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Characterization of ex post equilibrium in the VCG combinatorial auctions ☆

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Abstract

We prove that when the number of (potential) buyers is at least three, every ex post equilibrium in the Vickrey–Clarke–Groves combinatorial auction mechanisms is a bundling equilibrium and is symmetric. This complements a theorem proved by Holzman, Kfir-Dahav, Monderer, and Tennenholtz (2003), according to which, the symmetric bundling equilibria are precisely those defined by a quasi-field.

JEL classification: C72; D44; D78

1. Introduction

In a combinatorial auction, a number of goods are being offered for sale to a group of agents whose valuations for the various bundles of goods may not be separable (i.e., the utility that an agent derives from owning two of the goods need not be the sum of the utilities that he derives from owning each of them separately). A Vickrey–Clarke–Groves (VCG) mechanism for such an auction (Vickrey, 1961; Clarke, 1971; Groves, 1973) requires the agents to reveal their valuation functions (which are their private information), and based on the announced valuations it specifies an efficient allocation of the goods and the amount to be paid by each agent to the seller. The main feature of these

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mechanisms is that, thanks to a judicious choice of the monetary transfers, revealing one's true valuation function is a dominant strategy.

A major difficulty that arises in applying these mechanisms is the prohibitive communication complexity: when there are m goods, every agent has to communicate to the organizer 2^m numbers, his valuations for each and every bundle of goods. This cannot be helped if the agents are to use their dominant strategies. However, it was shown in (Holzman et al., 2003) that there exist other, non truth-telling strategies, which have a lower communication complexity and still possess a high degree of incentive compatibility.[1] Namely, each of these strategies induces a symmetric ex post equilibrium. This means that if an agent assumes that the other agents use this strategy, it is optimal for him to use it as well, regardless of the other agents' valuations and of the number of them who actually participate in the auction.

The strategies considered in (Holzman et al., 2003) were all of the following simple type: a certain subfamily Σ of the family of all 2^m bundles of goods is designated in advance, and the strategy f^{Σ} is to report only the (true) utilities the agent assigns to bundles in Σ, with the interpretation that his utility for any other bundle B equals the maximum of his reported utilities over all subsets C of B which lie in Σ. It was shown in (Holzman et al., 2003) that when there are at least three buyers, the strategy f^{Σ} induces a symmetric ex post equilibrium if and only if the subfamily Σ is a quasi-field, i.e., it is closed under complements and disjoint unions. This means that when Σ is a quasi-field—and only then—the agents will be willing to use the strategy f^{Σ} not only because it reduces the communication burden on the system but also because it is selfishly rational for them to do so. In this case, the resulting equilibrium was called a bundling equilibrium.

This provided a characterization of symmetric equilibrium strategies within the class of strategies of type f^{Σ}, but left open the possibility that there might exist other strategies, not of this type, which also induce symmetric ex post equilibria, and moreover that there might exist non-symmetric equilibrium profiles (with different buyers using different strategies). In this paper, we rule out these possibilities and obtain a complete characterization of all ex post equilibrium profiles in the VCG combinatorial auctions. All of them are symmetric and are bundling equilibria. Our result is proved under the assumptions of private values, no informational externalities, no allocative externalities, quasi-linear utilities, free disposal, and zero reserve prices.[2]

The reader should not confuse our model in which, in equilibrium, the buyers restrict their reports to a certain class of bundles, with a model in which the auction's organizer restricts the set of allowable reports. As was noted in (Holzman et al., 2003), in the latter case *every* class of bundles determines a truth-telling auction mechanism that uses the VCG scheme.

In this paper we do not touch the important question of how a certain equilibrium is picked up by the participants. This is a central question in models in economics with

[1] The use of these strategies typically entails a loss of economic efficiency. The tradeoff between economic efficiency and communication efficiency was investigated in detail in (Holzman et al., 2003).

[2] A characterization of symmetric bundling equilibria is given in (Holzman et al., 2003) for the case of non-zero reserve prices as well. We conjecture that with appropriate modifications, our main theorem can also be proved in this context.

multiple equilibria. The classical approaches (e.g., focal points or learning schemes) may be applied as usual.

The last section of this paper is devoted to a discussion of the somewhat related literature in economics and in computer science. In particular we discuss the seminal, and surprisingly not yet well known paper Roberts (1979).

2. The model and the result

In a combinatorial auction there is a seller, denoted by 0, who wishes to sell a set of m goods $A = \{a_1, \ldots, a_m\}$, $m \geqslant 1$, that she owns. We denote by 2^A the family of all bundles of goods (i.e., subsets of A). There is a set of n (potential[3]) buyers $N = \{1, \ldots, n\}$, $n \geqslant 1$. An *allocation* of the goods is an ordered partition $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ of A.[4] We denote by Γ the set of all allocations. We assume:

- no allocative externalities;
- free disposal;
- private values;
- no informational externalities;
- quasi-linear utilities.

That is, a buyer's *valuation function* is a function $v : 2^A \to \Re$, satisfying $v(\emptyset) = 0$ and

$$B \subseteq C, \quad B, C \in 2^A \quad \Rightarrow \quad v(B) \leqslant v(C).$$

When buyer i with the valuation function v_i receives the set of goods B, and pays a monetary transfer $c_i \in \Re$ his utility is $v_i(B) - c_i$. Every buyer knows his valuation function.

We denote by V the set of all possible valuation functions. The set V^N, the n-fold product of the set V, is the set of all profiles of valuations $\mathbf{v} = (v_1, \ldots, v_n)$, one for each buyer.

An *auction mechanism* $AM = (M, d, c)$ is defined by a message space M, by an allocation function $d : M^N \to \Gamma$, and by a transfer function $c : M^N \to \Re^N$. If for every $j \in N$ buyer j sends the message m_j, resulting in the message n-tuple $\mathbf{m} = (m_1, \ldots, m_n) \in M^N$, buyer i receives the set of goods $d_i(\mathbf{m})$, and pays a transfer $c_i(\mathbf{m})$ to the seller (here $d_i(\mathbf{m})$ and $c_i(\mathbf{m})$ are the i-th components of $d(\mathbf{m})$ and $c(\mathbf{m})$, respectively). His utility $u_i^{AM}(v_i, \mathbf{m})$ is then given by

$$u_i^{AM}(v_i, \mathbf{m}) = v_i\big(d_i(\mathbf{m})\big) - c_i(\mathbf{m}).$$

The behavior of buyer i in a mechanism AM is described by a *strategy* $b_i : V \to M$. Hence, the utility of i when he uses the strategy b_i, his valuation function is v_i, and the other buyers

[3] By this we mean that the set of buyers who actually participate in the auction will be some subset N' of N. For simplicity of presentation we introduce all concepts for the case $N' = N$, but we will later require the analogous concepts for arbitrary subsets N'.

