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Bundling equilibrium in combinatorial auctions ☆

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Abstract

This paper analyzes ex post equilibria in the VCG combinatorial auctions. If Σ is a family of bundles of goods, the organizer may restrict the bundles on which the participants submit bids, and the bundles allocated to them, to be in Σ. The Σ-VCG combinatorial auctions obtained in this way are known to be truth-telling mechanisms. In contrast, this paper deals with non-restricted VCG auctions, in which the buyers choose strategies that involve bidding only on bundles in Σ, and these strategies form an equilibrium. We fully characterize those Σ that induce an equilibrium in every VCG auction, and we refer to the associated equilibrium as a bundling equilibrium. The main motivation for studying all these equilibria, and not just the domination equilibrium, is that they afford a reduction of the communication complexity. We analyze the tradeoff between communication complexity and economic efficiency of bundling equilibrium.

JEL classification: C72; D44; D78

1. Introduction

The Vickrey–Clarke–Groves (VCG) mechanisms (Vickrey, 1961; Clarke, 1971; Groves, 1973) are central to the design of protocols with selfish participants (e.g., Ephrati and

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Rosenschein, 1991; Nisan and Ronen, 2001; Tennenholtz, 1999; Varian, 1995), and in particular for combinatorial auctions (e.g., Weber, 1983; Krishna and Perry, 1998; de Vries and Vohra, 2000; Wellman et al., 2001; Monderer and Tennenholtz, 2000; Lehmann et al., 1999), in which the participants submit bids, through which they can express preferences over bundles of goods. The organizer allocates the goods and collects payments based on the participants' bids.[1] These mechanisms allow to allocate a set of goods (or services, or tasks) in a socially optimal (surplus maximizing) manner, assuming there are no resource bounds on the agents' computational capabilities. There are at least two sources of computational issues, which arise when dealing with combinatorial auctions: winner determination–finding the optimal allocation (see, e.g., Rothkopf et al., 1998; Tennenholtz, 2000; Sandholm, 1999; Fujishima et al., 1999; Anderson et al., 2000; Sandholm et al., 2001; Hoos and Boutilier, 2000), and bid communication–the transfer of information, on which we focus in this paper.

The VCG mechanisms are designed in such a way that truthful revealing of the agents' private information[2] is a dominant strategy for them. They have been applied mainly in the context of games in informational form, where no probabilistic assumptions about agents' types are required.[3] Domination and equilibrium in such games have traditionally been referred to as ex post solutions because they have the property that if the players were told about the true state, after they chose their actions, they would not regret their actions.[4]

The revelation principle (see, e.g., Myerson, 1979) implies that the discussion of other, non-truth revealing equilibria of the VCG mechanisms may seem unneeded, and indeed it has been ignored by the literature. It can be proved that every mechanism with an ex post equilibrium is economically equivalent to another mechanism—a direct mechanism—in which every agent is required to submit his information. In this direct mechanism, revealing the true type is an ex post dominating strategy for every agent, and it yields the same economics parameters as the original mechanism. However, the two mechanisms differ in the set of inputs that the player submits in equilibrium. This difference may be crucial when we deal with communication complexity. Thus, two mechanisms that are equivalent from the economics point of view, may be considered different mechanisms from the CS point of view.

[1] Motivated by the FCC auctions (see, e.g., Cramton, 1995; McMillan, 1994; Milgrom, 1998) there is an extensive recent literature devoted to the design and analysis of multistage combinatorial auctions, in which the bidders express partial preferences over bundles at each stage. See, e.g., Wellman et al. (2001), Perry and Reny (1999), Ausubel (2000), Parkes (1999), Parkes and Ungar (2000), Ausubel and Milgrom (2001).

[2] This paper deals with the private-values model, in which every buyer knows his own valuations of bundles of goods. In contrast, in a correlated-values model, every buyer receives a signal (possibly about all buyers' valuation functions), and this signal does not completely reveal his own valuation function (see, e.g., Milgrom and Weber (1982), Jehiel and Moldovanu (2001), McAfee and Reny (1992), Dasgupta and Maskin (2000), Perry and Reny (1999, 1999a) for discussions of models in which valuations are correlated).

[3] A game in informational form is a pre-Bayesian game. That is, it has all the ingredients of a Bayesian game except for the specification of probabilities.

[4] Alternatively, ex post solutions may be called probability-independent solutions because, up to some technicalities concerning the concept of measurable sets, they form Bayesian solutions for every specification of probabilities.

Thus, tackling the VCG mechanisms from a computational perspective introduces a vastly different picture. While the revelation of the agents' types defines one equilibrium, there are other (in fact, over-exponentially many) equilibria for the VCG auctions. Moreover, these equilibria have different communication requirements.

The communication problem has motivated researchers in economics and in computer science to examine the properties of simpler auction mechanisms, in which rational buyers do not fully reveal valuations (see, e.g., Gul and Stacchetti, 2000; Parkes, 1999; Parkes and Ungar, 2000; Wellman et al., 2001; Ausubel and Milgrom, 2001; Bikhchandani et al., 2001). The main goal of the above papers was to characterize models in which the suggested auctions lead to efficient outcomes. Such models are very rare, and they assume various forms of substitution properties (see, e.g., Gul and Stacchetti, 1999).

In this paper we deal with unrestricted valuation functions, and analyze ex post equilibria in the VCG mechanisms. Let Σ be a family of bundles of goods. We characterize those Σ, for which the strategy of reporting the true valuation over the bundles in Σ is a player-symmetric ex post equilibrium. An equilibrium that is defined by such Σ is called a bundling equilibrium.[5] We prove that Σ induces a bundling equilibrium if and only if it is a quasi-field[6] of bundles. The class of bundling equilibria includes a natural subclass that consists of partition-based equilibria, in which the family Σ is a field (i.e., it is generated by a partition).

The main topic we study is the quantitative tradeoff between economic efficiency and communication complexity offered by the class of bundling equilibria.[7] In other words, we address the following question: How much economic efficiency needs to be sacrificed in order to keep the communication complexity at an acceptable level? The underlying assumption is that a VCG mechanism is used, and the buyers' strategies form an ex post equilibrium. We measure the (worst case) economic inefficiency of a given equilibrium by the supremum, taken over all profiles of valuations for any number of buyers, of the ratio between the optimal social surplus and the surplus obtained in that equilibrium. We measure the communication complexity of a given bundling equilibrium by the number of bundles in Σ.[8] Qualitatively, it is clear that as Σ becomes larger the economic inefficiency is reduced at the expense of higher communication complexity. Our main results give

[5] It is far from obvious, but true, that every ex post equilibrium in the VCG mechanisms must be of this form, i.e., associated with a family of bundles Σ in the above sense. In particular, every ex post equilibrium is player-symmetric. This is proved in a subsequent paper (Holzman and Monderer, 2002).

[6] A quasi-field is a non-empty set of sets that is closed under complements and under disjoint unions.

