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Core Equivalence Theorems for Infinite Convex Games*

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We show that the core of a continuous convex game on a measurable space of players is a von Neumann–Morgenstern stable set. We also extend the definition of the Mas–Colell bargaining set to games with a measurable space of players and show that for continuous convex games the core may be strictly included in the bargaining set but it coincides with the set of all countably additive payoff measures in the bargaining set. We provide examples which show that the continuity assumption is essential to our results. *Journal of Economic Literature* Classification Number: C71. © 1997 Academic Press

1. INTRODUCTION

Convex coalitional games were introduced by Shapley [25]. They include in particular any convex function of a measure and occur in many applications.

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For example, it was shown in [28] that the coalitional game modeling a producer and a set of potential consumers under decreasing costs is convex. The airport game [21, Section XI.4] and the bankruptcy game [3] are also convex. Several examples of convex games which arise from public good models were given in [7]. The core of a convex game with a finite set of players was studied in [25] and other solution concepts were investigated in [19]. In this work we study the equivalence between the core, von Neumann–Morgenstern stable sets, and the Mas–Colell bargaining set in convex coalitional games over a measurable space of players.

Stable sets for cooperative games were introduced by von Neumann and Morgenstern in their seminal book [29]. It was shown in [25] that the core of a finite convex game is a von Neumann–Morgenstern stable set. The result was extended to cooperative games without side payments in [22]. Stable sets for coalitional games with a finite set of players have been studied intensively (for a comprehensive survey see [17]). There are a few works concerning stable sets of games with an infinite set of players. Results from [4] on stable sets of symmetric simple games were extended in [5] to games with a continuum of players. Stable sets of market games with a continuum of players were studied in [15]. It was shown in [9] that the core of a non-atomic glove market game which is defined as the minimum of a finite number of non-atomic probability measures is a stable set. Such a game is usually not convex. The stability of the core in games with a countable set of players was studied in [10]. The core of games with an infinite set of players was investigated in many works (for a comprehensive survey see [16]). In this work we show that the core of a continuous convex game with a measurable space of players is its unique von Neumann–Morgenstern stable set.

The first definition of a bargaining set for cooperative games was given by Aumann and Maschler [2]. Recently, several new concepts of bargaining sets have been introduced [8, 13, 14 and 20]; for a comprehensive survey see [18]. All these sets contain the core of the game. However, there are important cases in which some of these sets coincide with the core. It is known that for convex coalitional games with a finite set of players these sets coincide with the core [8, 14, and 19]. The equivalence between bargaining sets and the core in simple games was studied in [11]. The Mas–Colell bargaining set was introduced in [20], where it was proved that in an atomless pure exchange economy it coincides with the set of competitive equilibria (and hence, by Aumann's [1] equivalence theorem, it also coincides with the core). It was shown in [26] that for a large class of both finite and mixed market games the Mas–Colell bargaining set coincides with the core. In this work we extend the definition of the Mas–Colell bargaining set to coalitional games with transferable utility which have a measurable space of players, and prove that for continuous convex games the core coincides with the set of countably additive measures in the bargaining set. We give an example which shows that the continuity assumption is essential. We also give an example which shows that the bargaining set of an infinite continuous convex game may contain non-countably additive measures, and thus may strictly include the core.

The class of games to which our results apply is very general and includes, in particular, games with a finite set of players, games with countably many players, non-atomic games, and mixed games. The known proofs of the corresponding results for finite games do not seem to admit an extension to games with a measurable space of players. Our approach is different, and thus in particular provides new proofs in the finite case. Our proofs employ Delbaen's [6] characterization of convex games, Schmeidler's [24] characterization of convex games in terms of the Choquet integral, and a general minmax theorem due to Sion [27].