[4] Note that the goods are allocated among the buyers and the seller. We assume, however, that the seller derives no utility from keeping any of the goods, and that she does not set strategic reserve prices.

send the $(n-1)$-tuple of messages $\mathbf{m}_{-i} \in M^{N\setminus\{i\}}$ is $u_i^{AM}(v_i, (b_i(v_i), \mathbf{m}_{-i}))$. A strategy b_i is a *dominant* strategy for i if for every $v_i \in V$, and for every $\mathbf{m}_{-i} \in M^{N\setminus\{i\}}$,

$$u_i^{AM}\big(v_i, \big(b_i(v_i), \mathbf{m}_{-i}\big)\big) \geqslant u_i^{AM}\big(v_i, (m_i, \mathbf{m}_{-i})\big) \quad \text{for every } m_i \in M.$$

A strategy profile $(b_1, \ldots, b_n)$ forms an *ex post equilibrium* if for every buyer i, and for every profile of valuations $\mathbf{v} = (v_1, \ldots, v_n) \in V^N$,

$$u_i^{AM}\big(v_i, \big(b_i(v_i), \mathbf{b}_{-i}(\mathbf{v}_{-i})\big)\big) \geqslant u_i^{AM}\big(v_i, \big(m_i, \mathbf{b}_{-i}(\mathbf{v}_{-i})\big)\big) \quad \text{for every } m_i \in M,$$

where $\mathbf{v}_{-i} = (v_j)_{j \neq i}$, and $\mathbf{b}_{-i}(\mathbf{v}_{-i}) = (b_j(v_j))_{j \neq i}$. The profile $(b_1, \ldots, b_n)$ is *symmetric* if $b_i = b_j$ for every two buyers $i, j \in N$.

It is helpful to understand the definition of ex post equilibrium in the following way. For a strategy b_i of i we denote by $b_i(V) \subseteq M$ the image of b_i. That is, $b_i(V)$ is the set $\{m \in M\colon b_i(v) = m \text{ for some } v \in V\}$ of all messages that i may send when he uses b_i. The strategy profile $(b_1, \ldots, b_n)$ forms an ex post equilibrium if and only if for every buyer i, and for every $v_i \in V$, sending the message $b_i(v_i)$ is a best response for i, when he has valuation function v_i, against any $(n-1)$-tuple of messages of the other buyers that belong to the images $b_j(V)$ of the respective strategies b_j, $j \neq i$. When the strategies are not onto M, this requirement on b_i is weaker than being dominant.

In a *direct* auction mechanism $M = V$, that is, every buyer is required to report a valuation function. A direct mechanism is *truth-telling* if for every buyer i, the strategy $b_i(v_i) = v_i$ of revealing the true valuation is a dominant strategy.

Given two auction mechanisms $AM^1 = (M^1, d^1, c^1)$ and $AM^2 = (M^2, d^2, c^2)$ and two strategy profiles $\mathbf{b}^1 = (b_1^1, \ldots, b_n^1)$ and $\mathbf{b}^2 = (b_1^2, \ldots, b_n^2)$ in AM^1 and AM^2 respectively, we say that the pairs $(AM^1, \mathbf{b}^1)$, $(AM^2, \mathbf{b}^2)$ are *economically equivalent* if they induce the same output functions. That is,

$$\big(d^1\big(\mathbf{b}^1(\mathbf{v})\big), c^1\big(\mathbf{b}^1(\mathbf{v})\big)\big) = \big(d^2\big(\mathbf{b}^2(\mathbf{v})\big), c^2\big(\mathbf{b}^2(\mathbf{v})\big)\big) \quad \text{for every } \mathbf{v} \in V^N,$$

where $\mathbf{b}^k(\mathbf{v}) = (b_1^k(v_1), \ldots, b_n^k(v_n))$ for $k = 1, 2$.

The following principle is well known:

The revelation principle. Given an auction mechanism AM^1 and an ex post equilibrium b^1 in this mechanism, one can construct a direct and truth-telling mechanism AM^2 so that the pair $(AM^2, \mathbf{b}^2)$, where $\mathbf{b}^2$ is the strategy profile in which every buyer reveals his true valuation, is economically equivalent to the pair $(AM^1, \mathbf{b}^1)$.[5]

Because of the revelation principle, the concept of ex post equilibrium in private values models has largely been ignored in the economics literature. However, as noted in (Holzman et al., 2003), being economically equivalent does not imply equivalence from the computer science point of view: The difference is due to the different communication or computational complexities induced by the two mechanisms and the respective strategy profiles. For example, a direct mechanism and a truth revealing strategy profile in it require each agent to communicate a valuation function, that is a vector of 2^m numbers, while

[5] This form of the revelation principle holds true due to our private values assumption.

an economically equivalent mechanism—strategy profile pair may induce less (or more) communication complexity.

For an allocation $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \Gamma$ and a profile of valuations $\mathbf{v} = (v_1, \ldots, v_n) \in V^N$ we denote by $S(\mathbf{v}, \gamma)$ the *total social surplus* of the buyers, that is,

$$S(\mathbf{v}, \gamma) = \sum_{i \in N} v_i(\gamma_i).$$

We also denote

$$S_{\max}(\mathbf{v}) = \max_{\gamma \in \Gamma} S(\mathbf{v}, \gamma),$$

and we refer to an allocation γ that achieves this maximum as an *optimal* allocation for $\mathbf{v}$.

A Vickrey–Clarke (VC) auction mechanism is a direct mechanism described as follows. Based on the reported valuations $\hat{\mathbf{v}} = (\hat{v}_1, \ldots, \hat{v}_n) \in V^N$ the mechanism selects an allocation $d(\hat{\mathbf{v}}) = (d_0(\hat{\mathbf{v}}), \ldots, d_n(\hat{\mathbf{v}})) \in \Gamma$, which is optimal for $\hat{\mathbf{v}}$. Because ties are possible, such an allocation may not be unique, and therefore there is more than one VC mechanism. Every function $d : V^N \to \Gamma$ satisfying $S(\hat{\mathbf{v}}, d(\hat{\mathbf{v}})) = S_{\max}(\hat{\mathbf{v}})$ for all $\hat{\mathbf{v}} \in V^N$ determines uniquely a VC mechanism, which we refer to as the VC mechanism d. This mechanism assigns to buyer i the bundle $d_i(\hat{\mathbf{v}})$ and makes him pay $c_i^d(\hat{\mathbf{v}})$ to the seller, where

$$c_i^d(\hat{\mathbf{v}}) = \max_{\gamma \in \Gamma} \sum_{j \neq i} \widehat{v_j}(\gamma_j) - \sum_{j \neq i} \widehat{v_j}\big(d_j(\hat{\mathbf{v}})\big).$$

This represents the loss to the other agents' total surplus caused by agent i's presence.

A Vickrey–Clarke–Groves (VCG) auction mechanism is a direct mechanism parametrized by a VC mechanism d, and by an n-tuple $\mathbf{h} = (h_1, \ldots, h_n)$ of functions $h_i : V^{N \setminus \{i\}} \to \Re$. The mechanism selects an allocation according to the allocation function d, and the transfer function of buyer i is

$$c_i^{d,\mathbf{h}}(\hat{\mathbf{v}}) = c_i^d(\hat{\mathbf{v}}) + h_i(\hat{\mathbf{v}}_{-i}).$$

Hence, a VC auction mechanism is a special type of VCG auction mechanism, in which h_i is the function that is identically equal to zero for every i. It is well known that every VCG auction mechanism is truth-telling. Obviously, every VCG mechanism is efficient, in the sense that truth-revealing behavior results in an optimal allocation.