[7] The same kind of tradeoff is studied in (Nisan and Segal, 2002) for a wider class of mechanisms. One difference between our approach and that of (Nisan and Segal, 2002) is that we look at equilibrium profiles only. Another difference, which is important for a comparison of our results and those of (Nisan and Segal, 2002), is that we view the set of goods as fixed and allow the number of buyers to vary in our worst-case analysis, whereas in (Nisan and Segal, 2002) the number of buyers is fixed and the bounds obtained are asymptotic as the number of goods becomes large. Communication complexity in equilibrium has been discussed also in (Shoham and Tennenholtz, 2001).

[8] In our context, this measure of communication complexity seems very intuitive, and therefore we do not delve into formal definitions of communication complexity that apply to more general situations. See (Nisan and Segal, 2002) for a discussion of two approaches to this concept: the dimensionality measure, suited for continuous settings and common in economics (e.g., Hurwicz, 1977), and the bit-count measure, suited for discrete settings and common in computer science (e.g., Kushilevitz and Nisan, 1997).

quantitative bounds on this tradeoff for partition-based equilibria, which are tight in infinitely many cases.

Thus, we are looking at a full spectrum of equilibria. At one end, we have the truth-telling equilibrium that yields the maximum social surplus but has prohibitive communication complexity. At the other end, we have the very simple but potentially highly inefficient equilibrium in which the buyers only bid on the bundle of all goods. As is common in a multiple equilibria setup, we do not know what equilibrium will be reached. However, the buyers will reach an equilibrium,[9] which reflects a reasonable compromise between complexity and efficiency considerations. Note that had the organizer known the chosen equilibrium (i.e., the family Σ) she could have restricted the allowed bids (and allocations) to bundles in Σ. In such case, this equilibrium becomes a domination equilibrium. However, the organizer may not wish to restrict the bids because she is handling many auctions with different types of users, and it is not reasonable for her to keep changing the rules of the auction. Alternatively, the organizer can recommend to the buyers to bid only on bundles in the family Σ that induces the selected bundling equilibrium. Each buyer is free to accept or reject the recommendation, but in view of the equilibrium property, it is rational for the buyers to follow the recommendation.

Section 2 provides the reader with a rigorous framework for general analysis of VCG mechanisms for combinatorial auctions. In Section 3 we introduce bundling equilibrium, and provide a full characterization of bundling equilibria for VCG mechanisms. Then we discuss bundling equilibrium that is generated by a partition, titled partition-based equilibrium. In Section 4 we deal with the social surplus of VCG mechanisms for combinatorial auctions when following partition-based equilibrium, exploring the spectrum between economic efficiency and communication efficiency.

2. Combinatorial auctions

In a combinatorial auction there is a seller, denoted by 0, who wishes to sell a set of m items $A = \{a_1, \ldots, a_m\}$, $m \geqslant 1$, that are owned by her. There is a set of buyers $N = \{1, 2, \ldots, n\}$, $n \geqslant 1$. Let Γ be the set of all allocations of the goods. That is, every $\gamma \in \Gamma$ is an ordered partition of A, $\gamma = (\gamma_i)_{i \in N \cup \{0\}}$. A *valuation function* of buyer i is a function $v_i : 2^A \to \Re$, where $\Re$ denotes the set of real numbers, with the normalization $v_i(\emptyset) = 0$, which satisfies: If $B \subseteq C$, $B, C \in 2^A$, then $v_i(B) \leqslant v_i(C)$.

Let V_i be the set of all possible valuation functions of i (obviously $V_i = V_j$ for all $i, j \in N$), and let $V = \times_{i \in N} V_i$. We assume each buyer knows his valuation function only.

A *mechanism* $M = (X, d, c)$ for allocating the goods is defined by sets of messages X_i, one set for each buyer i, and by a pair (d, c) with $d : X \to \Gamma$, and $c : X \to \Re^n$, where $X = \times_{i \in N} X_i$. d is called the allocation function and c the transfer function; if the buyers send the profile of messages $x \in X$, buyer i receives the set of goods $d_i(x)$ and pays $c_i(x)$ to the seller; his utility is $u_i(v_i, x) = v_i(d_i(x)) - c_i(x)$. In a general model of auctions the

[9] For example, the agents may reach the equilibrium by a process of learning (see, e.g., Hon-Snir et al., 1998).

seller may set reserve prices (in the form of a valuation function). We assume: *No reserve prices*.[10]

A *strategy* of i is a function $b_i : V_i \to X_i$. A profile of strategies $b = (b_1, \dots, b_n)$ is *player-symmetric* if $b_i = b_j$ for all $i, j \in N$. A profile of strategies $b = (b_1, \dots, b_n)$ is an *ex post equilibrium*, if for every agent i, for every $v_i \in V_i$, for every $v_{-i} \in V_{-i}$, and for every $x_i \in X_i$,

$$u_i\big(v_i, b_i(v_i), b_{-i}(v_{-i})\big) \geqslant u_i\big(v_i, x_i, b_{-i}(v_{-i})\big), \tag{2.1}$$

where $b_{-i}(v_{-i}) = (b_j(v_j))_{j \neq i}$. A strategy b_i of i is an *ex post dominant strategy* for i, if (2.1) holds for every b_{-i}, for every $v_i \in V_i$, for every $v_{-i} \in V_{-i}$, and for every $x_i \in X_i$.

Obviously, if b_i is an ex post dominant strategy for every i, b is an ex post equilibrium, but not necessarily vice versa. An ex post equilibrium b, in which every strategy b_i is ex post dominant is called an ex post *domination equilibrium*.

A mechanism (X, d, c) is called a *direct* mechanism if $X_i = V_i$ for every $i \in N$. That is, in a direct mechanism a buyer's message contains a full description of some valuation function. A direct mechanism is called *truth revealing* if for every buyer i, telling the truth $(b_i(v_i) = v_i)$ is an ex post dominant strategy.

For an allocation γ and a profile of valuations v, we denote by $S(v, \gamma)$ the *total social surplus* of the buyers, that is

$$S(v, \gamma) = \sum_{i \in N} v_i(\gamma_i).$$

We also denote:

$$S_{\max}(v) = \max_{\gamma \in \Gamma} S(v, \gamma).$$

Well-known truth revealing mechanisms are the VC (Vickrey–Clarke) mechanisms.[11] These mechanisms are parametrized by an allocation function d, that is socially optimal. That is, $S(v, d(v)) = S_{\max}(v)$ for every $v \in V$. The transfer functions are defined as follows:

$$c_i^d(v) = \max_{\gamma \in \Gamma} \sum_{j \neq i} v_j(\gamma_j) - \sum_{j \neq i} v_j\big(d_j(v)\big). \tag{2.2}$$

The mechanisms differ in the allocation they pick in cases in which there exist more than one socially optimal allocation, and therefore in the second term in (2.2). We will refer to a VC mechanism by the allocation function d that determines it. Note that $0 \leqslant c_i^d(v) \leqslant v_i(d_i(v))$ for every $i \in N$ and $v \in V$. This implies that a truth telling buyer always gets non-negative utility (this is called individual rationality), and the seller's revenue is always non-negative.