2. DEFINITIONS AND MAIN RESULTS

Let (T, Σ) be a measurable space, i.e., T is a set and Σ is a σ-field of subsets of T. We refer to the members of T as *players* and to those of Σ as *coalitions*. A *coalitional game*, or simply a *game* on (T, Σ), is a function $v: \Sigma \to \mathscr{R}_+$ with $v(\varnothing) = 0$. A game v on (T, Σ) is *continuous* at $S \in \Sigma$ if for all sequences $S_1 \subseteq S_2 \subseteq \cdots$ of coalitions such that $\bigcup_{n=1}^{\infty} S_n = S$, and all sequences $S_1 \supseteq S_2 \supseteq \cdots$ of coalitions such that $\bigcap_{n=1}^{\infty} S_n = S$, we have $v(S_n) \to v(S)$. The game v is *continuous* if it is continuous at each S in Σ. Observe that if Σ is finite then continuity is satisfied automatically.

A game v on (T, Σ) is *superadditive* if

$$v(A \cup B) \geqslant v(A) + v(B)$$

whenever A and B are disjoint coalitions in Σ. It is *convex* if

$$v(A \cup B) + v(A \cap B) \geqslant v(A) + v(B)$$

for every $A, B \in \Sigma$. Clearly, a convex game is superadditive. We note that by [23, Proposition 3.15] a convex game which is continuous at the grand coalition is continuous at every coalition.

We denote by $ba = ba(T, \Sigma)$ the Banach space of all bounded finitely additive measures on (T, Σ) with the variation norm. If μ is a countably additive measure on (T, Σ) we denote by $ba(\mu) = ba(T, \Sigma, \mu)$ the subspace of ba which consists of all bounded finitely additive measures on (T, Σ) which vanish on the μ-measure zero sets in Σ. The subspace of ba which consists of all bounded countably additive measures on (T, Σ) is denoted by $ca = ca(T, \Sigma)$. If μ is a measure in ca then $ca(\mu) = ca(T, \Sigma, \mu)$ denotes the set of all members of ca which are absolutely continuous with respect to μ. Finally, for a subset A of an ordered vector space, we denote by A_+ the set of all nonnegative members of A.

A *payoff measure* in a game v on (T, Σ) is a member ξ of ba (not necessarily nonnegative) which satisfies $\xi(T) \leqslant v(T)$. The *core* of the game v, denoted by Core(v), is the set of all payoff measures ξ such that $\xi(S) \geqslant v(S)$ for all $S \in \Sigma$. As observed by Schmeidler [23, Theorem 3.2], if v is continuous at T, then every member of Core(v) is countably additive. It is well known that the core of a convex game is non-empty (see [25] for finite games and [23] for games with a measurable space of players).

We proceed with some definitions leading to the concept of von Neumann–Morgenstern stable sets. In all these definitions, we assume that v is a superadditive game on (T, Σ). For any coalition $S \in \Sigma$, we define

$$\sigma_v(S) = \inf \sum_{i=1}^{\infty} v(S_i)$$

where the infimum is taken over all countable partitions $S_1, S_2, ...$ of S such that $S_i \in \Sigma$ for all i. It is easy to verify, using superadditivity, that $\sigma_v \in ca_+$. Clearly, $\sigma_v(S) \leqslant v(S)$ for all $S \in \Sigma$. Intuitively, $\sigma_v(S)$ is the amount that the members of S are guaranteed to obtain in v without cooperation. This permits to extend the notion of individual rationality from finite games. We say that a member ξ of ba is *individually rational* with respect to the game v if $\xi(S) \geqslant \sigma_v(S)$ for each $S \in \Sigma$. The set of all individually rational payoff measures in a superadditive game v on (T, Σ) is denoted by $I(v)$, i.e.,

$$I(v) = \{\xi \in ba_+ \mid \xi \text{ is individually rational and } \xi(T) \leqslant v(T)\}.$$

A coalition A in Σ is *inessential* in v if $\sigma_v(A) = v(A)$, and for each coalition B such that $A \cap B = \varnothing$ we have $v(A \cup B) = v(A) + v(B)$. A coalition that is not inessential is called *essential*.

We now define a dominance relation on $I(v)$. Let $\xi, \eta \in I(v)$ and let $A \in \Sigma$ be an essential coalition. Then η *dominates* ξ *via* A, denoted by $\eta \succ_A \xi$, if $\eta(A) \leqslant v(A)$ and $\eta(B) > \xi(B)$ for each $B \in \Sigma$ such that $B \subseteq A$ and B is essential. We say that η *dominates* ξ, denoted by $\eta \succ \xi$, if there exists an essential coalition $A \in \Sigma$ such that $\eta \succ_A \xi$.

