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The core and the bargaining set in glove-market games*

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Abstract. In a glove-market game, the worth of a coalition is defined as the minimum, over all commodities in the market, of the total quantity that the coalition owns of each commodity. We identify a subclass of these games for which the core and the bargaining set coincide with the set of competitive equilibrium outcomes. We present examples showing that these solution concepts differ outside that subclass. We also illustrate a peculiar behavior of the bargaining set with respect to replication of a glove-market. These examples provide a simple economic setting in which the merits of the various solution concepts may be discussed and compared.

Key words: Cooperative games, Glove-markets, Core, Bargaining set, Competitive equilibrium.

1. Introduction

In a glove-market there are m commodities and n traders. Each trader i holds a quantity $b^i_j \geq 0$ of the j-th commodity, $i = 1, \dots, n,\ j = 1, \dots, m$. There is demand only for equal quantities of each commodity, and these are valued at unit price. In the associated cooperative game, the players are the traders, and the worth of a coalition S is $v(S) = \min\{\sum_{i \in S} b^i_j | j = 1, \dots, m\}$. A game that can be represented in this way is called a glove-market game.

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The name is derived from the classical example with two commodities, left-hand and right-hand gloves. The case of one trader owning a left-hand glove and two traders owning a right-hand glove each, is traditionally used as a classroom example to illustrate basic ideas in cooperative game theory. Shapley (1959) studied von Neumann-Morgenstern solutions of such games. Postlewaite and Rosenthal (1974) presented a glove-market game that displays paradoxical behavior of the core with respect to syndication. Maschler (1976) used the same example to show an advantage of the bargaining set over the core.

In economic terms, the glove-market model describes perfect complementarity in the consumption of the goods. Alternatively, if the commodities are thought of as inputs in the production of a consumer-good, this model corresponds to production requiring fixed proportions of the various inputs. As such, the model represents an extreme case of interaction between commodities.

Even though the underlying economic assumption of glove-markets is very special, it turns out that the class of games that can be represented in this way is large. In fact, Kalai and Zemel (1982a) showed that every totally balanced game is a glove-market game (without actually using the glove-market terminology). By recalling this and some other representation results, we point out in Section 2 the equivalence between the class of glove-market games and three other classes of games associated with economic models: market games, linear production games and flow games.

This equivalence may be used to infer information about one of these models from facts known about another. Indeed, in this paper we gain information on a class of glove-market games by considering their flow game representations. But each of the economic models is worth studying on its own, since the economic context suggests looking at natural subclasses of games which may not be natural for another model.

Here we investigate the core and the bargaining set in glove-market games, and compare them with the set of competitive equilibrium outcomes. In these markets, any market-clearing price vector assigns a zero price to every commodity of which there is an oversupply, and thus competitive equilibria offer no reward for possession of these commodities. In Section 3 we identify a class of glove-market games for which the core and the bargaining set (as well as some related solution concepts) coincide with the set of competitive equilibrium outcomes. In these games, which we call unitary glove-market games, each trader holds one unit of one commodity.

In Section 4 we present examples showing that these results do not hold for some natural extensions of the class of unitary glove-market games. The examples that exhibit a difference between the core and the bargaining set are particularly instructive. In both examples, the bargaining set treats more fairly the owners of commodities of which there is an oversupply. One of the examples is borrowed from Maschler (1976). The other is new, and sheds more light on the distinctions between the economic principles leading to the core and the bargaining set, as discussed in Maschler's analysis of his example. The computation of the bargaining set for this new example turned out to be a complicated task (we are not aware of any other game of comparable complexity for which the bargaining set was successfully computed). We present this computation in an appendix.

A central idea in the theory of markets is that in a proper sense the difference between the various solution concepts should disappear as the market becomes large. In Section 5 we examine the extent to which this is true for glove-markets and the solution concepts considered here. An interesting example shows a peculiar behavior of the bargaining set with respect to replication.

2. Preliminaries

We denote by N the set of *players* or *traders*; usually $N = \{1, \dots, n\}$. We refer to subsets of N as *coalitions*. The set of all coalitions is the power set $\mathscr{P}(N)$.

In this paper, a *game* is a pair (N, v) where $v : \mathscr{P}(N) \to \Re_+$ (the *characteristic function*) satisfies $v(\emptyset) = 0$. We think of $v(S)$ as the worth of the coalition S. The assumption that the worth is always nonnegative is not essential, but is convenient for our context. The game (N, v) is *superadditive* if for every two disjoint coalitions S and T we have $v(S \cup T) \geq v(S) + v(T)$. A *subgame* of (N, v) is a game of the form (S, v_S), where $\emptyset \neq S \subseteq N$ and $v_S(T) = v(T)$ for all $T \subseteq S$.