More general mechanisms are the VCG mechanisms. Every VCG mechanism is obtained from some VC mechanism by changing the transfer functions: A VCG

[10] This is not an innocuous assumption. The general case is discussed in Section 3.3. For discussions of the importance of reserve prices see, e.g., (Myerson, 1981; Ausubel and Cramton, 1999a).

[11] By VC mechanisms we refer here to what is also known as Clarke mechanisms or the pivotal mechanism.

mechanism is defined by a socially optimal allocation function d and by a family of functions $h = (h_i)_{i\in N}$. The transfer functions are defined by

$$c_i^d(v) = \max_{\gamma\in\Gamma} \sum_{j\neq i} v_j(\gamma_j) - \sum_{j\neq i} v_j\bigl(d_j(v)\bigr) + h_i(v_{-i}).$$

Observe that a VCG mechanism is strategically equivalent to the VC mechanism it is based on, and moreover they require the same communication and result in identical allocations. For our purposes, therefore, we may (and will) restrict attention to VC mechanisms without loss of generality. Of course, the individual rationality and revenue non-negativity properties noted above for VC mechanisms do not carry over to general VCG mechanisms.

It is well known that in any VCG mechanism truth telling is an ex post dominant strategy for every buyer, and hence forms an ex post equilibrium. In the next section we discuss other ex post equilibria in the VCG mechanisms.

3. Bundling equilibrium

Let $\Sigma \subseteq 2^A$ be a family of bundles of goods. We deal only with such families Σ for which

- $\emptyset \in \Sigma$.

A valuation function v_i is a Σ*-valuation function* if

$$v_i(B) = \max_{C\in\Sigma, C\subseteq B} v_i(C), \quad \text{for every } B \in 2^A.$$

The set of all Σ-valuation functions in V_i is denoted by V_i^Σ. We further denote $V^\Sigma = \times_{i\in N} V_i^\Sigma$. For every valuation function v_i we denote by v_i^Σ its projection on V_i^Σ, that is:

$$v_i^\Sigma(B) = \max_{C\in\Sigma, C\subseteq B} v_i(C), \quad \text{for every } B \in 2^A.$$

Obviously $v_i^\Sigma \in V_i^\Sigma$, and for $v_i \in V_i^\Sigma$, $v_i^\Sigma = v_i$. In particular $(v_i^\Sigma)^\Sigma = v_i^\Sigma$ for every $v_i \in V_i$. Let $f^\Sigma : V_i \to V_i^\Sigma$ be the projection function defined by

$$f^\Sigma(v_i) = v_i^\Sigma.$$

An allocation γ is a Σ*-allocation* if $\gamma_i \in \Sigma$ for every buyer $i \in N$. The set of all Σ-allocations is denoted by Γ^Σ.

The following theorem is proved in (Holzman and Monderer, 2002).

Theorem (Holzman and Monderer). *Let N be a set of buyers, with $n = |N| \geqslant 3$. Let $b = (b_1, \dots, b_n)$ be a profile of strategies such that for every $N' \subseteq N$ the restriction of b to N' forms an ex post equilibrium in every VCG mechanism with buyer set N'. Then b is player-symmetric, and moreover there exists a $\Sigma \subseteq 2^A$ such that $b_i = f^\Sigma$ for every $i \in N$.*

Here we are interested in the following question: For which Σ, do we have that $(f^\Sigma, \ldots, f^\Sigma)$ is an ex post equilibrium in every VCG mechanism (with any number of buyers)? In such a case we call f^Σ a *bundling equilibrium* for the VCG mechanisms and say that Σ induces a bundling equilibrium. The next example shows that not every Σ induces a bundling equilibrium.

Before we present the example we need the following notation. Let $B \in 2^A$, we denote by w_B the following valuation function:

$$\text{If } B \neq \emptyset, \quad w_B(C) = \begin{cases} 1 & \text{if } B \subseteq C, \\ 0 & \text{otherwise.}^{12} \end{cases}$$

$$\text{If } B = \emptyset, \quad w_B(C) = 0 \quad \text{for all } C \in 2^A.$$

Example 1. Let A contain four goods a, b, c, d. Let

$$\Sigma = \{a, d, bcd, abc, A, \emptyset\}.^{13}$$

Let $v_2 = w_a$, $v_3 = w_d$. Consider buyer 1 with $v_1 = w_{bc}$. Note that $v_i \in V_i^\Sigma$ for $i = 2, 3$. If buyer 1 uses f^Σ he declares $v_1'(bcd) = v_1'(abc) = v_1'(A) = 1$ and $v_1'(C) = 0$ for all other C, and there exists a VC mechanism that allocates a to 2, d to 3, and bc to the seller. In this mechanism the utility of 1 from using f^Σ is zero. On the other hand, if agent 1 reports the truth (w_{bc}), he receives (in every VC mechanism) bc and pays nothing. Hence, his utility would be 1. Therefore f^Σ is not in equilibrium in this VC mechanism, and hence Σ does not induce a bundling equilibrium.

3.1. A characterization of bundling equilibria

$\Sigma \subseteq 2^A$ is called a *quasi-field* if it satisfies the following properties:[14]

- $B \in \Sigma$ implies that $B^c \in \Sigma$, where $B^c = A \setminus B$.
- $B, C \in \Sigma$ and $B \cap C = \emptyset$ imply that $B \cup C \in \Sigma$.[15]

Theorem 1. *Σ induces a bundling equilibrium if and only if it is a quasi-field.*

Proof. *Suppose Σ is a quasi-field.* Consider a VC mechanism with an allocation function d. We show that $(f^\Sigma, \ldots, f^\Sigma)$ is an ex post equilibrium in this VC mechanism.

Assume that every buyer j, $j \neq i$, uses the strategy $b_j = f^\Sigma$. Let $v_{-i} \in V_{-i}$. We have to show that for buyer i with valuation v_i, v_i^Σ is a best reply to v_{-i}^Σ. As truth revealing is a dominant strategy in every VC mechanism, it suffices to show that buyer i's utility when submitting v_i^Σ is the same as when submitting v_i. That is, we need to show that

$$S\big((v_i, v_{-i}^\Sigma), \gamma\big) - \alpha = S_{\max}\big(v_i, v_{-i}^\Sigma\big) - \alpha,$$

[12] For $B \neq \emptyset$, a valuation function of the form w_B is called a unanimity TU game in cooperative game theory. An agent with such a valuation function (up to scaling) is called by Lehmann et al. (1999) a single-minded agent.

[13] We omit braces and commas when writing subsets of A.

[14] Recall our assumption that we deal only with Σ such that $\emptyset \in \Sigma$.

[15] Equivalently, the union of any number of pairwise disjoint sets in Σ is also in Σ.

where $\gamma = d(v_i^\Sigma, v_{-i}^\Sigma)$, and α is the first term in (2.2) which depends only on v_{-i}^Σ.