We note a connection between the core concept and the dominance relation. If $\xi \in$ Core(v) then there is no $\eta \in I(v)$ such that $\eta \succ \xi$ (indeed, $\eta \succ_A \xi$ implies that $\xi(A) < \eta(A) \leqslant v(A)$). On the other hand, if $\xi \in I(v) \setminus$ Core(v) then there is $\eta \in I(v)$ such that $\eta \succ \xi$, provided that the game v

satisfies the following (mild) assumption: there is a member μ of ba_+ such that $\mu(B) > 0$ for every essential $B \in \Sigma$. Indeed, if $\xi \in I(v) \setminus \text{Core}(v)$ then there is a (necessarily essential) coalition A such that $\xi(A) < v(A)$. Taking $\varepsilon > 0$ sufficiently small, we observe that the measure η defined by $\eta(S) = \xi(S \cap A) + \varepsilon\mu(S \cap A) + \sigma_v(S \setminus A)$ is a member of $I(v)$ and satisfies $\eta \succ_A \xi$. It follows that under the above assumptions Core(v) could be alternatively defined as the set of undominated members of $I(v)$.

We come now to the definition of a von Neumann–Morgenstern stable set:

Let v be a superadditive game on (T, Σ). A set $V \subseteq I(v)$ is a *von Neumann–Morgenstern stable set* (or simply a *stable set*) of the game v if:

(1) V is *internally stable*, i.e., if $\xi \in V$ then there is no $\eta \in V$ such that $\eta \succ \xi$.

(2) V is *externally stable*, i.e., if $\xi \in I(v) \setminus V$ then there is $\eta \in V$ such that $\eta \succ \xi$.

Our first result is:

THEOREM A. *Let v be a continuous convex game on (T, Σ). Then the core of v is its unique von Neumann–Morgenstern stable set.*

For finite games, this result was proved by Shapley [25]. A version of this result for games with countably many players was proved by Einy and Shitovitz [10]. Their assumptions on the field of coalitions Σ were different from those made here, and hence their result is not implied by Theorem A, but it also does not imply Theorem A in the countable case.

We now give a definition of the Mas–Colell bargaining set, which extends the definition given in [20] for finite games. For this definition, the game v need not be superadditive.

Let v be a game on (T, Σ) and $\xi \in ba$ be a payoff measure in v. An *objection* to ξ is a pair (A, η) such that $A \in \Sigma$ and $\eta \in ba$ satisfy $\eta(A) \leqslant v(A)$, $\eta(A) > \xi(A)$ and $\eta(B) \geqslant \xi(B)$ for all $B \in \Sigma$ with $B \subset A$. A *counter objection* to the objection (A, η) is a pair (C, ζ) such that:

(1) $C \in \Sigma$, $\zeta \in ba$ and $\zeta(C) \leqslant v(C)$.

(2) If $B \in \Sigma$ satisfies $B \subseteq A \cap C$ then $\zeta(B) \geqslant \eta(B)$, and if $D \in \Sigma$ satisfies $D \subseteq C \setminus A$ then $\zeta(D) \geqslant \xi(D)$.

(3) $\zeta(C) > \eta(A \cap C) + \xi(C \setminus A)$.

An objection to a payoff measure is called *justified* if there is no counter objection to it. The *Mas–Colell bargaining set* of the game v, denoted by

$MB(v)$, is the set of all payoff measures in v which have no justified objection.

We come now to our second result.

THEOREM B. *Let v be a continuous convex game on (T, Σ). Then*

$$\text{Core}(v) = MB(v) \cap ca.$$

The following corollary is an immediate consequence of Theorem B.

COROLLARY B. *Let Σ be finite, and let v be a convex game on (T, Σ). Then*

$$\text{Core}(v) = MB(v).$$

Note that if T is finite and Σ is the set of all subsets of T, then this result was proved by Dutta, Ray, Sengupta and Vohra [8, Proposition 3.3].

The following example shows that the continuity assumption in Theorems A and B cannot be removed.