For a non-empty coalition S, we denote by $\Re^S$ the $|S|$-dimensional Euclidean space with coordinates indexed by the players in S. If $x \in \Re^S$ and $T \subseteq S$, we write $x(T)$ for $\sum_{i \in T} x_i$. The set of *imputations* in (N, v) is the set of efficient and individually rational payoff vectors, i.e., the set:

$$\mathscr{X} = \mathscr{X}(N, v) = \{x \in \Re^N | x(N) = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}$$

For an imputation x and a coalition S in a game (N, v), the *excess* of S at x is defined by:

$$e(S, x) = v(S) - x(S)$$

The *core* of (N, v) is the set:

$$\mathscr{C}(N, v) = \{x \in \mathscr{X} | e(S, x) \leq 0 \text{ for all } S \subseteq N\}$$

The game (N, v) is *totally balanced* if the core of every subgame of (N, v) is non-empty.

We proceed to present some economic models and the games associated with them. In a *pure exchange economy* there are m commodities. Each trader $i \in N$ is endowed with an initial bundle $b^i = (b^i_1, \dots, b^i_m) \in \Re^m_+$, and has a continuous concave utility function $u_i : \Re^m_+ \to \Re_+$. The characteristic function of the game associated with the economy is defined by:

$$v(S) = \max\left\{\sum_{i \in S} u_i(a^i) | \sum_{i \in S} a^i = \sum_{i \in S} b^i, a^i \in \Re^m_+ \text{ for all } i \in S\right\}$$

A game (N, v) that can be represented in this way is called a *market game*.

A game (N, v) that can be represented by a pure exchange economy in which the utility functions u_i are all equal to the function $u(a_1, \dots, a_m) = \min\{a_1, \dots, a_m\}$, is called a *glove-market game*. Such a game is fully determined by the initial bundles $b^i, i \in N$. We find it convenient to write this information in the form of an $m \times n$ matrix B, whose columns are $b^1, \dots, b^n$. We say that B is a glove-market representation of (N, v). The characteristic function may be rewritten in the simpler form:

$$v(S) = \min\left\{\sum_{i\in S} b_j^i | j = 1,\ldots,m\right\}$$

In a *linear production model* (Owen 1975) there are m inputs and p outputs. Each trader $i \in N$ is endowed with an input bundle $b^i = (b_1^i,\ldots,b_m^i) \in \Re_+^m$. The production technology is given by an $m \times p$ matrix A, with the interpretation that the production of an output bundle $y = (y_1,\ldots,y_p) \in \Re_+^p$ requires the input bundle Ay. The matrix A has nonnegative entries and no zero column. The output prices are given by a vector $c \in \Re^p$. The characteristic function of the game associated with this model is defined by:

$$v(S) = \max\left\{c^T y | Ay \le \sum_{i\in S} b^i, y \ge 0\right\}$$

A game (N,v) that can be represented in this way is called a *linear production game*.

A *network-flow model* (we give the original definition of Kalai and Zemel (1982a); several variants exist in the literature) is described by a finite directed graph $G = (V,E)$ with two distinguished nodes – a source and a sink. For every arc $e \in E$, its capacity $cap(e) \in \Re_+$ and its owner $own(e) \in N$ are specified. In the associated game, $v(S)$ is defined for $S \subseteq N$ to be the maximum flow-value from the source to the sink, subject to the capacity constraints and using only arcs owned by members of S. A game (N,v) that can be represented in this way is called a *flow game*.

The above-mentioned classes of games may appear to be different, but turn out to be identical.

Theorem 2.1. *Let (N,v) be a game. The following are equivalent*:

(*a*) *(N,v) is totally balanced.*
(*b*) *(N,v) is a market game.*
(*c*) *(N,v) is a glove-market game.*
(*d*) *(N,v) is a linear production game.*
(*e*) *(N,v) is a flow game.*

Proof: The equivalence of (a) and (b) was established by Shapley and Shubik (1969). The equivalence of (a), (c) and (e) was proved by Kalai and Zemel (1982a). Owen (1975) observed that (d) implies (b). Finally, (c) implies (d): starting from a glove-market representation B of (N,v), consider the linear production model in which the inputs are the commodities of B, there is one output, the input bundles are as in B, the production technology is given by the $m \times 1$ matrix of ones, and the output price is one. ■

Next, we make some observations about the core of a glove-market game (N,v). For a glove-market representation B of (N,v), we denote by $b_1,\ldots,b_m$ the rows of B, and define:

$$J = \left\{j | \sum_{i\in N} b_j^i \le \sum_{i\in N} b_{j'}^i \text{ for all } j' = 1,\ldots,m\right\}$$

Thus, J is the set of (indices of) commodities of which the total quantity in the market is minimal. Given the nature of the common utility function

$\min\{a_1, \ldots, a_m\}$, we can say that there is an oversupply of any commodity not in J. By the definition of the associated game, we have $b_j \in \mathscr{C}(N, v)$ for every $j \in J$. This makes it very easy to find a core element of a game, given its glove-market representation. Moreover, the convexity of the core implies the following fact (we denote by Conv(X) the convex hull of the set X).