Hence, we have to show that

$$S\big((v_i, v_{-i}^\Sigma), \gamma\big) = S_{\max}\big(v_i, v_{-i}^\Sigma\big). \tag{3.1}$$

Obviously,

$$S\big((v_i, v_{-i}^\Sigma), \gamma\big) \leqslant S_{\max}\big(v_i, v_{-i}^\Sigma\big). \tag{3.2}$$

As $v_i(B) \geqslant v_i^\Sigma(B)$ for every $B \in 2^A$,

$$S\big((v_i, v_{-i}^\Sigma), \gamma\big) \geqslant S\big((v_i^\Sigma, v_{-i}^\Sigma), \gamma\big) = S_{\max}\big(v_i^\Sigma, v_{-i}^\Sigma\big). \tag{3.3}$$

Let $\xi = d(v_i, v_{-i}^\Sigma)$. For $j \neq i$ and $j \neq 0$, let $\xi_j^\Sigma \in \Sigma$ be such that $\xi_j^\Sigma \subseteq \xi_j$ and $v_j^\Sigma(\xi_j^\Sigma) = v_j^\Sigma(\xi_j)$. Let $\xi_i^\Sigma = (\cup_{j\neq 0,i}\xi_j^\Sigma)^c$, and let $\xi_0^\Sigma = \emptyset$.

Because Σ is a quasi-field, $\xi_i^\Sigma \in \Sigma$, and hence $\xi^\Sigma \in \Gamma^\Sigma$. As $\xi_i \subseteq \xi_i^\Sigma$, ξ^Σ is also optimal for (v_i, v_{-i}^Σ). Hence

$$S_{\max}\big(v_i, v_{-i}^\Sigma\big) = S\big((v_i, v_{-i}^\Sigma), \xi^\Sigma\big) = S\big((v_i^\Sigma, v_{-i}^\Sigma), \xi^\Sigma\big) \leqslant S_{\max}\big(v_i^\Sigma, v_{-i}^\Sigma\big). \tag{3.4}$$

Combining (3.2), (3.3), and (3.4) yields (3.1).

Suppose Σ induces a bundling equilibrium. We first show that if $B \in \Sigma$, then $B^c \in \Sigma$. If $B = A$ then by definition $B^c = \emptyset \in \Sigma$. Let $B \subset A$. Assume, for the sake of contradiction, that $B^c \notin \Sigma$. Let $v_2 = w_B$ and $v_1 = w_{B^c}$. Note that $v_2^\Sigma = v_2$. Thus, if buyer 2 uses f^Σ, he declares v_2. If buyer 1 uses f^Σ, he declares v_1^Σ, where $v_1^\Sigma(B^c) = 0$. Hence, there exists a VC mechanism d, that allocates B to agent 2 and B^c to the seller. However, if buyer 1 deviates and declares his true valuation, then this VC mechanism allocates B^c to him, and he pays nothing. Hence, there is a profitable deviation from f^Σ, a contradiction.

Next, we show that if $B, C \in \Sigma$ are disjoint then $B \cup C \in \Sigma$. By the first part of the proof, it suffices to show that $(B \cup C)^c \in \Sigma$. Clearly, we may assume that the sets B, C, and $(B \cup C)^c$ are all non-empty. Assume, for the sake of contradiction, that $(B \cup C)^c \notin \Sigma$. Consider three buyers with valuations $v_1 = w_{(B\cup C)^c}$, $v_2 = w_B$, $v_3 = w_C$. Proceeding as in the first part of the current part of the proof yields a similar contradiction. □

It may be useful to note that if f^Σ is a buyer-symmetric equilibrium for a fixed set of buyers, then Σ is not necessarily a quasi-field. For example, if there is only one buyer, every Σ such that $A \in \Sigma$ induces an equilibrium. In the case of two buyers, being closed under complements is necessary and sufficient for Σ to induce an equilibrium. However, it can be deduced from the proof of the only if part of Theorem 1, that for a fixed set of buyers N, if $n = |N| \geqslant 3$, then Σ must be a quasi-field if it induces an equilibrium for the set of buyers N.

3.2. *Partition-based equilibrium*

Let $\pi = \{A_1, \ldots, A_k\}$ be a partition of A into non-empty parts. That is, $A_i \neq \emptyset$ for every $A_i \in \pi$, $\cup_{i=1}^k A_i = A$, and $A_i \cap A_j = \emptyset$ for every $i \neq j$. Let Σ_π be the field generated by π. That is, Σ_π contains all the sets of goods of the form $\cup_{i\in I} A_i$, where $I \subseteq \{1, \ldots, k\}$. To avoid confusion: $\emptyset \in \Sigma_\pi$. For convenience, we will use f^π to denote f^{Σ_π}. A corollary of Theorem 1 is

Corollary 1. *f^{π} is a bundling equilibrium.*

Proof. As Σ_π is a field it is in particular a quasi-field. Hence, the proof follows from Theorem 1. □

A bundling equilibrium of the form f^{π}, where π is a partition, will be called a *partition-based equilibrium*. Thus, a partition-based equilibrium is a bundling equilibrium f^{Σ} that is based on a field $\Sigma = \Sigma_\pi$. It is important to note that there exist quasi-fields, which are not fields. For example, let $A = \{a, b, c, d\}$. $\Sigma = \{ab, cd, ac, bd, A, \emptyset\}$ is a quasi-field, which is not a field. We note, however, that when $m = |A| \leqslant 3$, the notions of quasi-field and field coincide.

3.3. Reserve prices

The seller may wish to prevent the sale of goods at a price that she considers too low. This is modeled by assuming that the seller has a valuation function v_0. There are two common methods to modify the VCG auction's rules in order to allow the seller to express her preferences.[16] In one of them, the seller submits a bid v_r, which is not known to the buyers. Of course, the buyers must believe that the seller cannot change her bid after they make their bids. This is the case when the seller and the organizer are not the same entity, or if the auction is controlled by a trusty third party. In such a case, the seller is just an additional player in the auction game. However, in the associated $(n+1)$-person game, the payoff of the additional player is not determined by the VCG scheme, and our results do not hold for such games.

In the second approach the seller announces v_r as part of the auction's rules. In such a case our results are easily generalized as follows:

Theorem 1_r. *Σ induces a bundling equilibrium if and only if it is a quasi-field and $v_r = v_r^{\Sigma}$.*

Hence, the seller may keep all bundling equilibria alive by introducing a reserve price which is a $\{\emptyset, A\}$-valuation function, and she can kill all bundling equilibria except for the domination equilibrium by submitting an arbitrary valuation function with a positive price for each good. We do not attempt in this paper to analyze the design stage, in which the seller decides strategically which v_r to announce.

4. Economic efficiency and communication complexity

Let $\Sigma \subseteq 2^A$. If every buyer uses f^{Σ}, then in every VCG mechanism, the total surplus generated when the valuations of the buyers are given by $v \in V$ is $S_{\max}(v^{\Sigma})$.

[16] Both methods appear in eBay auctions.