Definition. Fix a set of goods A, and a set of potential buyers $N = \{1, \ldots, n\}$. Let $(b_1, \ldots, b_n)$ be a profile of strategies, i.e., $b_i : V \to V$ for each $i \in N$. We say that $(b_1, \ldots, b_n)$ forms an *ex post equilibrium in the VCG mechanisms*, if for every subset $N' \subseteq N$, the profile $(b_i)_{i \in N'}$ is an ex post equilibrium in every VCG mechanism for the buyers in N'.

There are two non-standard aspects of this definition that require some elaboration. First, the equilibrium property is required to hold not only for the entire profile of strategies, but also for its restrictions to arbitrary subsets of the set of potential buyers.[6] Conceptually,

[6] It is interesting to note that this requirement makes a difference for our main result (see Section 5.2). We also remark that in the symmetric case, i.e., when considering equilibria of the form $(b, \ldots, b)$, one can take care

this strengthens the notion of ex post equilibrium in the following sense: A strategy in such a profile is a best response regardless of the subset of potential buyers who actually participate in the auction. In other words, a buyer is justified in using his strategy if he believes that each of the other potential buyers either uses his strategy in the profile or stays out. On a technical level, the main result of this paper is valid also without this strengthening, if one adds the following technical condition on the strategies used: $0 \in b_i(V)$ for all i, where 0 is the valuation function that is identically zero. This is a very mild condition: if a buyer's true valuation function is 0, it seems hard to come up with a reason for him to report any valuation $v \neq 0$.

The second non-standard aspect of the above definition is the requirement that the profile should form an ex post equilibrium not just in a given VCG mechanism but in every VCG mechanism. It would make no difference if we focused on the VC mechanisms[7] rather than the more general VCG mechanisms, because the additional part **h** in the transfer functions does not affect the equilibrium property. However, the particular choice of the allocation function d may matter. The intuitive meaning of insisting that the profile be an equilibrium regardless of d, is that we do not want the property of being in equilibrium to hinge on the particular way in which the allocation function breaks ties.

A special type of strategies was considered in (Holzman et al., 2003). A *bundling strategy* for buyer i is parametrized by a subfamily Σ_i of 2^A such that $\emptyset \in \Sigma_i$, and is denoted by f^{Σ_i}. It maps every $v \in V$ to $v^{\Sigma_i} \in V$ defined by

$$v^{\Sigma_i}(B) = \max_{C \subseteq B,\, C \in \Sigma_i} v(C) \quad \text{for every } B \in 2^A.$$

This has the effect of pretending that the agent cares only about bundles in Σ_i (for which he announces his true valuation), and derives his valuation for other bundles by maximizing over the bundles in Σ_i that they contain. If the profile of strategies $(f^{\Sigma_1}, \ldots, f^{\Sigma_n})$ forms an ex post equilibrium in the VCG mechanisms it is called a *bundling equilibrium*. If in addition $\Sigma_i = \Sigma$ for every $i \in N$, we say that the corresponding equilibrium is a *symmetric* bundling equilibrium in the VCG mechanisms (induced by the bundling strategy f^{Σ}).

For what choices of $\Sigma \subseteq 2^A$ does the profile of strategies $(f^{\Sigma}, \ldots, f^{\Sigma})$ form an ex post equilibrium? This question was answered in (Holzman et al., 2003). A subfamily Σ of 2^A such that $\emptyset \in \Sigma$ is a *quasi-field* if it satisfies the following two conditions:

$$B \in \Sigma \quad \Rightarrow \quad A \setminus B \in \Sigma,$$
$$B, C \in \Sigma \quad \text{and} \quad B \cap C = \emptyset \quad \Rightarrow \quad B \cup C \in \Sigma.$$

Theorem 1 (Holzman et al., 2003). *Let $n \geqslant 3$. Let Σ be a subfamily of 2^A such that $\emptyset \in \Sigma$. The bundling strategy f^{Σ} induces a symmetric bundling equilibrium in the VCG mechanisms if and only if Σ is a quasi-field.*

of the variable participation issue by focusing on strategies b with the property that $(b, \ldots, b)$ is an equilibrium for every number of buyers n. This was the approach taken in (Holzman et al., 2003). It has no natural analogue in the non-symmetric case.

[7] Indeed, in our proofs we restrict attention to VC mechanisms.

In (Holzman et al., 2003) only symmetric profiles of bundling strategies were considered. Here, by contrast, we look at arbitrary (not necessarily symmetric) profiles of arbitrary (not necessarily bundling) strategies. Our result is that for $n \geqslant 3$, every ex post equilibrium in the VCG mechanisms is of the form discovered in (Holzman et al., 2003), i.e., a symmetric bundling equilibrium defined by a quasi-field.

Theorem 2. *Let $n \geqslant 3$. The profile of strategies $(b_1, \dots, b_n)$ forms an ex post equilibrium in the VCG mechanisms if and only if there exists a quasi-field $\Sigma \subseteq 2^A$ such that $b_i = f^{\Sigma}$ for every $i \in N$.*

Our proof of Theorem 2 is divided into two parts. In Section 3 we stay in the realm of bundling strategies, and extend the treatment of (Holzman et al., 2003) from the symmetric case to the general case. In Section 4 we examine equilibria made of arbitrary strategies.

3. Bundling strategies

In this section we show that for $n \geqslant 3$ every bundling equilibrium is symmetric. While doing this, we also re-prove Theorem 1, showing that $(f^{\Sigma}, \dots, f^{\Sigma})$ is a bundling equilibrium if and only if Σ is a quasi-field. This renders the current paper self-contained.

Proposition 1. *Let $n \geqslant 3$. For each $i \in N$, let Σ_i be a subfamily of 2^A such that $\emptyset \in \Sigma_i$. The profile of strategies $(f^{\Sigma_1}, \dots, f^{\Sigma_n})$ is a bundling equilibrium if and only if there exists a quasi-field $\Sigma \subseteq 2^A$ such that $\Sigma_i = \Sigma$ for every $i \in N$.*

The key concept needed for the proof of Proposition 1 is the following condition on n-tuples $(\Sigma_1, \dots, \Sigma_n)$ of subfamilies of 2^A. We say that such an n-tuple satisfies the *partition property* if for every $i \in N$ and for every ordered partition $(B_1, \dots, B_n)$ of A,

$$B_j \in \Sigma_j \quad \text{for every } j \in N \setminus \{i\} \quad \Rightarrow \quad B_i \in \Sigma_i.$$

We also need to introduce a simple type of valuation functions. For $B \subseteq A$, $B \neq \emptyset$, let w_B be the following valuation function:[8]

$$w_B(C) = \begin{cases} 1 & \text{if } B \subseteq C \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

For $B = \emptyset$, let w_B be the zero valuation, i.e., $w_B(C) = 0$ for all $C \subseteq A$.