Observation 2.2. *If B is a glove-market representation of (N, v) then*

$$\mathscr{C}(N, v) \supseteq \ Conv(\{b_j | j \in J\}).$$

In order to place this observation in context, we recall the concept of a competitive equilibrium outcome. Let (N, v) be a market game, represented by a pure exchange economy in which each trader $i \in N$ has an initial bundle $b^i \in \Re^m_+$ and a utility function $u_i : \Re^m_+ \to \Re_+$. An imputation x is a *competitive equilibrium outcome* if there exist a price vector $p = (p_1, \ldots, p_m) \in \Re^m_+$, $p \neq 0$, and bundles $a^i \in \Re^m_+$ for each $i \in N$, so that the following conditions are satisfied:

$$a^i \in \operatorname{argmax}\left\{ u_i(a) | a \in \Re^m_+, \sum_{j=1}^m p_j a_j \leq \sum_{j=1}^m p_j b^i_j \right\} \quad \text{for all } i \in N$$

$$\sum_{i \in N} a^i = \sum_{i \in N} b^i$$

$$x_i = u_i(a^i) \quad \text{for all } i \in N$$

In the case of glove-market games, the following characterization of competitive equilibrium outcomes is known (and easy to check).

Observation 2.3. *If B is a glove-market representation of (N, v) then the set of competitive equilibrium outcomes is $Conv(\{b_j | j \in J\})$.*

Thus, for the case of glove-markets Observation 2.2 expresses the well-known containment relation between the core and the set of competitive equilibrium outcomes.

We conclude this section with the definitions of two other solution concepts: the bargaining set (Aumann and Maschler 1964, Davis and Maschler 1967) and the kernel (Davis and Maschler 1965). Let (N, v) be a game, let x be an imputation, and let k and l be distinct players. An *objection* of k against l at x is a pair (C, y), where C is a coalition containing k but not l, and y is in $\Re^C$ and satisfies $y(C) = v(C)$ and $y_i > x_i$ for all $i \in C$. Let (C, y) be an objection of k against l at x. A *counter-objection* to this objection is a pair (D, z), where D is a coalition containing l but not k, and z is in $\Re^D$ and satisfies $z(D) = v(D),\ z_i \geq y_i$ for all $i \in D \cap C$ and $z_i \geq x_i$ for all $i \in D \setminus C$. An objection is *justified* if there is no counter-objection to it. The *bargaining set* of (N, v) is the set:

$$\mathscr{M}_1^{(i)}(N, v) = \{x \in \mathscr{X} | \text{ no player has a justified objection at } x \text{ against any other player}\}$$

Let (N, v) be a game, let x be an imputation, and let k and l be distinct players. The *surplus* of k against l at x is defined as:

$$s_{k,l}(x) = \max\{e(S,x)|S \subseteq N, k \in S, l \notin S\}$$

The *kernel* of (N, v) is the set:

$$\mathscr{K}(N,v) = \{x \in \mathscr{X} | \; s_{k,l}(x) \leq s_{l,k}(x) \text{ for any two distinct players } k \text{ and } l \text{ such that } x_l > v(\{l\})\}$$

It is well-known that the bargaining set contains both the core and the kernel.

3. Unitary glove-market games

We have seen that glove-market games are as general as arbitrary market games, and therefore we cannot expect to be able to say more about the behavior of solution concepts on this class than is generally known for market games. However, it is interesting to study the consequences of imposing additional conditions on the glove-market representation.

A *unitary* glove-market game is a game (N, v) that has a glove-market representation B in which each trader holds one unit of one commodity, i.e., every column of B contains one 1 and $m - 1$ 0's. In such a game, the set of players N may be partitioned into types $N_1, \ldots, N_m$ according to the commodity held, and the characteristic function assumes the form:

$$v(S) = \min\{|S \cap N_j| \; |j = 1, \ldots, m\}$$

We refer to such a representation as a type representation of (N, v).

Unitary glove-markets have the following divisibility property: If a coalition S owns the total initial bundle $\sum_{i \in S} b^i \in \Re^m_+$, then for every bundle $b \in \Re^m_+$ with integer coordinates satisfying $b \leq \sum_{i \in S} b^i$ there exists a subcoalition $T \subseteq S$ that owns exactly $\sum_{i \in T} b^i = b$. This property is reminiscent of properties of non-atomic games. It is well-known (see Section 5) that various solution concepts that differ on finite games become equivalent when applied to non-atomic games. This provides motivation for studying unitary glove-market games.

While an arbitrary glove-market game has a network-flow representation (see Theorem 2.1), a unitary glove-market game may be represented by a network-flow model with additional properties. A game (N, v) is called a *simple* flow game if it has a network-flow representation in which all capacities are 1 and each arc is owned by a different player.