We denote

$$S_{\Sigma\text{-}\max}(v) = \max_{\gamma \in \Gamma^{\Sigma}} S(v, \gamma).$$

Obviously,

$$S_{\Sigma\text{-}\max}(v) = S_{\Sigma\text{-}\max}(v^{\Sigma}) = S_{\max}(v^{\Sigma}), \quad \text{for every } v \in V.$$

For convenience we denote $S_{\Sigma\text{-}\max}$ by S_{Σ}, and we call S_{Σ} the *Σ-optimal surplus function* (note that $S_{2^A} = S_{\max}$). When Σ is a field generated by a partition π we write S_{π} for $S_{\Sigma_{\pi}}$.

If Σ is a quasi-field we say that the *communication complexity* of the equilibrium f^{Σ} is the number of bundles in Σ, that is $|\Sigma|$. Notice that this is a natural definition because a buyer who is using f^{Σ} has to submit a vector of $|\Sigma|$ numbers to the seller.[17] Thus, if π is a partition, the communication complexity is $2^{|\pi|}$. If $\Sigma_1 \subseteq \Sigma_2$, then $S_{\Sigma_1}(v) \leqslant S_{\Sigma_2}(v)$ for every $v \in V$. So, Σ_2 induces more surplus (a proxy for economic efficiency) than Σ_1, but Σ_2 also induces higher communication complexity.[18] Hence, there is a tradeoff between economic efficiency and computational complexity.

For every family of bundles Σ with $A \in \Sigma$ we define

$$r_{\Sigma}^{n} = \sup_{v \in V, v \neq 0} \frac{S_{\max}(v)}{S_{\Sigma}(v)}, \tag{4.1}$$

where $V = V_1 \times \cdots \times V_n$. Thus, r_{Σ}^{n} is a worst-case measure of the economic inefficiency that may result from using the strategy f^{Σ} when there are n buyers. Obviously $r_{\Sigma}^{n} \geqslant 1$, and equality holds for $\Sigma = 2^A$. A standard argument using homogeneity and continuity of $S_{\max}/S_{\Sigma}$ shows that the supremum in (4.1) is attained, i.e., it is a maximum.

The following remark gives a simple upper bound on the inefficiency associated with Σ.

Remark 1. For every $\Sigma \subseteq 2^A$ with $A \in \Sigma$, and for every $v \in V$,

$$S_{\max}(v) \leqslant n S_{\Sigma}(v),$$

where n is the number of buyers. Consequently,

$$r_{\Sigma}^{n} \leqslant n.$$

Proof. Let $\gamma = d(v)$, where d is any VC mechanism.

$$S_{\max}(v) = S(v, \gamma) = \sum_{i \in N} v_i(\gamma_i) \leqslant \sum_{i \in N} v_i(A) = \sum_{i \in N} v_i^{\Sigma}(A) \leqslant n S_{\Sigma}(v). \qquad \square$$

[17] A discussion of the way this can be extended to deal with the introduction of concise bidding languages (Nisan, 2000; Boutilier and Hoos, 2001) is beyond the scope of this paper.

[18] Incidentally, unlike social surplus, the revenue of the seller is not a monotone function of the quasi-field Σ, as can be seen from simple examples. Yet, it is commonly believed that social optimality is a good proxy for revenue. This was proved to be asymptotically correct when the number of buyers is large, and the organizer has a Bayesian belief over the distribution of valuation functions, which assumes independence across buyers (see Monderer and Tennenholtz, 2000). It was also proved to be correct in models that assume the possibility of a resale (Ausubel and Cramton, 1999).

However, we are interested mainly in upper bounds on the economic inefficiency that are independent of the number of buyers. For every family of bundles Σ with $A \in \Sigma$ we define

$$r_{\Sigma} = \sup_{n \geqslant 1} r_{\Sigma}^{n}. \tag{4.2}$$

It is easy to see that, since any allocation assigns non-empty bundles to at most $m = |A|$ buyers, the supremum in (4.2) is attained for some $n \leqslant m$. When $\Sigma = \Sigma_\pi$ for a partition π, we write r_π instead of r_{Σ_π}.

In the following subsection we characterize and estimate r_π, thereby obtaining a quantitative form of the tradeoff between communication and economic efficiency in partition-based equilibria.

4.1. Communication efficiency vs. economic efficiency in partition-based equilibria

We first express r_π in terms of the partition $\pi = \{A_1, \ldots, A_k\}$ only. A *feasible* family for π is a family $\Delta = (H_i)_{i=1}^{s}$ of (not necessarily distinct) subsets of $\{1, \ldots, k\}$ satisfying the following two conditions:

- $H_i \cap H_j \neq \emptyset$ for every $1 \leqslant i, j \leqslant s$.
- $|\{i : l \in H_i\}| \leqslant |A_l|$ for every $1 \leqslant l \leqslant k$.

We write $s = s(\Delta)$ for the number of sets in the family Δ (counted with repetitions).

Theorem 2. *For every partition π,*

$$r_\pi = \max s(\Delta),$$

where the maximum is taken over all families Δ that are feasible for π.

Proof. We first prove that $r_\pi \leqslant \max s(\Delta)$. It suffices to show that for every $v \in V$ there exists a feasible family Δ for π such that

$$S_{\max}(v) \leqslant s(\Delta) \cdot S_\pi(v).$$

Let $v \in V$. Let γ be a socially optimal allocation. That is,

$$S_{\max}(v) = \sum_{i \in N} v_i(\gamma_i).$$

For every γ_i let γ_i^π be the minimal set in Σ_π that contains γ_i. That is $\gamma_i^\pi = \cup_{l \in J_i} A_l$ where $J_i = \{l \in \{1, \ldots, k\}\colon\ A_l \cap \gamma_i \neq \emptyset\}$.

Let ξ be a partition of N to r subsets, such that for every $i, j \in I \in \xi$, $i \neq j$, $\gamma_i^\pi \cap \gamma_j^\pi = \emptyset$. Assume r is the *minimal* cardinality of such a partition. For every $I \in \xi$ let $H_I = \bigcup_{i \in I} J_i$. That is, each H_I is a set of indices of parts A_l in π that should be allocated to the buyers in I in order for each of them to get the goods they received in the optimal allocation γ. Note that if $I \neq J$, $H_I \cap H_J \neq \emptyset$, otherwise we can join I and J together in contradiction to the minimality of the cardinality of ξ. Hence, $\Delta = (H_I)_{I \in \xi}$ is a family of subsets of $\{1,\dots,k\}$ that satisfies that any two subsets in Δ intersect. Furthermore, the second condition for feasibility is also satisfied, because for any given $l \in \{1,\dots,k\}$ there are at most $|A_l|$ buyers i with $\gamma_i \cap A_l \neq \emptyset$, and hence at most $|A_l|$ parts $I \in \xi$ such that $l \in H_I$. Thus, Δ is a feasible family for π with $s(\Delta) = r$.