EXAMPLE 2.1. Let T be the set of natural numbers and Σ be the set of all subsets of T. As was done in [10, Example 3.5], one can show that the convex game

$$v(S) = \begin{cases} 1 & \text{if} \quad T \backslash S \text{ is finite} \\ 0 & \text{otherwise} \end{cases}$$

does not have a von Neumann–Morgenstern stable set. Therefore the continuity assumption is needed in Theorem A.

For Theorem B, consider the same game and the measure ξ defined by $\xi(S) = \Sigma_{i \in S} 2^{-i}$. Then $\xi \in ca$ and $\xi \notin \text{Core}(v)$. We will show that $\xi \in MB(v)$. Let (A, η) be any objection to ξ. Then $T \backslash A$ is finite. Let $i \in A$. Then $\eta(\{i\}) > 0$. Now it is easy to construct a counter objection of the form $(A \backslash \{i\}, \zeta)$ to (A, η), and thus $\xi \in MB(v)$.

The following example shows that for infinite games the bargaining set may strictly include the core even when the game is continuous and convex, and thus the restriction to countably additive measures in Theorem B is necessary.

EXAMPLE 2.2. Let T be the set of natural numbers and Σ be the set of all subsets of T. Define a game v on (T, Σ) by $v(S) = \Sigma_{i \in S} 2^{-i}$ for each $S \in \Sigma$. It is clear that v is continuous and convex. In fact, v is a countably additive measure, and hence $\text{Core}(v) = \{v\}$. Let $F = \{S \in \Sigma \mid T \backslash S \text{ is finite}\}$.

Then F is a filter in Σ. Let F_0 be a maximal filter which contains F. Define a measure ξ on Σ by $\xi(S)=1$ if $S \in F_0$ and $\xi(S)=0$ otherwise. We show that $\xi \in MB(v)$. Let (A, η) be an objection to ξ. Then $A \neq T$. Let $i \in T \backslash A$. Since $\xi(\{i\})=0$ and $v(\{i\})>0$, $(\{i\}, v)$ is a counter objection to (A, η), and thus $\xi \in MB(v)$.

3. PROOFS

Let v be a game on (T, Σ). A coalition C is a *carrier* of v if $v(S)=v(S \cap C)$ for all $S \in \Sigma$. A coalition S is *null* in v if $T \backslash S$ is a carrier of v. Observe that a null coalition is inessential, but not vice versa.

The following fact is a consequence of [23, Theorem 3.10] and is recorded here for later use.

PROPOSITION 3.1. *Let v be a continuous convex game on (T, Σ). Then there exists a measure $\mu \in ca_+$ such that a coalition $S \in \Sigma$ is null in v iff $\mu(S)=0$. Moreover,* $\operatorname{Core}(v) \subseteq ca_+(\mu)$.

We state and prove now a lemma concerning continuous convex games which constitutes the main part of the proofs of our equivalence theorems.

LEMMA 3.2. *Let v be a continuous convex game on (T, Σ). Assume that $\xi \in ca$ satisfies $\xi(S)<v(S)$ for some $S \in \Sigma$. Then there exist $A \in \Sigma$ and $\eta \in \operatorname{Core}(v)$ such that*

$$v(A)-\xi(A)=\max\{v(C)-\xi(C) \mid C \in \Sigma\}. \tag{3.1}$$

$$\begin{aligned} &\eta(A)=v(A)>\xi(A) && \textit{and} \\ &\eta(B) \geqslant \xi(B) && \textit{for all} \quad B \in \Sigma \quad \textit{with} \quad B \subset A. \end{aligned} \tag{3.2}$$

Proof. We need the following notation: If $\xi \in ca$ and f is a ξ-integrable function then the integral $\int_T f\, d\xi$ will be denoted by $\xi(f)$.

Let $\mu \in ca_+$ be a measure as guaranteed in Proposition 3.1. Let $\mathbf{B}$ be the unit ball of $L_\infty(\mu)=L_\infty(T, \Sigma, \mu)$. The proof proceeds in several steps.

Step 1. We extend v to a function $\bar{v}$ defined on $\mathbf{B}_+$.