Lemma 3.1. *Every unitary glove-market game is a simple flow game.*

Proof: Let (N, v) be a unitary glove-market game with type representation $N_1, \ldots, N_m$. We construct a directed graph G with node-set $V = \{v_0, v_1, \ldots, v_m\}$, taking v_0 as the source and v_m as the sink. For $j = 1, \ldots, m$, we draw $|N_j|$ parallel arcs from v_{j-1} to v_j, all with unit capacity and each owned by a different player in N_j. The game associated with this network-flow model is precisely (N, v). ■

We show now that for unitary glove-market games, the containment relation in Observation 2.2 is actually an equality. This is easy to prove directly, but we give an argument based on the reduction to simple flow games.

Theorem 3.2. *If B is a unitary glove-market representation of* (N, v) *then*

$$\mathscr{C}(N, v) = Conv(\{b_j | j \in J\}).$$

Proof: Kalai and Zemel (1982b) proved that the extreme points of the core of a simple flow game are the characteristic vectors of those coalitions whose arc-set forms a minimum cut in the network-flow representation. For the construction given in the proof of Lemma 3.1, these coalitions are the N_j's of minimum cardinality, and their characteristic vectors are b_j, $j \in J$. ∎

We show next that for unitary glove-market games, the containment relation between the bargaining set and the core is actually an equality. This result, too, can be proved directly (though not as easily as the previous one), but we invoke again an analogous result for simple flow games. We remark that the coincidence of the bargaining set and the core has been shown for several special classes of games, starting with convex games (Maschler et al. 1972). For a survey of such results and a unified approach to proving them, see Solymosi (1999).

Theorem 3.3. *Let* (N, v) *be a unitary glove-market game. Then*

$$\mathscr{M}_1^{(i)}(N, v) = \mathscr{C}(N, v).$$

Proof: By Lemma 3.1, it suffices to prove that the equality holds for simple flow games. This was proved by Reijnierse et al. (1996). We remark that their Theorem 7.9 has an additional assumption of superadditivity, which is needed because they consider a wider class of network-flow models allowing for public arcs. For the flow games considered here, superadditivity is automatic. ∎

We conclude this section with comments on some other solution concepts. A number of variants of the bargaining set have been introduced, which are known to contain the core and be contained in the classical bargaining set. We mention in particular the reactive bargaining set (Granot 1994) and the semi-reactive bargaining set (Sudhölter and Potters 2001). By Theorem 3.3, these solution concepts also coincide with the core for unitary glove-market games.

A notable exception among the variants is the Mas-Colell (1989) bargaining set, which *contains* the classical one for superadditive games and this containment is often strict (see Holzman 2001). Nevertheless, we have been able to show that for unitary glove-market games the Mas-Colell bargaining set, too, coincides with the core (we omit the details). This implies the same conclusion for the consistent bargaining set (Dutta et al. 1989). The variants obtained from the above-mentioned solution concepts by renouncing individual rationality (known as prebargaining sets) retain the property of being contained in the Mas-Colell bargaining set, and hence these variants, too, coincide with the core for unitary glove-market games.

As for the kernel, it can be shown (directly, or using a result of Granot and Granot 1992) that for unitary glove-market games with at most one type of size one the kernel contains the core, and since it is always contained in the bargaining set, it must coincide with both for such games.

Finally, we mention two recent papers in which the class of unitary glove-market games was studied with respect to other solution concepts. Rosenmüller and Shitovitz (2000) characterized the polyhedral von Neumann-Morgenstern solutions of unitary glove-market games. Rosenmüller and Sudhölter (2002) computed the modiclus for games in this class. Unlike all the above-mentioned solution concepts, these two reward also the traders of oversupply-types.

4. Examples

In the definition of unitary glove-market games we imposed two conditions: (a) a trader may hold a positive quantity of only one commodity (let us call this condition *singularity*), and (b) a trader may hold only one unit – or zero – of any commodity (let us call this condition *uniformity*). In this section we give examples showing that neither of these conditions by itself is sufficient for the results of Section 3 to hold.

Example 4.1. A singular glove-market game whose core is larger than the set of competitive equilibrium outcomes:

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

This example appears in Owen (1975). The unique competitive equilibrium outcome is $(1,0)$, whereas the core is the segment $[(1,0),(0,1)]$.

Example 4.2. A uniform glove-market game whose core is larger than the set of competitive equilibrium outcomes:

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Here the set of competitive equilibrium outcomes is the convex hull of the rows of B. The core is the set

$$\{x \in \Re^N_+ | x(N) = 2, x_k + x_l \leq 1 \text{ for } k = 1,2,3, l = 4,5,6\}$$

including, e.g., the imputation $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0)$ which is not a convex combination of rows of B.