Every H_I, $I \in \xi$ defines a Σ_π-allocation. In this allocation every $i \in I$ receives γ_i^π, and the seller receives all other goods. Therefore $\sum_{i\in I} v_i(\gamma_i^\pi) \leqslant S_\pi(v)$ for every $I \in \xi$. Hence,

$$S_{\max}(v) \leqslant \sum_{i\in N} v_i\left(\gamma_i^\pi\right) = \sum_{I\in\xi}\sum_{i\in I} v_i\left(\gamma_i^\pi\right) \leqslant \sum_{I\in\xi} S_\pi(v) = r S_\pi(v).$$

Next, we prove that $r_\pi \geqslant \max s(\Delta)$. It suffices to show that for every feasible family Δ for π there exists a profile of valuations $v = (v_1,\dots,v_n) \neq 0$ for some number n of buyers satisfying

$$S_{\max}(v) \geqslant s(\Delta)\cdot S_\pi(v).$$

Let $\Delta = (H_i)_{i=1}^s$ be a feasible family for π. By the second condition of feasibility, we can associate with each H_i a set of goods B_i containing one good from each A_l such that $l \in H_i$, in such a way that the sets B_i are pairwise disjoint. By the first condition of feasibility, for every $1 \leqslant i, j \leqslant s$ there can be no two disjoint sets $C_i, C_j \in \Sigma_\pi$ such that $B_i \subseteq C_i$, $B_j \subseteq C_j$.

Now, we take $n = s$ buyers, and let buyer i have the valuation $v_i = w_{B_i}$. Then $S_{\max}(v) = s$ whereas $S_\pi(v) = 1$. □

Theorem 2 reduces the determination of the economic inefficiency measure r_π to a purely combinatorial problem. However, this combinatorial problem does not admit an easy solution.[19] Nevertheless, we will use Theorem 2 to calculate r_π in some special cases, and to obtain a general upper bound for it which is tight in infinitely many cases.

The following proposition determines r_π for partitions π with a small number of parts. We use the notations $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ for the lower and upper integer rounding functions, respectively.

Proposition 1. *Let $|A| = m$, and let $\pi = \{A_1,\dots,A_k\}$ be a partition of A into k non-empty sets.*

- *If $k = 1$ then $r_\pi = m$.*
- *If $k = 2$ then $r_\pi = \max\{|A_1|, |A_2|\}$. Consequently, the minimum of r_π over all partitions of A into 2 parts is $\lceil m/2\rceil$.*
- *If $k = 3$ then $r_\pi = \max\{|A_1|, |A_2|, |A_3|, \lfloor m/2\rfloor\}$. Consequently, the minimum of r_π over all partitions of A into 3 parts is $\lfloor m/2\rfloor$.*

Proof. In each case, we determine the maximum of $s(\Delta)$ over all families Δ that are feasible for π.

[19] The special case of this problem, in which $|A_i| = |A_j|$ for all $A_i, A_j \in \pi$, has been treated in the combinatorial literature using a different but equivalent terminology (see, e.g. Füredi, 1990). But even in this case, a precise formula for $\max s(\Delta)$ seems out of reach.

For $k = 1$, a feasible family consists of at most $|A_1| = m$ copies of $\{1\}$, and therefore $\max s(\Delta) = m$.

A feasible family for $k = 2$ cannot contain two sets, H_i and H_j, such that $1 \notin H_i$ and $2 \notin H_j$, because such sets would be disjoint. Hence, for any feasible family Δ, either all sets contain 1 or all of them contain 2. Therefore, $s(\Delta) \leqslant \max\{|A_1|, |A_2|\}$. On the other hand, feasible families of size $|A_1|, |A_2|$ trivially exist.

Suppose $k = 3$, and denote

$$\beta_l = |A_l| \quad \text{for } l = 1, 2, 3.$$

We first show that $s(\Delta) \leqslant \max\{\beta_1, \beta_2, \beta_3, \lfloor m/2 \rfloor\}$ for every feasible family Δ. If Δ contains some singleton $\{l\}$, then all sets in Δ must contain l, and hence $s(\Delta) \leqslant \beta_l$. Otherwise, Δ consists of s_{12} copies of $\{1, 2\}$, s_{13} copies of $\{1, 3\}$, s_{23} copies of $\{2, 3\}$, and s_{123} copies of $\{1, 2, 3\}$, for some non-negative integers $s_{12}, s_{13}, s_{23}, s_{123}$. We have the following inequalities:

$$s_{12} + s_{13} + s_{123} \leqslant \beta_1, \qquad s_{12} + s_{23} + s_{123} \leqslant \beta_2, \qquad s_{13} + s_{23} + s_{123} \leqslant \beta_3.$$

Upon adding these inequalities we obtain

$$2(s_{12} + s_{13} + s_{23}) + 3s_{123} \leqslant m,$$

which implies

$$s(\Delta) = s_{12} + s_{13} + s_{23} + s_{123} \leqslant \lfloor m/2 \rfloor.$$

We show next that there exists a feasible family Δ with $s(\Delta) = \max\{\beta_1, \beta_2, \beta_3, \lfloor m/2 \rfloor\}$. If this maximum is one of the β_l's, this is trivial. So assume that $\beta_l < \lfloor m/2 \rfloor$ for $l = 1, 2, 3$. If m is even then the family Δ that consists of

$$s_{12} = \frac{\beta_1 + \beta_2 - \beta_3}{2} \quad \text{copies of } \{1, 2\}, \qquad s_{13} = \frac{\beta_1 + \beta_3 - \beta_2}{2} \quad \text{copies of } \{1, 3\},$$
$$s_{23} = \frac{\beta_2 + \beta_3 - \beta_1}{2} \quad \text{copies of } \{2, 3\},$$

is feasible (note that the prescribed numbers are non-negative because $\beta_l < \lfloor m/2 \rfloor$ for $l = 1, 2, 3$, and they are integers because $\beta_1 + \beta_2 + \beta_3 = m$ is even). The size of this family is $s(\Delta) = s_{12} + s_{13} + s_{23} = m/2$. If m is odd, we make slight changes in the values of s_{12}, s_{13}, s_{23}: we add $1/2$ to one of them and subtract $1/2$ from the other two. In this way we get a family Δ with $s(\Delta) = \lfloor m/2 \rfloor$. □

We see from Proposition 1 that if we use partitions into two parts (entailing a communication complexity of 4), the best we can do in terms of economic efficiency is $r_\pi = \lceil m/2 \rceil$, and this is achieved by partitioning A into equal or nearly equal parts. Allowing for three parts (and therefore a communication complexity of 8) permits only a small gain in r_π (in fact, no gain at all when m is even).

We will now state the two parts of our main result.

Theorem 3. *Let $\pi = \{A_1, \ldots, A_k\}$ be a partition of A into k non-empty sets of maximum size $\beta(\pi)$. (That is, $\beta(\pi) = \max\{|A_1|, \ldots, |A_k|\}$.) Then*

$$r_\pi \leqslant \beta(\pi) \cdot \varphi(k), \quad \textit{where } \varphi(k) = \max_{j=1,\ldots,k} \min\{j, k/j\}.$$

The proof of Theorem 3 is given in the following subsection.