Lemma 1. *Let $n \geqslant 1$. For each $i \in N$, let Σ_i be a subfamily of 2^A such that $\emptyset \in \Sigma_i$. The profile of strategies $(f^{\Sigma_1}, \dots, f^{\Sigma_n})$ is a bundling equilibrium if and only if $(\Sigma_1, \dots, \Sigma_n)$ satisfies the partition property.*

[8] For $B \neq \emptyset$, a valuation function of the form w_B is called a unanimity TU game in cooperative game theory. An agent with such a valuation function (up to scaling) is called by Lehmann et al. (1999) a single-minded agent.

Proof. Suppose first that $(\Sigma_1, \ldots, \Sigma_n)$ satisfies the partition property. Observe that for every $N' \subseteq N$, the restriction of $(\Sigma_1, \ldots, \Sigma_n)$ to N' also satisfies the partition property, since $\emptyset \in \Sigma_i$ for all $i \in N$. Thus, in order to prove that $(f^{\Sigma_1}, \ldots, f^{\Sigma_n})$ is a bundling equilibrium, it suffices to show that it is an ex post equilibrium in every VC (and hence in every VCG) mechanism for the entire set of buyers N.

Consider a VC mechanism d, a buyer $i \in N$, and a profile of valuations $\mathbf{v} = (v_1, \ldots, v_n) \in V^N$. According to the strategies used in $(f^{\Sigma_1}, \ldots, f^{\Sigma_n})$, the profile of announced valuations is $\hat{\mathbf{v}} = (v_1^{\Sigma_1}, \ldots, v_n^{\Sigma_n})$. Let

$$t = \max_{\gamma \in \Gamma} \sum_{j \neq i} v_j^{\Sigma_j}(\gamma_j).$$

Let $\mathbf{v}'$ be the profile of announced valuations consisting of an arbitrary announcement v_i' of buyer i and the fixed announcements $v_j^{\Sigma_j}$ of the buyers $j \in N \setminus \{i\}$. Suppose that in the allocation $d(\mathbf{v}')$ each buyer j receives the bundle B_j'. Then the utility of buyer i is

$$u_i^d(v_i, \mathbf{v}') = v_i(B_i') - c_i^d(\mathbf{v}') = v_i(B_i') + \sum_{j \neq i} v_j^{\Sigma_j}(B_j') - t.$$

For each $j \in N \setminus \{i\}$, let $B_j'' \subseteq B_j'$ be a bundle in Σ_j so that $v_j^{\Sigma_j}(B_j') = v_j^{\Sigma_j}(B_j'')$; the existence of such a bundle follows from the definition of $v_j^{\Sigma_j}$. Let $B_i'' = A \setminus (\bigcup_{j \neq i} B_j'')$. It follows from the partition property that $B_i'' \in \Sigma_i$. Since $B_i' \subseteq B_i''$, we have $v_i(B_i') \leqslant v_i(B_i'') = v_i^{\Sigma_i}(B_i'')$. Hence

$$u_i^d(v_i, \mathbf{v}') \leqslant v_i^{\Sigma_i}(B_i'') + \sum_{j \neq i} v_j^{\Sigma_j}(B_j'') - t.$$

However, if buyer i announces $v_i^{\Sigma_i}$, the resulting allocation $d(\hat{\mathbf{v}})$ gives each buyer j some bundle B_j, and the utility of buyer i is

$$u_i^d(v_i, \hat{\mathbf{v}}) = v_i(B_i) + \sum_{j \neq i} v_j^{\Sigma_j}(B_j) - t \geqslant v_i^{\Sigma_i}(B_i) + \sum_{j \neq i} v_j^{\Sigma_j}(B_j) - t.$$

By the optimality of the allocation $d(\hat{\mathbf{v}})$ for $\hat{\mathbf{v}}$, it follows that $u_i^d(v_i, \hat{\mathbf{v}}) \geqslant u_i^d(v_i, \mathbf{v}')$. Thus $(f^{\Sigma_1}, \ldots, f^{\Sigma_n})$ is an ex post equilibrium.

In the other direction, suppose that $(\Sigma_1, \ldots, \Sigma_n)$ does not satisfy the partition property. Let $(B_1, \ldots, B_n)$ be an ordered partition of A and let i be a buyer such that $B_j \in \Sigma_j$ for every $j \in N \setminus \{i\}$ but $B_i \notin \Sigma_i$. Note that $B_i \neq \emptyset$.

Consider the profile of valuations $\mathbf{w} = (w_{B_1}, \ldots, w_{B_n})$. According to the strategies used in $(f^{\Sigma_1}, \ldots, f^{\Sigma_n})$, buyer i reports that his valuation for B_i is zero, and every other buyer reports his true valuation function w_{B_j}. It is easy to see that given these reports, the allocation $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ in which $\gamma_j = B_j$ for $j \in N \setminus \{i\}$, $\gamma_i = \emptyset$, and $\gamma_0 = B_i$, is optimal. Hence there exists a VC mechanism d that selects the allocation γ in this situation. This results in a utility level of zero to buyer i.

However, buyer i is better off reporting his true valuation function w_{B_i}, because then any VC mechanism gives each buyer j his B_j with no charges, and in particular buyer i gets a utility level of one. Therefore $(f^{\Sigma_1}, \ldots, f^{\Sigma_n})$ is not an ex post equilibrium in d. □

Proposition 1 is an immediate consequence of the previous and the next lemma.

Lemma 2. *Let $n \geqslant 3$. For each $i \in N$, let Σ_i be a subfamily of 2^A such that $\emptyset \in \Sigma_i$. The n-tuple $(\Sigma_1, \ldots, \Sigma_n)$ satisfies the partition property if and only if there exists a quasi-field $\Sigma \subseteq 2^A$ such that $\Sigma_i = \Sigma$ for every $i \in N$.*

Proof. Suppose first that all Σ_i are equal to some fixed quasi-field Σ. Let $i \in N$, and let $(B_1, \ldots, B_n)$ be an ordered partition of A, such that $B_j \in \Sigma$ for every $j \in N \setminus \{i\}$. We have to show that $B_i \in \Sigma$ as well. By repeated applications of the closedness of Σ with respect to disjoint unions, we obtain that $\bigcup_{j \in N \setminus \{i\}} B_j \in \Sigma$. By the closedness of Σ with respect to complements, it follows that $B_i \in \Sigma$.