Example 4.3. A singular glove-market game whose bargaining set is larger than the core:

$$B = \begin{pmatrix} 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

This example is due to Postlewaite and Rosenthal (1974), and its bargaining set was computed and compared to the core by Maschler (1976). The core consists of the unique imputation $(0,0,1,1,1)$, whereas the bargaining set is the segment:

$$\left[\left(\frac{3}{2},\frac{3}{2},0,0,0\right),(0,0,1,1,1)\right]$$

Granot and Maschler (1997) further analyzed this example and found that the reactive bargaining set and the kernel coincide with the bargaining set.

Example 4.4. A uniform glove-market game whose bargaining set is larger than the core:

$$B=\begin{pmatrix}1&1&1&1&0&0&0&0&0\\1&1&0&0&1&1&0&0&0\\0&0&1&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\end{pmatrix}$$

Here the core consists of the unique imputation $(0,0,0,0,0,0,1,1,1)$, whereas the bargaining set (as well as the kernel) is the segment:

$$\left[\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0\right),(0,0,0,0,0,0,1,1,1)\right]$$

The proofs of these facts are given in the appendix. We remark that the Mas-Colell bargaining set in this example is much larger than the classical one. We have not fully computed it, but we know it contains the imputation $(\frac{2}{5},\frac{3}{5},\frac{2}{5},\frac{3}{5},\frac{2}{5},\frac{3}{5},0,0,0)$ in its interior (relative to the hyperplane $x(N)=v(N)$).

Maschler (1976) argued that the bargaining set in Example 4.3 is more intuitive than the core. It is interesting to compare Example 4.4 to Maschler's example, and to see the extent to which his arguments apply to this example as well.

Maschler pointed out that while the core outcome $(0,0,1,1,1)$ in Example 4.3 is driven by the oversupply of the commodity held by traders 1 and 2, these players are not helpless. For one thing, each of these players realizes that without him, the rest of the players are worth less than with him, and therefore he is unlikely to settle for zero. In other words, while there is an oversupply of the first commodity, there is no oversupply of traders holding it.

We note that this distinction does not apply to Example 4.4. Not only is there an oversupply of the first three commodities, but each of the first six traders can leave the market without affecting the worth of the rest (though of course they do have positive marginal contributions to the worths of smaller coalitions). From this point of view, the core outcome giving zero to these players seems less implausible than in Maschler's example.

In order to understand the justification for the higher payoffs offered by the bargaining set to players $1,\dots,6$ in Example 4.4, one has to look more closely, as Maschler did for his example, into the options they have for countering threats made against them. Suppose that the imputation $(\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,1-2\alpha,1-2\alpha,1-2\alpha)$ is proposed, where $0<\alpha\le\frac{1}{2}$. A typical threat against player 1, in an attempt to convince him to yield some of his payoff, would be by player 7, say, who would threaten to form the coalition $\{2,\dots,9\}$ and share the extra α between its members, or perhaps to form a coalition such as $\{2,4,6,7,8\}$ and share the extra α between its members. Player 1 can counter this by pointing out that he can form the coalition $\{1,3,5,8,9\}$, which also has an extra α to share. This makes it possible to match any offers made by player 7 to those of his prospective partners whose consent is also required for player 1's plan.

5. Large glove-market games

A traditional approach to studying solution concepts for large market games is to consider a sequence of r-fold replications of a fixed economy, and analyze the asymptotic behavior as $r \to \infty$. Debreu and Scarf (1963) proved that for such a sequence the core may be viewed as a set of imputations in the original game (the "equal treatment property"), and as such it shrinks to the set of competitive equlibrium outcomes. This applies of course to glove-markets as well, and for these games it means that any difference between the core and $\mathrm{Conv}(\{b_j | j \in J\})$ – which may exist if the original game is not unitary – must vanish as $r \to \infty$. Owen (1975) showed that in the case of linear production games satisfying a certain non-degeneracy condition, the difference disappears already after a finite number of replications. When applied to glove-market games, the non-degeneracy condition becomes $|J| = 1$. Owen's result was extended in various ways by Rosenmüller (1982) and Samet and Zemel (1984).

As for the bargaining set, Shapley and Shubik (see Appendix B of Shubik 1984) proved that it approaches the core (in a sense which we do not make precise here) for sequences of replications of markets with smooth utility functions. Note, however, that the function $\min\{a_1, \ldots, a_m\}$ is *not* smooth. Indeed, Shapley (1992) gave an example of a sequence of replications of a glove-market in which the kernel of the associated game does not converge, and remarked that this can be used to show non-convergence of the bargaining set as well. We present here a very similar example, consisting of replications of the glove-market analyzed by Maschler (Example 4.3).