Note that

$$\varphi(k) \leqslant \sqrt{k}.$$

In particular, if all sets in π have equal size m/k, we obtain the upper bound

$$r_\pi \leqslant \frac{m}{\sqrt{k}}.$$

Now, consider the case when, for some non-negative integer q, we have

$$k = q^2 + q + 1, \tag{4.3}$$

$$|A_i| = q + 1 \quad \text{for } i = 1, \ldots, k. \tag{4.4}$$

In this case

$$\varphi(k) = \frac{q^2 + q + 1}{q + 1},$$

and hence the upper bound of Theorem 3 takes the form

$$r_\pi \leqslant k.$$

The second part of our main result implies that in infinitely many of these cases this upper bound is tight.

Theorem 4. *Let $\pi = \{A_1, \ldots, A_k\}$ be a partition that satisfies* (4.3) *and* (4.4) *for some q which is either* 0 *or* 1 *or of the form p^l where p is a prime number and l is a positive integer. Then*

$$r_\pi = k.$$

We prove Theorems 3 and 4 in the following subsection.

4.2. Proofs of Theorems 3 and 4

We begin with some preparations. Let $\Delta = (H_i)_{i=1}^s$ be a family of (not necessarily distinct) subsets of $\{1, \ldots, k\}$. A vector of non-negative numbers $\delta = (\delta_i)_{i=1}^s$ is called a *semi balanced*[20] vector for Δ if for every $l \in \{1, \ldots, k\}$,

$$\sum_{i:\, l \in H_i} \delta_i \leqslant 1.$$

[20] This concept is equivalent to what is called a fractional matching in combinatorics. We chose the term semi balanced, because balanced vectors, defined by requiring equality instead of weak inequality, are a familiar concept in game theory (see, e.g., Shapley, 1967).

Proposition 2. *Let $\Delta = (H_i)_{i=1}^s$ be a family of (not necessarily distinct) subsets of $\{1,\dots,k\}$ such that $H_i \cap H_j \neq \emptyset$ for every $1 \leqslant i, j \leqslant s$. Let $\delta = (\delta_i)_{i=1}^s$ be a semi balanced vector for Δ. Then*

$$\sum_{i=1}^{s} \delta_i \leqslant \varphi(k), \quad \textit{where } \varphi(k) = \max_{j=1,\dots,k} \min\{j, k/j\}.$$

Proof. Assume without loss of generality that $h = |H_1|$ is the minimal number of elements in a member of Δ. The proposition will be proved if we prove the following two claims:

Claim 1. $\sum_{i=1}^s \delta_i \leqslant h$.

Claim 2. $\sum_{i=1}^s \delta_i \leqslant k/h$.

Proof of Claim 1. Let $z = \sum_{l \in H_1} \sum_{i: l \in H_i} \delta_i$. As every H_i intersects H_1, every δ_i appears in z at least once. Therefore, $z \geqslant \sum_{i=1}^s \delta_i$. Because δ is semi balanced, $\sum_{i:\, l \in H_i} \delta_i \leqslant 1$ for every l, and in particular for $l \in H_1$. Hence, $z \leqslant \sum_{l \in H_1} 1 = h$. $\square$

Proof of Claim 2. Let $w = \sum_{l=1}^k \sum_{i:\, l \in H_i} \delta_i$. Every δ_i appears in w exactly $|H_i|$ times. Since $|H_i| \geqslant h$ for every i, we have $w \geqslant h \sum_{i=1}^s \delta_i$. On the other hand, as in the proof of Claim 1, we obtain $w \leqslant \sum_{l=1}^k 1 = k$. Combining the two inequalities, we get $\sum_{i=1}^s \delta_i \leqslant k/h$. $\square$

Therefore,

$$\sum_{i=1}^{s} \delta_i \leqslant \min\{h, k/h\} \leqslant \varphi(k). \qquad \square$$

We are now ready for the proof of Theorem 3.

Proof of Theorem 3. Let $\pi = \{A_1, \dots, A_k\}$ be a partition of A into k non-empty sets of maximum size $\beta(\pi)$. We have to prove that $r_\pi \leqslant \beta(\pi) \cdot \varphi(k)$. By Theorem 2, it suffices to show that for every feasible family Δ for π, we have

$$s(\Delta) \leqslant \beta(\pi) \cdot \varphi(k).$$

Let $\Delta = (H_i)_{i=1}^s$ be such a family. Consider the vector $\delta = (\delta_i)_{i=1}^s$ with

$$\delta_i = \frac{1}{\beta(\pi)}, \quad i = 1, \dots, s.$$

By the second condition of feasibility, this vector is semi balanced. Hence we may apply Proposition 2 and conclude that

$$\sum_{i=1}^{s} \delta_i \leqslant \varphi(k), \quad \text{or equivalently,} \quad \frac{s}{\beta(\pi)} \leqslant \varphi(k),$$

as required. □

In order to prove Theorem 4 we invoke a result about finite geometries (see, e.g., Dembowski, 1968). A *finite projective plane* of order q is a system consisting of a set Π of points and a set Λ of lines (in this abstract setting, a line is just a set of points, i.e., $L \subseteq \Pi$ for every $L \in \Lambda$), satisfying the following conditions:

- $|\Pi| = |\Lambda| = q^2 + q + 1$.
- Every point is incident to $q + 1$ lines and every line contains $q + 1$ points.
- There is exactly one line containing any two points, and there is exactly one point common to any two lines.

Such a system does not exist for every q. However, it trivially exists for $q = 0$ (a single point) and for $q = 1$ (a triangle) and it is known to exist for every q of the form $q = p^l$, where p is a prime number and l is a positive integer. The first non-trivial example, corresponding to $q = 2$, is called the *Fano plane*:

$$\Pi = \{1, 2, 3, 4, 5, 6, 7\}, \qquad \Lambda = \{124, 235, 346, 457, 561, 672, 713\}.$$

Proof of Theorem 4. Let $\pi = \{A_1, \ldots, A_k\}$ be a partition that satisfies (4.3) and (4.4) for some q which is either 0 or 1 or of the form p^l where p is a prime number and l is a positive integer. As $r_\pi \leqslant k$ follows from Theorem 3 (see the discussion preceding the statement of Theorem 4), we need to prove only that $r_\pi \geqslant k$. By Theorem 2, it suffices to show that there exists a family Δ with $s(\Delta) = k$ which is feasible for π. Such a family is given by the system of lines of a projective plane of order q, when the points are identified with $1, \ldots, k$. □

4.3. More on the ranking of equilibria

The tradeoff between communication complexity and economic efficiency, as delineated above, may be made concrete by the following scenario. Suppose that a set A of m goods is given, and we are in a position to recommend to the buyers an equilibrium strategy. Assume further that a certain level M of communication complexity is considered the maximum acceptable level. If we are going to recommend a partition-based equilibrium f^π, then the number of parts in π should be at most $k = \lfloor \log_2 M \rfloor$. From the viewpoint of economic efficiency, we would like to choose such a partition π with r_π as low as possible. Which partition should it be?

