains $T_\beta \cup \bigcup\{K_\gamma^\beta : \gamma \leqslant \beta\}$, and let $T_\alpha = \mathrm{cl}(S_\alpha)$. Next, if α is a limit ordinal, we let $T_\alpha = \bigcup\{T_\gamma : \gamma < \alpha\}$.

The sequence T_α $(\alpha \leqslant \theta)$ defined above is immediately seen to satisfy our conditions 1, 3 and 4. For the other conditions, we need the following.

Assertion 2.2c. *The set T_α is closed* $(\alpha \leqslant \theta)$.

Proof. The only case that requires attention is when α is a limit ordinal. Let $A \in \mathscr{A}$, $A \nsubseteq T_\alpha$. Let $R = A \cap T_\alpha$, and let $\gamma < \alpha$ be the first such that $R \subseteq T_\gamma$. Then γ is not a limit ordinal and so T_γ is closed. Hence $\tau(h(\mathscr{A}, T_\gamma, R)) = \aleph_\zeta$ for some $\zeta > \gamma$ and $R \in \mathscr{W}_\gamma$. Referring to the definition of the system K_γ^β, we see that a set $M_R = \bigcup\{M_R^\beta : \gamma \leqslant \beta < \zeta\}$ which covers $h(\mathscr{A}, T_\gamma, R)$ was introduced there. Since $M_R^\beta \subseteq K_\gamma^\beta \subseteq T_{\beta+1}$ whenever $\gamma \leqslant \beta < \zeta$ it follows that if we had $\zeta \leqslant \alpha$ then we would have $M_R \subseteq T_\alpha$ and so T_α would have to meet $A \backslash R$, which it does not. Hence $\zeta > \alpha$ and so $\tau(h(\mathscr{A}, T_\gamma, R)) > \aleph_\alpha$; thus, since $|T_\alpha| = \aleph_\alpha$, it follows that $\tau(h(\mathscr{A}, T_\alpha, R)) > \aleph_\alpha$.

By setting $\alpha = \theta$ in Assertion 2.2c, it follows that $A \subseteq T_\theta$ for all $A \in \mathscr{A}$, and so Condition 2 holds. To verify Condition 5, suppose that $\alpha < \theta$ and $A \in \mathscr{A}^\alpha$, and let $R = A \cap T_\alpha$. Since T_α is closed, $\tau(h(\mathscr{A}, T_\alpha, R)) > \aleph_\alpha$. Referring to the definition of the system K_α^β, we see that a matching of size $\aleph_{\alpha+1}$ in $h(\mathscr{A}, T_\alpha, R)$ (namely $\mathscr{M}_R^\alpha$) was thrown into K_α^α and therefore into $T_{\alpha+1}$. This implies that $\tau(h(\mathscr{A}^\alpha, T_\alpha, R)) \geqslant \aleph_{\alpha+1}$; since $|T_{\alpha+1}| = \aleph_{\alpha+1}$ we have equality.

Having constructed the sequence T_α $(\alpha \leqslant \theta)$ it is easy to define a good bijection ϕ. By Conditions 1 to 3,

$$\mathscr{A} = \mathscr{A}_0 \cup \bigcup\{\mathscr{A}^\alpha : \alpha < \theta\}, \quad \text{where } \mathscr{A}_0 = \{A \in \mathscr{A} : A \subseteq T_0\}$$

and the union is disjoint. We define ϕ on $\mathscr{A}_0$ to be a good bijection of $\mathscr{A}_0$ onto itself (which exists by Case I or the finite case). Next, for each $\alpha < \theta$, we define ϕ on $\mathscr{A}^\alpha$ as follows. Let $\mathscr{R}_\alpha = \{A \cap T_\alpha : A \in \mathscr{A}^\alpha\}$. By the induction hypothesis Theorem 2.2 holds for $\alpha < \theta$. Hence there exists a good bijection $\phi_\alpha : \mathscr{R}_\alpha \to \mathscr{R}_\alpha$. (We could use here also an inductive assumption on k, since $|R| < k$ for all $R \in \mathscr{R}_\alpha$.) By Condition 5, $\tau(h(\mathscr{A}^\alpha, T_\alpha, R)) = \aleph_{\alpha+1}$ for each $R \in \mathscr{R}_\alpha$. By Lemma 2.4 this implies that for each $R \in \mathscr{R}_\alpha$ there exists a bijection $\mu_R : h(\mathscr{A}^\alpha, T_\alpha, R) \to h(\mathscr{A}^\alpha, T_\alpha, \phi_\alpha(R))$ such that $B \cap \mu_R(B) = \varnothing$ for all $B \in h(\mathscr{A}^\alpha, T_\alpha, R)$. For each set $A \in \mathscr{A}^\alpha$ let $R = R(A) = A \cap T_\alpha$ and let $\phi(A) = \phi_\alpha(R) \cup \mu_R(A \backslash R)$. Then

$$A \backslash \phi(A) = (R \backslash \phi_\alpha(R)) \cup (A \backslash T_\alpha). \tag{2.1}$$