Example 5.1. A sequence of replications of a glove-market for which the bargaining set of the associated game does not converge:

$$B^{(r)} = \begin{pmatrix} \overbrace{2 \quad 2}^{2r} & \overbrace{0 \quad 0}^{3r} \\ \cdots & \cdots \\ 0 \quad 0 & 1 \quad 1 \end{pmatrix}$$

We denote by $N_1^{(r)}$ the set consisting of the first $2r$ players and by $N_2^{(r)}$ the set consisting of the remaining $3r$ players. In our analysis of solution concepts for the game associated with $B^{(r)}$ we consider only equal-treatment imputations, i.e., those that give the same payoff to players with equal resources; this has to be the case for imputations in the core or the kernel, but the bargaining set may conceivably contain other imputations as well. We refer to any such imputation as if it were an imputation in the original game, writing it in the form $x_\alpha = (\alpha, \alpha, 1 - \frac{2\alpha}{3}, 1 - \frac{2\alpha}{3}, 1 - \frac{2\alpha}{3})$, where $0 \le \alpha \le \frac{3}{2}$.

For all r, the core consists of the unique imputation $x_0 = (0, 0, 1, 1, 1)$. Now, consider any $0 < \alpha \le \frac{3}{2}$, any player k in $N_2^{(r)}$ and any player l in $N_1^{(r)}$. If r is odd, then the surpluses of k and l against each other at x_α are attained at coalitions containing $\frac{3r-1}{2}$ members of $N_1^{(r)}$ and $3r - 1$ members of $N_2^{(r)}$. This implies that $s_{k,l}(x_\alpha) = s_{l,k}(x_\alpha)$, and so x_α is in the kernel and, therefore, in the bargaining set. On the other hand, if r is even then k has a justified objection against l at x_α: take $C = C_1 \cup N_2^{(r)}$, where $C_1 \subseteq N_1^{(r)} \setminus \{l\}$ and $|C_1| = \frac{3r}{2}$, and define $y \in \Re^C$ by

$$y_i = \begin{cases} \alpha + \epsilon & \text{if } i \in C_1 \\ 1 - \frac{2\alpha}{3} + \epsilon & \text{if } i = k \\ 1 - \frac{(9r-4)\alpha}{18r-6} - \frac{(3r+2)\epsilon}{6r-2} & \text{if } i \in N_2^{(r)} \setminus \{k\} \end{cases}$$

for some small enough $\epsilon > 0$. It is easy to verify that this is a justified objection, and so x_α is not in the bargaining set, nor in the kernel.

We have seen that for odd r, all x_α, $0 \le \alpha \le \frac{3}{2}$, are in the bargaining set and the kernel, while for even r, only x_0 is in these solution sets. In addition to being a counterexample to convergence, we find this example disturbing from an intuitive point of view. It is hard to accept an economic model of large markets whose predictions are so sensitive to the parity of the number of traders. Of course, in view of the above-mentioned result of Shapley and Shubik, this anomaly may be ascribed to the non-smoothness of glove-markets rather than to an inadequacy of the solution concepts being applied.

Another approach to modeling large markets is to consider markets with a continuum of traders. Aumann (1964) established the equivalence between the core and the competitve equilibrium outcomes in this model. Billera and Raanan (1981) specialized this to linear production games, proving in effect that the non-atomic analog of Observation 2.2 always holds with equality.

The classical bargaining set is not defined in the continuum model. However, Mas-Colell (1989) defined his bargaining set for this model, and proved that it coincides with the core and the set of competitive equilibrium outcomes. Strictly speaking, glove-markets are not covered by Mas-Colell's result because they do not satisfy all of his assumptions. But Shitovitz (1999) observed that one can, in fact, deduce from Mas-Colell's theorem that the Mas-Colell bargaining set of a non-atomic glove-market game coincides with the core and the set of competitive equilibrium outcomes.

A Appendix: Proofs for Example 4.4

In this appendix we determine the core, the kernel and the bargaining set of the game (N, v) associated with the glove-market:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Assertion A.1. $\mathscr{C}(N, v) = \{(0,0,0,0,0,0,1,1,1)\}$

Proof: By Observation 2.2, the indicated imputation is in the core. Conversely, suppose $x \in \mathscr{C}(N, v)$. Since $v(N \setminus \{i\}) = v(N)$ for $i = 1, \dots, 6$, we must have $x_1 = \cdots = x_6 = 0$. Since $v(\{1, 3, i\}) = 1$ for $i = 7, 8, 9$, we must have $x_i \ge 1$ for $i = 7, 8, 9$. It follows that $x = (0,0,0,0,0,0,1,1,1)$. ■

For $0 \le \alpha \le \frac{1}{2}$, we introduce the notation:

$$x_\alpha = (\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, 1 - 2\alpha, 1 - 2\alpha, 1 - 2\alpha)$$

Assertion A.2. $\mathscr{K}(N,v) \supseteq \{x_\alpha | 0 \leq \alpha \leq \frac{1}{2}\}$

Proof: Due to symmetry, it is enough to show that $s_{1,7}(x_\alpha) = s_{7,1}(x_\alpha)$ for every α. Indeed, each of these surpluses is easily seen to be equal to α. ■

Since the bargaining set contains the kernel, the above suffices in order to conclude that the bargaining set in this example is larger than the core. For the purpose of fully determining the bargaining set and the kernel, we need to show that no imputation other than the x_α's is in the bargaining set. This requires much more work.