According to Theorem 3, we obtain the lowest guarantee on r_π by making the maximum size of a part in π as small as possible, which means splitting A into k equal (or nearly equal, depending on divisibility) parts. This leads to the question whether, for given m and k, the lowest value of r_π itself (not of our upper bound) over all partitions π of A into k parts is achieved at an equipartition, i.e., a partition $\pi = \{A_1, \ldots, A_k\}$ such that $\lfloor m/k \rfloor \leqslant |A_i| \leqslant \lceil m/k \rceil$, $i = 1, \ldots, k$.

While Proposition 1 gives an affirmative answer for $k = 1, 2, 3$, it turns out, somewhat surprisingly, that this is not always the case. This is shown in the following example.

Example 2. Let $m = 21$ and $k = 7$. If π is an equipartition of the 21 goods into 7 triples then, by Theorem 4, $r_\pi = 7$. Consider now a partition $\pi' = \{A_1, \dots, A_7\}$ in which

$$|A_1| = 2, |A_2| = 4, |A_3| = \cdots = |A_7| = 3.$$

We claim that $r_{\pi'} \leqslant 6$.

In order to prove this, it suffices to show that there exists no feasible family of 7 sets for π'. Suppose, for the sake of contradiction, that $\Delta = (H_i)_{i=1}^7$ is such a family. Let H_i be an arbitrary set in Δ. It follows from the second condition of feasibility that if H_i contains the element 1 then it shares it with at most one other set in Δ. Similarly, if H_i contains the element 2 then it shares it with at most three other sets in Δ. For $l = 3, \dots, 7$, if H_i contains the element l then it shares it with at most two other sets in Δ. This implies that H_i must contain at least three elements (because it must share an element with every other set, and $3 + 2 < 6$). Moreover, if H_i contains exactly three elements and one of them is 1, then it also contains 2 (since $1 + 2 + 2 < 6$). On the other hand, we have

$$\sum_{i=1}^{7} |H_i| = \sum_{i=1}^{7} \sum_{l \in H_i} 1 = \sum_{l=1}^{7} \sum_{i:\ l \in H_i} 1 = \sum_{l=1}^{7} \bigl|\{i:\ l \in H_i\}\bigr| \leqslant \sum_{l=1}^{7} |A_l| = 21.$$

Since every H_i has at least three elements, it follows that every H_i has exactly three elements, and all the weak inequalities $|\{i:\ l \in H_i\}| \leqslant |A_l|$ must in fact hold as equalities. In particular, there exist two sets in Δ, say H_i and H_j, that contain the element 1. By the above, they both contain 2 as well. Let l be the third element of H_i. Then among the remaining five sets in Δ, the set H_i shares the element 1 with none of them, it shares the element 2 with two of them, and the element l with at most two of them. This contradicts the fact that H_i intersects every other set in Δ.

It can be checked that in fact $r_{\pi'} = 6$ and this is the lowest achievable value among all partitions of 21 goods into 7 sets. We omit the detailed verification of this.

The tradeoff between communication complexity and economic efficiency was quantitatively analyzed above only for partition-based equilibria. It is natural to ask whether it is possible to beat this tradeoff using the more general bundling equilibria. The answer is, in a sense made precise below: sometimes yes, but not by much.

Example 3. Assume that the number of goods m is even, and let the set of goods A be partitioned into two equal parts B and C. Consider $\Sigma \subseteq 2^A$ defined by

$$\Sigma = \bigl\{D \subseteq A:\ |D \cap B| = |D \cap C|\bigr\}.$$

It is easy to check that Σ is a quasi-field, and hence it induces a bundling equilibrium. The communication complexity is

$$|\Sigma| = \sum_{j=0}^{m/2} \binom{m/2}{j}^2 = \sum_{j=0}^{m/2} \binom{m/2}{j}\binom{m/2}{m/2 - j} = \binom{m}{m/2}.$$

We claim that $r_\Sigma = 2$. That $r_\Sigma \geqslant 2$ can be seen by taking two buyers with valuations w_B and w_C, respectively. To see that $r_\Sigma \leqslant 2$, suppose that v is a profile of valuations for a set of buyers N, and let γ be an optimal allocation. Split the set N into two sets:

$$N_B = \{i \in N\colon |\gamma_i \cap B| \geqslant |\gamma_i \cap C|\}, \qquad N_C = \{i \in N\colon |\gamma_i \cap B| < |\gamma_i \cap C|\}.$$

Note that the sets of goods γ_i, $i \in N_B$, can be expanded to pairwise disjoint sets of goods that belong to Σ. In other words, there exists a Σ-allocation ξ such that $\gamma_i \subseteq \xi_i$ for every $i \in N_B$. Similarly, there exists a Σ-allocation η such that $\gamma_i \subseteq \eta_i$ for every $i \in N_C$. Hence

$$S_{\max}(v) = \sum_{i \in N} v_i(\gamma_i) = \sum_{i \in N_B} v_i(\gamma_i) + \sum_{i \in N_C} v_i(\gamma_i) \leqslant \sum_{i \in N} v_i(\xi_i) + \sum_{i \in N} v_i(\eta_i) \leqslant 2S_\Sigma(v).$$

Thus, $r_\Sigma \leqslant 2$.

We claim further that if a partition π of A satisfies $r_\pi \leqslant 2$ then $|\Sigma_\pi| \geqslant 2^{m-2}$. Indeed, suppose $\pi = \{A_1, \ldots, A_k\}$. It is easy to find a feasible family of 3 sets for π if one of the A_l's has three or more elements, or if three of the A_l's have two elements each. Therefore, $r_\pi \leqslant 2$ implies that at most two of the sets $A_1, \ldots, A_k$ have two elements and the rest are singletons. Thus $k \geqslant m - 2$ and $|\Sigma_\pi| \geqslant 2^{m-2}$.

Since $\binom{m}{m/2} < 2^{m-2}$ for all even $m \geqslant 10$, we have the following conclusion: If $m \geqslant 10$ then every partition-based equilibrium that matches the economic efficiency of f^Σ has a higher communication complexity than f^Σ. In other words, the quasi-field Σ offers an efficiency–complexity combination that cannot be achieved or improved upon (in the Pareto sense) by any field.

The above example notwithstanding, the efficiency–complexity combinations which arise from arbitrary quasi-fields are still subject to a tradeoff that is not much better than for fields. This is the content of our final remark.

Remark 2. Let $m = |A|$ and let k be a positive integer. Any quasi-field $\Sigma \subseteq 2^A$ with $r_\Sigma \leqslant m/k$ must contain a partition of A into k non-empty parts, and therefore must satisfy $|\Sigma| \geqslant 2^k$.

Proof. Let there be m buyers, each with valuation w_a for a distinct $a \in A$. For this v we have $S_{\max}(v) = m$. If $r_\Sigma \leqslant m/k$ then we must have $S_\Sigma(v) \geqslant k$. Hence an optimal Σ-allocation has to assign non-empty bundles of goods to at least k buyers. Thus Σ contains k pairwise disjoint non-empty sets of goods, and therefore, being a quasi-field, also a partition of A into k non-empty parts. □

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