Conversely, suppose that $(\Sigma_1, \ldots, \Sigma_n)$ satisfies the partition property. By considering ordered partitions of A in which all but two of the sets are empty, it follows that for all $i \neq j$ and all $B \subseteq A$ we have the implication $B \in \Sigma_j \Rightarrow A \setminus B \in \Sigma_i$. Using the fact that $n \geqslant 3$, and applying the above twice, we conclude that all Σ_i are equal. Let Σ be the common subfamily of 2^A, i.e., $\Sigma_i = \Sigma$ for every $i \in N$. By the above, Σ is closed with respect to complements. Now, suppose that $B, C \in \Sigma$ and $B \cap C = \emptyset$. Applying the partition property to the ordered partition $(B, C, A \setminus (B \cup C), \emptyset, \ldots, \emptyset)$ it follows that $A \setminus (B \cup C) \in \Sigma$, and therefore, by the closedness of Σ with respect to complements, also $B \cup C \in \Sigma$. Thus Σ is a quasi-field. □

We note that the assumption $n \geqslant 3$ is needed in Proposition 1 because of its use in Lemma 2, but Lemma 1 does not depend on this assumption and yields characterizations of bundling equilibrium also for the cases $n = 1, 2$. For $n = 1$ the only requirement is $\emptyset, A \in \Sigma_1$. More interestingly, for $n = 2$ the profile $(f^{\Sigma_1}, f^{\Sigma_2})$ is a bundling equilibrium if and only if the bundles in Σ_2 are precisely the complements of those in Σ_1. In particular, Σ_1 and Σ_2 may be different in a bundling equilibrium for $n = 2$.

4. Arbitrary strategies

In this section we consider an arbitrary ex post equilibrium in the VCG mechanisms for some $n \geqslant 3$ and prove that it necessarily consists of bundling strategies. In view of Proposition 1, this will establish Theorem 2.

Let $(b_1, \ldots, b_n)$, with $n \geqslant 3$, form an ex post equilibrium in the VCG mechanisms. It is convenient to denote $b_i(v)$ by $\hat{v}^i$, and to denote the image of b_i, $b_i(V)$, by $\widehat{V}^i$.

We define the following subfamilies Σ_i, $i \in N$, of 2^A:

$$\Sigma_i = \{C \subseteq A\colon \exists w \in \widehat{V}^i \text{ such that } w(C) > w(D)\ \forall D \subsetneq C\}.$$

Note that $\emptyset \in \Sigma_i$ holds trivially.

Proposition 2. *For every buyer i, we have $\hat{v}^i(C) = v(C)$ for all $v \in V$ and all $C \in \Sigma_i$.*

We defer the proof of Proposition 2 and show first that Theorem 2 follows from it. Indeed, we show that Proposition 2 implies that the b_i are the bundling strategies corresponding to Σ_i, that is, for every buyer i

$$\hat{v}^i(B) = v^{\Sigma_i}(B) \quad \text{for all } v \in V \text{ and all } B \in 2^A.$$

This equality obviously follows from Proposition 2 when $B \in \Sigma_i$. Suppose now that $B \in 2^A \setminus \Sigma_i$, and let $v \in V$. By the definition of Σ_i, it follows that for every $w \in \widehat{V}^i$ there exists $D \subsetneq B$ such that $w(B) = w(D)$. Applying this to $\hat{v}^i \in \widehat{V}^i$, and iterating if necessary, we can find a subset $D \subsetneq B$ such that $D \in \Sigma_i$ and $\hat{v}^i(B) = \hat{v}^i(D)$. This implies that

$$\hat{v}^i(B) \leqslant \max_{C \subseteq B,\, C \in \Sigma_i} \hat{v}^i(C).$$

As the reverse weak inequality follows from monotonicity of $\hat{v}^i$, we obtain

$$\hat{v}^i(B) = \max_{C \subseteq B,\, C \in \Sigma_i} \hat{v}^i(C).$$

By Proposition 2 we can rewrite this as

$$\hat{v}^i(B) = \max_{C \subseteq B,\, C \in \Sigma_i} v(C),$$

which is just the required equality $\hat{v}^i(B) = v^{\Sigma_i}(B)$.

It remains to prove Proposition 2. This will be achieved through a sequence of lemmas. We will sometimes look at a subset $K \subseteq N$, and consider auctions in which only the buyers in K participate. In this context, the standard terminology (profiles, allocations) will refer to the restricted set of buyers. Recall that our assumption that $(b_1, \ldots, b_n)$ forms an ex post equilibrium in the VCG mechanisms means, by definition, that the restricted profile $(b_i)_{i \in K}$ is an ex post equilibrium in every such auction.

Lemma 3. *Let $1 \leqslant k \leqslant n$, and let $K \subseteq N$ be a subset of k buyers. Let $i \in K$, let $v_i \in V$ and let $w_j \in \widehat{V}^j$, $j \in K \setminus \{i\}$. Suppose that the allocation $\gamma^\star$ is optimal for the profile of valuations $(\hat{v}_i^i, (w_j)_{j \in K \setminus \{i\}})$. Then $\gamma^\star$ is also optimal for the profile $(v_i, (w_j)_{j \in K \setminus \{i\}})$.*

Proof. Without loss of generality assume that $K = \{1, \ldots, k\}$ and $i = 1$. Let d be a VC mechanism that selects the allocation $\gamma^\star$ at the profile $(\hat{v}_1^1, w_2, \ldots, w_k)$. Consider a situation where agent 1's true valuation is v_1 and the other agents' announced valuations (according to the strategies $b_2, \ldots, b_k$) are $w_2, \ldots, w_k$, respectively. Let

$$t = \max_{\gamma \in \Gamma} \sum_{j=2}^{k} w_j(\gamma_j).$$

If agent 1 announces a valuation v_1', the profile of announced valuations will be $(v_1', w_2, \ldots, w_k)$, which we denote by $\mathbf{w}'$. The resulting utility of agent 1 will be

$$\begin{aligned} u_1^d(v_1, \mathbf{w}') &= v_1\big(d_1(\mathbf{w}')\big) - c_1^d(\mathbf{w}') = v_1\big(d_1(\mathbf{w}')\big) - \left[t - \sum_{j=2}^{k} w_j\big(d_j(\mathbf{w}')\big) \right] \\ &= v_1\big(d_1(\mathbf{w}')\big) + \sum_{j=2}^{k} w_j\big(d_j(\mathbf{w}')\big) - t = S\big(\mathbf{w}, d(\mathbf{w}')\big) - t, \end{aligned}$$

where $\mathbf{w} = (v_1, w_2, \ldots, w_k)$. For the truthful announcement $v_1' = v_1$ this becomes

$$u_1^d(v_1, \mathbf{w}) = S\big(\mathbf{w}, d(\mathbf{w})\big) - t = S_{\max}(\mathbf{w}) - t.$$

It follows that an announcement v_1' of agent 1 is a best response if and only if $d(\mathbf{w}')$ is an optimal allocation for $\mathbf{w}$. By the equilibrium requirement, $\hat{v}_1^1$ should be a best response, and hence $d(\hat{v}_1^1, w_2, \ldots, w_k)$ must be an optimal allocation for $\mathbf{w}$. This is what had to be proved. □

A special type of valuation functions will play a role in the sequel. They are parametrized by a non-empty subset $B \subseteq A$ and a non-negative constant s, and defined by

$$v_B^s(C) = \begin{cases} s & \text{if } B \subseteq C \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4. *For every buyer i, we have $\widehat{v_A^s}^{\,i} = v_A^s$ for all $s \geqslant 0$.*