It is clear that $\phi \mid \mathscr{A}^\alpha$ is a bijection of $\mathscr{A}^\alpha$ onto itself. Hence ϕ is a bijection of $\mathscr{A}$ onto itself, and it remains to show that it is good.

If $A \in \mathscr{A}^\alpha$, $B \in \mathscr{A}^\beta$ and, say $\beta < \alpha$, then by (2.1) $B \setminus \phi(B) \subseteq T_{\beta+1} \subseteq T_\alpha$ while $A \setminus \phi(A) \supseteq A \setminus T_\alpha \neq \varnothing$, hence $A \setminus \phi(A) \neq B \setminus \phi(B)$. A similar argument holds if $A \in \mathscr{A}^\alpha$ and $B \in \mathscr{A}_0$. So, assume that $A, B \in \mathscr{A}^\alpha$ for some α, and $A \neq B$. Then, either $A \cap T_\alpha \neq B \cap T_\alpha$, in which case $(A \setminus \phi(A)) \cap T_\alpha \neq (B \setminus \phi(B)) \cap T_\alpha$, since ϕ_α is a good bijection; or $A \setminus T_\alpha \neq B \setminus T_\alpha$, in which case $(A \setminus \phi(A)) \setminus T_\alpha \neq (B \setminus \phi(B)) \setminus T_\alpha$, by (2.1).

We conclude this section with some remarks on other sufficient conditions for a family $\mathscr{A}$ to be good. Clearly, if $|\mathscr{A}| = \tau(\mathscr{A}) = \kappa$ and all sets in $\mathscr{A}$ have size less than κ then $\mathscr{A}$ is good (by an application of Lemma 2.4). In particular, if $\mathscr{A}$ is a family of finite sets and each element of the ground set belongs to finitely many sets in $\mathscr{A}$ then $\mathscr{A}$ is good. If $\mathscr{A}$ is uncountable then the latter conditions can be suitably weakened.

We do not know any example of a non-good family $\mathscr{A}$ of finite sets which does not contain an infinite ascending chain. We are inclined to guess that the absence of such chains is a sufficient condition for a family $\mathscr{A}$ of finite sets to be good.

3. *Differences formed by two families*

Instead of looking at the differences formed by a family $\mathscr{A}$ with itself, one may look at $\mathscr{A} \sim \mathscr{B}$ for two families $\mathscr{A}$ and $\mathscr{B}$. Now, $|\mathscr{A} \sim \mathscr{B}|$ may be very small even if $|\mathscr{A}|$ and $|\mathscr{B}|$ are large (for example, if $A \subseteq B$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$). Yet, Ahlswede and Daykin proved the following result which generalizes Theorem 0.1.

THEOREM 3.1 [2]. *Let $\mathscr{A}$ and $\mathscr{B}$ be finite families satisfying*

$$\forall A \in \mathscr{A} \quad \exists B \in \mathscr{B} \quad \text{such that } A \supseteq B. \tag{3.1}$$

Then $|\mathscr{A} \sim \mathscr{B}| \geqslant |\mathscr{A}|$.

Here we point out that a weaker condition than (3.1) already suffices.

THEOREM 3.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be finite families satisfying*

$$\forall A_1, A_2 \in \mathscr{A} \quad \exists B \in \mathscr{B} \quad \text{such that } (A_1 \setminus A_2) \cap B = \varnothing. \tag{3.2}$$

Then $|\mathscr{A} \sim \mathscr{B}| \geqslant |\mathscr{A}|$.