Assertion A.3. $\mathscr{M}_1^{(i)}(N,v) \subseteq \{x_\alpha | 0 \leq \alpha \leq \frac{1}{2}\}$

Proof: Let $x = (x_1, \ldots, x_9)$ be an imputation which is not of the form x_α. We assume w.l.o.g. that the following weak inequalities hold:

$$x_1 \leq x_2, \; x_3 \leq x_4, \; x_5 \leq x_6, \; x_1 \leq x_3 \leq x_5, \; x_7 \leq x_8 \leq x_9 \tag{1}$$

Let $i_0 \in \{2,4,6\}$ be such that

$$x_{i_0} = \max\{x_i | i = 1, \ldots, 6\} \tag{2}$$

and, in case of a tie, i_0 is the largest such index.

The argument below is organized into cases, subcases, etc., depending on certain inequalities which may or may not be satisfied by $x_1, \ldots, x_9$. In each case we find a justified objection (C,y) of one of the players against another one, thereby showing that $x \notin \mathscr{M}_1^{(i)}(N,v)$. In the interest of keeping this appendix from becoming even longer than it already is, we omit the details needed to verify that (C,y) is, indeed, a justified objection in each case. Whenever we define y, the quantity ϵ should be understood as positive and small enough.

Since x is not of the form x_α, we have:

either $x_1 < x_{i_0}$ or $x_7 < x_9$

Case 1. $x_1 < x_{i_0}$

In this case, we consider three coalitions:

$$S_1 = \{1,3,7\}, \; S_2 = \{1,3,5,7,8\}, \; S_3 = N \setminus \{i_0\}$$

Note that $v(S_p) = p$. Among these three coalitions, let C be one that has the largest excess, i.e.,

$$e(C,x) = \max\{e(S_p,x) | p = 1,2,3\}$$

and, in case of a tie, C is the largest such coalition. Note that $e(C,x) \geq e(S_3,x) = x_{i_0} > 0$. We will construct a justified objection of some player against another via C. The argument splits into subcases according to which of the three coalitions C is.

Case 1.1. $C = S_1 = \{1,3,7\}$

The assumption of this subcase translates into the following two inequalities:

$$x_5 + x_8 > 1 \tag{3}$$

$$x_2 + x_4 + x_5 + x_6 - x_{i_0} + x_8 + x_9 > 2 \tag{4}$$

We construct a justified objection (C, y) of player 7 against player i_0 by letting

$$y_i = \begin{cases} x_7 + \epsilon & \text{if } i = 7 \\ a & \text{if } i = 1 \\ b & \text{if } i = 3 \end{cases}$$

where a and b satisfy $a + b = 1 - x_7 - \epsilon,\ a > x_1,\ b > x_3$, and

$$a + x_4 + x_6 + x_8 + x_9 > 2,$$

$$b + x_2 + x_6 + x_8 + x_9 > 2.$$

(These two conditions are designed so as to prevent a counter-objection using one of two specific coalitions. As mentioned above, we suppress the detailed argument showing that no other coalition can provide a counter-objection.) In order to check that such a and b exist, one has to verify that the following holds:

$$\max\{x_1, 2 - x_4 - x_6 - x_8 - x_9\} + \max\{x_3, 2 - x_2 - x_6 - x_8 - x_9\} < 1 - x_7$$

This may be replaced by four inequalities, each of which may be verified using (1), (2), (3) and (4).

Case 1.2. $C = S_2 = \{1, 3, 5, 7, 8\}$

This subcase occurs when the following two inequalities hold:

$$x_5 + x_8 \leq 1 \tag{5}$$

$$x_2 + x_4 + x_6 - x_{i_0} + x_9 > 1 \tag{6}$$

We further distinguish two possibilities.

Case 1.2.1. $x_{i_0 - 1} = x_{i_0}$ and $x_7 = x_8 = x_9$

By the choice of i_0, this implies $i_0 = 6$. We claim that there exists $j_0 \in \{2, 4\}$ satisfying:

$$x_{j_0} > x_{j_0 - 1} \tag{7}$$

Suppose this is false, so $x_1 = x_2 = \alpha$ and $x_3 = x_4 = \beta$, and by our further assumption $x_7 = x_8 = x_9 = \gamma$. As the total payoff is 3, this implies $\alpha + \beta + \gamma \leq 1$. On the other hand, (6) yields $\alpha + \beta + \gamma > 1$, a contradiction.