Proof. In the first part of the proof, we show that for every i

$$\widehat{v_A^s}^{\,i}(C) = 0 \quad \text{for all } C \subsetneq A.$$

Assume, for the sake of contradiction, that this is not true. Without loss of generality suppose that there is a counterexample for $i = 1$, and let C be an inclusion-minimal such counterexample. Then $C \neq \emptyset, A$ and $\widehat{v_A^s}^{\,1}(C) > 0$. Let $B = A \setminus C$. Choose two constants b and h such that $b \geqslant \widehat{v_A^s}^{\,1}(A)$ and $0 < h < \widehat{v_A^s}^{\,1}(C)$, and consider the valuation function v defined by

$$v(D) = \begin{cases} b + h & \text{if } D = A, \\ b & \text{if } B \subseteq D \subsetneq A, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\gamma^\star = (\gamma_0^\star, \gamma_1^\star, \gamma_2^\star)$ be an optimal allocation for agent 1 and agent 2's profile $(\widehat{v_A^s}^{\,1}, \hat{v}^2)$. Applying Lemma 3 with $k = 2$, $w_1 = \widehat{v_A^s}^{\,1}$, and $v_2 = v$ we deduce that $\gamma^\star$ is also optimal for the profile $(\widehat{v_A^s}^{\,1}, v)$. This entails, first, that $B \subseteq \gamma_2^\star$, for otherwise we would have

$$\widehat{v_A^s}^{\,1}\big(\gamma_1^\star\big) + v\big(\gamma_2^\star\big) = \widehat{v_A^s}^{\,1}\big(\gamma_1^\star\big) + 0 \leqslant \widehat{v_A^s}^{\,1}(A) \leqslant b < b + h = v(A) = \widehat{v_A^s}^{\,1}(\emptyset) + v(A)$$

contradicting the optimality of $\gamma^\star$ for $(\widehat{v_A^s}^{\,1}, v)$. Secondly, it entails that $\gamma_2^\star = B$, for otherwise we would have $\gamma_1^\star \subsetneq C$ and hence, by the minimality of C,

$$\widehat{v_A^s}^{\,1}\big(\gamma_1^\star\big) + v\big(\gamma_2^\star\big) = 0 + v\big(\gamma_2^\star\big) \leqslant b + h < \widehat{v_A^s}^{\,1}(C) + b = \widehat{v_A^s}^{\,1}(C) + v(B),$$

again contradicting the optimality of $\gamma^\star$ for $(\widehat{v_A^s}^{\,1}, v)$.

Another application of Lemma 3 yields that $\gamma^\star$ is also optimal for the profile $(v_A^s, \hat{v}^2)$. Since

$$v_A^s\big(\gamma_1^\star\big) + \hat{v}^2\big(\gamma_2^\star\big) = 0 + \hat{v}^2(B) \leqslant v_A^s(\emptyset) + \hat{v}^2(A),$$

the optimality implies that $\hat{v}^2(B) = \hat{v}^2(A)$. This means that the allocation $\gamma = (\gamma_0, \gamma_2) = (C, B)$ is optimal for the single agent (buyer 2) profile $(\hat{v}^2)$. By yet another application of Lemma 3, it follows that γ is also optimal for (v). This is a contradiction, since $v(B) = b < b + h = v(A)$.

In the second part of the proof, we show that for every buyer i,

$$\widehat{v_A^s}^i(A) = s.$$

Let us denote $g_i(s) = \widehat{v_A^s}^i(A)$. We show first that the system of functions $g_1, \dots, g_n$ must satisfy a condition that we call mean value exclusion: for all $i \neq j$ and all $s, t, y \geqslant 0$,

$$s < y \leqslant g_i(s) \quad \text{or} \quad g_i(s) \leqslant y < s \quad \Rightarrow \quad y \neq g_j(t).$$

Indeed, suppose for example that $s < g_2(t) \leqslant g_1(s)$. Consider agent 1 and agent 2's profile $(\widehat{v_A^s}^1, \widehat{v_A^t}^2)$ which, by the first part of the proof, is just $(v_A^{g_1(s)}, v_A^{g_2(t)})$. The allocation $\gamma^\star = (\gamma_0^\star, \gamma_1^\star, \gamma_2^\star) = (\emptyset, A, \emptyset)$ is optimal for this profile but not for the profile $(v_A^s, v_A^{g_2(t)})$. This contradicts Lemma 3. A similar contradiction is obtained for any pair of buyers $i \neq j$ and also in the case with reversed inequalities.

We complete the proof by showing that the condition of mean value exclusion implies that $g_i(s) = s$ for all $i \in N$ and all $s \geqslant 0$. Suppose first that $g_i(s) > s$ for some i and s. Choose some t such that $s < t < g_i(s)$. If we have $g_j(t) \geqslant t$ for some $j \neq i$, then mean value exclusion is violated by the pair i, j (possibly with their roles interchanged). Therefore we must have $g_j(t) < t$ for every $j \neq i$. But then, as $n \geqslant 3$, we may consider two distinct $j_1, j_2 \neq i$ and note that the pair j_1, j_2 provides a violation of mean value exclusion. An analogous argument may be used to rule out the possibility that $g_i(s) < s$ for some i and s. □

Lemma 5. *Let $1 \leqslant k < n$, and let $K \subset N$ be a subset of k buyers. Let $i \in K$, let $v_i \in V$ and let $w_j \in \widehat{V}^j$, $j \in K \setminus \{i\}$. Suppose that the allocation $\gamma^\star = (\gamma_j^\star)_{j \in K \cup \{0\}}$ is optimal for the profile of valuations $(\hat{v}_i^i, (w_j)_{j \in K \setminus \{i\}})$. Then $\hat{v}_i^i(\gamma_i^\star) = v_i(\gamma_i^\star)$.*

Proof. Without loss of generality $K = \{1, \dots, k\}$ and $i = 1$. We show that each of the two possible inequalities between $\hat{v}_1^1(\gamma_1^\star)$ and $v_1(\gamma_1^\star)$ leads to a contradiction. Assume first that $\hat{v}_1^1(\gamma_1^\star) < v_1(\gamma_1^\star)$. Choose a number s such that

$$\hat{v}_1^1(\gamma_1^\star) + \sum_{j=2}^{k} w_j(\gamma_j^\star) < s < v_1(\gamma_1^\star) + \sum_{j=2}^{k} w_j(\gamma_j^\star),$$

and consider the $(k+1)$-agent profile $(\hat{v}_1^1, w_2, \dots, w_k, v_A^s)$. Note that, by Lemma 4, $v_A^s \in \widehat{V}^{k+1}$, and therefore we may apply Lemma 3 to this profile.