Proof. Our proof, like the one in [2], essentially imitates the proof of the Marica–Schönheim inequality described at the beginning of this paper. We state the argument briefly. If $\mathscr{A} = \varnothing$ then $\mathscr{A} \sim \mathscr{B} = \varnothing$. If $\mathscr{A} = \{\varnothing\}$ then $\mathscr{B} \neq \varnothing$ by (3.2) and so $\mathscr{A} \sim \mathscr{B} = \{\varnothing\}$. So the result holds if $V = \varnothing$ (where V is the ground set of $\mathscr{A}$). If $V \neq \varnothing$, we form the families $\mathscr{A}^+$, $\mathscr{A}^-$, $\overline{\mathscr{A}}$, $\underline{\mathscr{A}}$ and similarly $\mathscr{B}^+$, $\mathscr{B}^-$, $\overline{\mathscr{B}}$, all with respect to a fixed $x \in V$. It is easy to verify that each of the pairs $\overline{\mathscr{A}}, \overline{\mathscr{B}}$ and $\underline{\mathscr{A}}, \mathscr{B}^-$ satisfies (3.2). By the induction hypothesis and the counting argument preceding (0.2), we obtain

$$|\mathscr{A}| = |\overline{\mathscr{A}}| + |\underline{\mathscr{A}}| \leqslant |\overline{\mathscr{A}} \sim \overline{\mathscr{B}}| + |\underline{\mathscr{A}} \sim \mathscr{B}^-| \leqslant |\mathscr{A} \sim \mathscr{B}|.$$

Clearly, (3.1) implies (3.2), and (3.2) holds in many examples where (3.1) fails (for example, when $A \cap B = \varnothing$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$). Admittedly, (3.2) looks a bit awkward, but we should perhaps mention that we have been able to apply Theorem 3.2 in a natural way to obtain an alternative proof of a theorem of Chvátal [3]. We omit the details.

One could ask for conditions that guarantee $|\mathscr{A} \sim \mathscr{B}| \geqslant |\mathscr{B}|$. Such a condition can be obtained from (3.2), upon passing to complements ($|\mathscr{A} \sim \mathscr{B}| \geqslant |\mathscr{B}|$ is equivalent to $|\mathscr{B}^c \sim \mathscr{A}^c| \geqslant |\mathscr{B}^c|$).

We conclude with a version of Theorem 3.2 which extends it to infinite families as well.

THEOREM 3.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be (possibly infinite) families of sets satisfying* (3.2). *If $|A| < \infty$ for all $A \in \mathscr{A}$ then there exists a mapping $\phi: \mathscr{A} \to \mathscr{B}$ so that $A_1 \backslash \phi(A_1) \neq A_2 \backslash \phi(A_2)$ for all $A_1, A_2 \in \mathscr{A}$, $A_1 \neq A_2$.*

Proof. For each $A \in \mathscr{A}$, let $\mathscr{D}_A = \{A\} \sim \mathscr{B}$. Our task is to show that the family $\{\mathscr{D}_A\}$, indexed by $A \in \mathscr{A}$, has a system of distinct representatives. Since each $\mathscr{D}_A$ is finite ($|\mathscr{D}_A| \leqslant 2^{|A|}$), we need only check Hall's condition: for every finite subfamily $\mathscr{F} \subseteq \mathscr{A}$, $|\bigcup\{\mathscr{D}_A : A \in \mathscr{F}\}| \geqslant |\mathscr{F}|$. But $\bigcup\{\mathscr{D}_A : A \in \mathscr{F}\} = \mathscr{F} \sim \mathscr{B}$, so the inequality holds by Theorem 3.2.

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References

1. R. AHLSWEDE and D. E. DAYKIN, 'An inequality for the weights of two families of sets, their unions and intersections', *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 43 (1978) 183–185.
2. R. AHLSWEDE and D. E. DAYKIN, 'The number of values of combinatorial functions', *Bull. London Math. Soc.* 11 (1979) 49–51.
3. V. CHVÁTAL, 'Intersecting families of edges in hypergraphs having the hereditary property', *Hypergraph seminar*, Lecture Notes in Mathematics 411 (eds. C. Berge and D. Ray-Chaudhuri, Springer-Verlag, Berlin, 1974) 61–66.
4. D. E. DAYKIN and L. LOVÁSZ, 'The number of values of a Boolean function', *J. London Math. Soc.* (2) 12 (1976) 225–230.
5. J. MARICA and J. SCHÖNHEIM, 'Differences of sets and a problem of Graham', *Canad. Math. Bull.* 12 (1969) 635–637.

Department of Mathematics
Technion-Israel Institute of Technology
Haifa 32000
Israel

Department of Applied Mathematics
and Computer Science
The Weizmann Institute of Science
Rehovot 76100
Israel