We denote by j_0 one player among players 2 and 4 for whom (7) holds, and denote by k_0 the remaining one. We define:

$$c = \max\{x_{k_0 - 1}, 1 - x_{j_0} - x_9\}$$

The following is a justified objection (via C) of player 7 against player j_0:

$$y_i = \begin{cases} x_i + \epsilon & \text{if } i \in \{i_0 - 1, j_0 - 1, 7\} \\ c + \epsilon & \text{if } i = k_0 - 1 \\ 2 - x_{i_0 - 1} - x_{j_0 - 1} - x_7 - c - 4\epsilon & \text{if } i = 8 \end{cases}$$

Case 1.2.2. $x_{i_0-1} < x_{i_0}$ or $x_7 < x_9$

We denote by j_0 and k_0 the two players in $\{2,4,6\} \setminus \{i_0\}$. We define:

$$d = \max\{x_{j_0-1}, 1 - x_{i_0} - x_9\}$$

$$e = \max\{x_{k_0-1}, 1 - x_{i_0} - x_9\}$$

The following is a justified objection (via C) of player 7 against player i_0:

$$y_i = \begin{cases} x_i + \epsilon & \text{if } i \in \{i_0 - 1, 7\} \\ d + \epsilon & \text{if } i = j_0 - 1 \\ e + \epsilon & \text{if } i = k_0 - 1 \\ 2 - x_{i_0-1} - x_7 - d - e - 4\epsilon & \text{if } i = 8 \end{cases}$$

Case 1.3. $C = S_3 = N \setminus \{i_0\}$

This occurs when the following two inequalities hold:

$$x_2 + x_4 + x_5 + x_6 - x_{i_0} + x_8 + x_9 \leq 2 \tag{8}$$

$$x_2 + x_4 + x_6 - x_{i_0} + x_9 \leq 1 \tag{9}$$

We further distinguish three possibilities.

Case 1.3.1. $x_{i_0-1} = x_{i_0}$, $x_7 = x_8 = x_9$ and (9) holds with equality

By the choice of i_0, this implies $i_0 = 6$. The equality in (9) takes the form:

$$x_2 + x_4 + x_9 = 1$$

As in Case 1.2.1, we claim that there exists $j_0 \in \{2,4\}$ for whom (7) holds. Indeed, if we assume otherwise, then we obtain $x_1 + x_3 + x_8 = x_2 + x_4 + x_9 = 1$ and $x_5 + x_6 + x_7 > x_1 + x_3 + x_8$ (the latter using $x_6 > x_1$ from the assumption of Case 1). This yields a contradiction to the fact that the total payoff is 3.

We need to handle separately the case when $j_0 = 2$ satisfies (7) and the case when it does not (but $j_0 = 4$ does).

Case 1.3.1.1. $x_1 < x_2$

In this case, the following is a justified objection (via C) of player 1 against player 6:

$$y_i = \begin{cases} x_i + \epsilon & \text{if } i \in N \setminus \{2,6\} \\ x_2 + x_6 - 7\epsilon & \text{if } i = 2 \end{cases}$$

Case 1.3.1.2. $x_1 = x_2$ and $x_3 < x_4$

We define:

$$f = \max\{x_1, 1 - x_6 - x_9\}$$

The following is a jusitified objection (via C) of player 3 against player 6:

$$y_i = \begin{cases} x_i + \epsilon & \text{if } i \in \{3,5,7,8,9\} \\ f + \epsilon & \text{if } i \in \{1,2\} \\ 3 - x_3 - x_5 - 3x_9 - 2f - 7\epsilon & \text{if } i = 4 \end{cases}$$

Case 1.3.2. $x_{i_0-1} < x_{i_0}$ or $x_7 < x_8$ or (9) holds with strict inequality

In this case, the following is a justified objection (via C) of player 7 against player i_0:

$$y_i = \begin{cases} x_i + \epsilon & \text{if } i \in N \setminus \{i_0, 8, 9\} \\ x_i + \frac{x_{i_0}}{2} - 3\epsilon & \text{if } i \in \{8,9\} \end{cases}$$

Case 1.3.3. $x_7 = x_8 < x_9$

A variant of the previous y, in which $y_8 = x_8 + \frac{x_{i_0}}{2}$ and $y_9 = x_9 + \frac{x_{i_0}}{2} - 6\epsilon$, works in this case.

Case 2. $x_1 = \cdots = x_6$ and $x_7 < x_9$

In this case, player 7 has a justified objection against player 9, namely (C, y) where $C = \{1,3,5,7,8\}$ and y is defined by:

$$y_i = \begin{cases} x_i + \epsilon & \text{if } i \in \{1,3,5,7\} \\ 2 - 3x_1 - x_7 - 4\epsilon & \text{if } i = 8 \end{cases} \qquad \blacksquare$$

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