Now, we claim that the allocation

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_k, \gamma_{k+1}) = (\emptyset, \emptyset, \dots, \emptyset, A)$$

is optimal for $(\hat{v}_1^1, w_2, \dots, w_k, v_A^s)$. Indeed, this allocation has a total social surplus of s. If $\delta = (\delta_0, \delta_1, \dots, \delta_k, \delta_{k+1})$ is any other allocation then $\delta_{k+1} \subsetneq A$ and hence $v_A^s(\delta_{k+1}) = 0$;

therefore, its total social surplus can be bounded from above as follows:

$$\hat{v}_1^1(\delta_1) + \sum_{j=2}^{k} w_j(\delta_j) \leqslant \hat{v}_1^1(\gamma_1^\star) + \sum_{j=2}^{k} w_j(\gamma_j^\star) < s.$$

It follows by Lemma 3 that γ should also be optimal for the profile $(v_1, w_2, \ldots, w_k, v_A^s)$. This, however, is not the case, because the allocation $(\gamma_0^\star, \gamma_1^\star, \ldots, \gamma_k^\star, \emptyset)$ has a higher total social surplus for this profile:

$$v_1(\gamma_1^\star) + \sum_{j=2}^{k} w_j(\gamma_j^\star) > s.$$

Assume now that $v_1(\gamma_1^\star) < \hat{v}_1^1(\gamma_1^\star)$. In this case, choose a number s such that

$$v_1(\gamma_1^\star) + \sum_{j=2}^{k} w_j(\gamma_j^\star) < s < \hat{v}_1^1(\gamma_1^\star) + \sum_{j=2}^{k} w_j(\gamma_j^\star).$$

As in the previous case, a contradiction arises from an application of Lemma 3: it is easy to check that the allocation $(\gamma_0^\star, \gamma_1^\star, \ldots, \gamma_k^\star, \emptyset)$ is optimal for the profile $(\hat{v}_1^1, w_2, \ldots, w_k, v_A^s)$ but is inferior to the allocation $(\emptyset, \emptyset, \ldots, \emptyset, A)$ at the profile $(v_1, w_2, \ldots, w_k, v_A^s)$. □

Lemma 6. *For every buyer i, we have $\hat{v}^i(A) = v(A)$ for all $v \in V$.*

Proof. The lemma follows directly from the case $k = 1$ of Lemma 5. □

Lemma 7. *Let $i, j \in N$, $i \neq j$. Let $C \in \Sigma_j$, $C \neq A$, and let $B = A \setminus C$. Then for any sufficiently large constant s we have $\widehat{v_B^s}^i(E) = s$ for all E such that $B \subseteq E \subseteq A$.*

Proof. Without loss of generality $i = 1$, $j = 2$. It suffices to prove the statement of the lemma for $E = B$, because the general statement will then follow using monotonicity and Lemma 6:

$$s = \widehat{v_B^s}^1(B) \leqslant \widehat{v_B^s}^1(E) \leqslant \widehat{v_B^s}^1(A) = v_B^s(A) = s.$$

Recalling the definition of Σ_2, we can find a valuation function $w_2 \in \widehat{V}^2$ such that $w_2(C) > w_2(D)$ for all $D \subsetneq C$. We will show that the statement of the lemma holds for every $s > w_2(A)$.

Let $\gamma^\star = (\gamma_0^\star, \gamma_1^\star, \gamma_2^\star)$ be an optimal allocation for the profile of valuations $(\widehat{v_B^s}^1, w_2)$. By Lemma 3, $\gamma^\star$ is also optimal for the profile (v_B^s, w_2). This implies, first, that $B \subseteq \gamma_1^\star$, for otherwise we would have

$$v_B^s(\gamma_1^\star) + w_2(\gamma_2^\star) = 0 + w_2(\gamma_2^\star) \leqslant w_2(A) < s = v_B^s(A) + w_2(\emptyset)$$

contradicting the optimality of $\gamma^\star$ for (v_B^s, w_2). Moreover, it implies that $\gamma_1^\star = B$, for otherwise we would have $\gamma_2^\star \subsetneq C$ and therefore

$$v_B^s(\gamma_1^\star) + w_2(\gamma_2^\star) = s + w_2(\gamma_2^\star) < s + w_2(C) = v_B^s(B) + w_2(C),$$

again contradicting the optimality of $\gamma^\star$ for (v_B^s, w_2).

Finally, from Lemma 5 (which may be applied here with $k = 2$, since $n \geqslant 3$) and the fact that $\gamma_1^\star = B$ we obtain that $\widehat{v_B^s}^1(B) = v_B^s(B) = s$, as required. □

Proof of Proposition 2. Let $i \in N$. Let $v \in V$ and let $C \in \Sigma_i$. We have to show that $\hat{v}^i(C) = v(C)$. Without loss of generality $i = 1$. If $C = A$ this follows from Lemma 6, so we assume that $C \neq A$. Let $B = A \setminus C$. Choose a constant s that satisfies $s > \hat{v}^1(A)$ and is sufficiently large so that, by Lemma 7, $\widehat{v_B^s}^2(E) = s$ for all E such that $B \subseteq E \subseteq A$.

Let $\gamma^\star = (\gamma_0^\star, \gamma_1^\star, \gamma_2^\star)$ be an optimal allocation for the profile of valuations $(\hat{v}^1, \widehat{v_B^s}^2)$. It follows from Lemma 5 that $\widehat{v_B^s}^2(\gamma_2^\star) = v_B^s(\gamma_2^\star)$. If $B \nsubseteq \gamma_2^\star$ this implies that $\widehat{v_B^s}^2(\gamma_2^\star) = 0$ and therefore

$$\hat{v}^1(\gamma_1^\star) + \widehat{v_B^s}^2(\gamma_2^\star) = \hat{v}^1(\gamma_1^\star) \leqslant \hat{v}^1(A) < s = \widehat{v_B^s}^2(A) = \hat{v}^1(\emptyset) + \widehat{v_B^s}^2(A).$$

This contradicts the optimality of $\gamma^\star$ for $(\hat{v}^1, \widehat{v_B^s}^2)$. Thus, we conclude that $B \subseteq \gamma_2^\star$ and therefore $\gamma_1^\star \subseteq C$. From this and Lemma 7 it follows that

$$\hat{v}^1(\gamma_1^\star) + \widehat{v_B^s}^2(\gamma_2^\star) \leqslant \hat{v}^1(C) + \widehat{v_B^s}^2(A) = \hat{v}^1(C) + \widehat{v_B^s}^2(B).$$

Since $\gamma^\star$ is an optimal allocation for $(\hat{v}^1, \widehat{v_B^s}^2)$, this implies that the allocation $\gamma = (\gamma_0, \gamma_1, \gamma_2) = (\emptyset, C, B)$ is optimal for the same profile. Now, it follows from Lemma 5 that $\hat{v}^1(C) = v(C)$, as required. □

5. Comments

5.1. *Supermodular valuations suffice*

There are various kinds of restrictions that one might want to impose on the domain of valuation functions. For instance, a valuation function v is called *supermodular* if it satisfies

$$v(B \cup C) + v(B \cap C) \geqslant v(B) + v(C) \quad \text{for all } B, C \subseteq A.$$

This can be interpreted as a strong form of complementarity across goods.

Our proof of Theorem 2 still goes through if it is assumed that only supermodular valuation functions are possible. However, if the agents' reports are also required to be supermodular, some of the bundling strategies are no longer available. It can be shown that, for a quasi-field Σ, the mapping $v \mapsto v^\Sigma$ preserves supermodularity if and only if Σ is a *field*, i.e., it is closed under complements and arbitrary (not just disjoint) unions. An equivalent and more explicit description of a field is that it is based on a partition of A and consists of all unions of any number of blocks of the partition. The bundling equilibria that arise in this way, called *partition-based equilibria*, were the main objects of study in (Holzman et al., 2003). Thus, when all buyers have supermodular valuation functions, and reports are restricted to be supermodular valuation functions, our characterization takes the following form:

For $n \geqslant 3$, every ex post equilibrium in the VCG mechanisms is a symmetric partition-based equilibrium.

5.2. A counterexample under fixed participation

Our definition of ex post equilibrium in the VCG mechanisms allowed for the possibility of partial participation by demanding that the profile of strategies remain in equilibrium when restricted to any subset of the set of potential buyers. Theorem 2 does not hold true if the equilibrium property is required only for the fixed set of buyers $N = \{1, \ldots, n\}$, as shown by the following counterexample.

Let $n \geqslant 1$, and consider the profile of strategies $(b_1, \ldots, b_n)$ in which $b_i(v) = \hat{v}^i$ where $\hat{v}^i(B) = 0$ for all $i \in N$, all $v \in V$ and all $B \subsetneq A$, and $\hat{v}^i(A)$ is defined as follows:

$$\hat{v}^i(A) = \begin{cases} 1 & \text{if } i = 1 \text{ and } v(A) \leqslant 1, \\ 0 & \text{if } i \geqslant 2 \text{ and } v(A) \leqslant 1, \\ v(A) & \text{otherwise.} \end{cases}$$

The profile $(b_1, \ldots, b_n)$ is not symmetric and the strategies in it are not bundling strategies. Yet it is straightforward to check that $(b_1, \ldots, b_n)$ forms an ex post equilibrium in every VCG mechanism for the entire set of buyers $N = \{1, \ldots, n\}$.

6. Related literature

As we said, the concept of ex post equilibrium in models of private values was largely ignored in the economics literature. Hence, this section mainly deals with partially related results concerning uniqueness of truth-telling mechanisms.

Some early characterizations of truth-telling mechanisms were carried out in the context of a society that has to choose one out of a number of possible social states. In this model, every agent has an arbitrary valuation function defined over the set of social states, and the mechanism has to select a social state and specify monetary transfers. Our combinatorial auctions model may be made comparable to this more abstract model by viewing the possible allocations of goods as the social states. However, our assumption that an agent's utility depends only on the bundle allocated to him (and not on the entire allocation), and moreover this dependence is monotone, imposes severe restrictions on the domain of valuation functions.

In order to lift these restrictions, one may modify our model as follows. We say that the buyers have *allocative externalities* if the valuation of every buyer i depends on the whole allocation of goods, that is, $v_i : \Gamma \to \Re$. For example, in the case of one good and two buyers, there are three possible allocations, including that in which the seller keeps the good. A buyer may assign to the latter allocation a different utility from that assigned to the allocation that gives the good to the other buyer. We say that the domain of valuation functions is *complete* if $V = \Re^{\Gamma}$ (i.e., V is not a proper subset).

In this modified model of combinatorial auctions we may state a theorem of Roberts (1979), which was originally formulated in the abstract social states model. First we need some definitions. Let $AM = (V, d, c)$ be a direct and truth-telling auction mechanism. We say that AM satisfies the condition of *non-imposition* if for every allocation $\gamma \in \Gamma$ there exists a profile of announced valuations $\hat{\mathbf{v}} \in V^N$ such that $d(\hat{\mathbf{v}}) = \gamma$. The mechanism AM is a *weighted generalized VCG mechanism* if there exist non-negative weights $k_1, \ldots, k_n$, not all of them equal to zero, a function $F : \Gamma \to \Re$, and functions $h_i : V^{N \setminus \{i\}} \to \Re$ for $i \in N$, such that for every profile of announced valuations $\hat{\mathbf{v}} \in V^N$ we have:

$$d(\hat{\mathbf{v}}) \in \arg\max_{\gamma \in \Gamma} \sum_{i \in N} k_i \hat{v}_i(\gamma) + F(\gamma),$$

and for all $i \in N$ such that $k_i > 0$,

$$c_i(\hat{\mathbf{v}}) = -\frac{\sum_{j \neq i} k_j \hat{v}_j(d(\hat{\mathbf{v}})) + F(d(\hat{\mathbf{v}}))}{k_i} + h_i(\hat{\mathbf{v}}_{-i}).$$

Roberts' Theorem. *Consider a private values model of combinatorial auctions with allocative externalities and a complete domain of valuation functions. Assume that there are at least two goods or at least two buyers. Then every direct and truth-telling auction mechanism that satisfies non-imposition is a weighted generalized VCG mechanism.*

Our Theorem 2 is related in principle to Roberts' Theorem because every ex post equilibrium in any auction mechanism gives rise, according to the revelation principle, to an economically equivalent direct and truth-telling mechanism. So one might hope to learn about the structure of ex post equilibria by invoking the revelation principle and Roberts' Theorem. There are, however, two difficulties with this program. First, Roberts' Theorem requires allocative externalities and a complete domain of valuation functions, which are absent (and quite unrealistic) in our model. Secondly, note that the mechanisms generated by this scheme from our ex post equilibria will in general violate the condition of non-imposition. This is so because, when the buyers use the bundling strategy f^{Σ}, essentially only allocations giving the buyers bundles in Σ may be optimal with respect to the announced valuations, and other allocations will never be selected by the mechanism. It follows, in particular, that the direct truth-telling mechanisms generated from our ex post equilibria are not weighted generalized VCG mechanisms.

It is natural to ask whether an analogue of Roberts' Theorem holds true in the standard combinatorial auctions model, i.e., without allocative externalities and with the free disposal assumption. This question is addressed in a recent work by Lavi et al. (2003). They found examples of direct and truth-telling auction mechanisms that satisfy non-imposition but are not weighted generalized VCG mechanisms. Yet, subject to some further conditions and qualifications, they did prove a version of Roberts' Theorem for standard combinatorial auctions.

Roberts' Theorem was preceded by a result of Green and Laffont (1977) of a more limited scope. Working in essentially the same model as Roberts, they proved that a mechanism that is direct, truth-telling and efficient is necessarily a VCG mechanism. In this result, the allocation function d is presumed to be of a specific form (given by efficiency), and the object of the characterization is the transfer function c.[9] It is well known (although

[9] This phenomenon, by which the allocation function together with the property of incentive compatibility determine the transfer function, has been shown to hold in several other contexts (see, e.g., Myerson, 1981).

we are not aware of a published proof of this) that Green and Laffont's Theorem holds true also for standard combinatorial auctions.

A growing body of recent literature deals with uniqueness-impossibility theorems in models that not only have allocative externalities, but in particular informational externalities. In such models every buyer receives a signal (possibly about all buyers' valuation functions), and this signal does not necessarily completely reveal his own valuation function (see, e.g., Milgrom and Weber, 1982; Perry and Reny, 1999; Dasgupta and Maskin, 2000; Jehiel and Moldovanu, 2001; Bergemann and Välimäki, 2002; Meyer-ter-Vehn and Moldovanu, 2002; Jehiel et al., 2002). It is interesting to note that Meyer-ter-Vehn and Moldovanu (2002) uses Roberts' Theorem extensively.

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