For each $f \in \mathbf{B}_+$ let $\bar{v}(f)=\min\{\zeta(f) \mid \zeta \in \operatorname{Core}(v)\}$ (the minimum exists because $\operatorname{Core}(v)$ is a weak*-compact non-empty subset of $ba(\mu)$, which is the norm-dual of $L_\infty(\mu)$). Since v is convex, by [24, Proposition 3] (see also [12, Theorem 2.2]), for each $f \in \mathbf{B}_+$ we have

$$\bar{v}(f)=\int_0^1 v(\{t \in T \mid f(t) \geqslant x\}, dx,$$

where the integral is a Riemann integral and is known as the Choquet integral of f with respect to v. In particular for each $S \in \Sigma$ we have $\bar{v}(\chi_s) = v(S)$.

In Steps 2–5 we treat the special case when $\xi \in ca_+(\mu)$.

Step 2. We prove that $\bar{v} - \xi$ attains a maximum on $\mathbf{B}_+$.

As $\text{Core}(v) \subseteq ca_+(\mu)$, by the Radon–Nikodym theorem for each $\zeta \in \text{Core}(v)$ the function $\zeta(f) = \int_T f\, d\zeta$ is continuous on $\mathbf{B}_+$ with respect to the weak*-topology which is induced by $L_\infty(\mu)$ on $\mathbf{B}_+$. Therefore $\bar{v}$ is weak*-upper semicontinuous on $\mathbf{B}_+$, as it is the minimum of weak*-continuous functions on $\mathbf{B}_+$. As $\xi \in ca_+(\mu)$, $\bar{v} - \xi$ is a weak*-upper semicontinuous function on $\mathbf{B}_+$. Now by Alaoglu's theorem $\mathbf{B}_+$ is compact in the weak*-topology. Therefore $\bar{v} - \xi$ attains a maximum on $\mathbf{B}_+$.

Step 3. We prove that the maximum in Step 2 is attained at a coalition A.

Let $M = \max_{f \in \mathbf{B}_+} (\bar{v} - \xi)(f)$. Let $f^* \in \mathbf{B}_+$ be such that $M = (\bar{v} - \xi)(f^*)$. For each $0 \leqslant x \leqslant 1$ let $T_x = \{t \in T \mid f^*(t) \geqslant x\}$. Then $M = \int_0^1 (v - \xi)(T_x)\, dx$ and thus $\int_0^1 [M - (v - \xi)(T_x)]\, dx = 0$. As $M \geqslant (v - \xi)(T_x)$ for each $0 \leqslant x \leqslant 1$, there is $0 \leqslant x_0 \leqslant 1$ such that $M = (v - \xi)(T_{x_0})$. Let $A = T_{x_0}$. Then $v(A) - \xi(A) \geqslant \bar{v}(f) - \xi(f)$ for all $f \in \mathbf{B}_+$.

It is clear that the coalition A found in Step 3 satisfies (3.1). Moreover, since $v(S) > \xi(S)$, we have $v(A) > \xi(A)$. It remains to find $\eta \in \text{Core}(v)$ which satisfies (3.2).

Step 4. We prove that for each $f \in \mathbf{B}_+$ with $f \leqslant \chi_A$ there exists $\eta \in \text{Core}(v)$ with $\eta(A) = v(A)$ such that $\eta(f) \geqslant \xi(f)$.

Let $f \in \mathbf{B}_+$ be such that $f \leqslant \chi_A$. Denote $g = \chi_A - f$, and for each $0 \leqslant x \leqslant 1$ let $A_x = \{t \in A \mid g(t) \geqslant x\}$. It is clear that if $x, y \in [0, 1]$ and $x \leqslant y$ then $A_x \supseteq A_y$. Thus $\{A_x\}_{0 \leqslant x \leqslant 1}$ is a chain in Σ. As v is convex, by [6, Corollary 3], there exists $\eta \in \text{Core}(v)$ such that $\eta(A_x) = v(A_x)$ for each $0 \leqslant x \leqslant 1$. As $A_0 = A$, we have $\eta(A) = v(A)$. Now for each $x > 0$ we have $\{t \in T \mid g(t) \geqslant x\} = A_x$, and so

$$\eta(g) = \int_0^1 \eta(A_x)\, dx = \int_0^1 v(A_x)\, dx = \bar{v}(g).$$

Thus $\eta(A) - \xi(A) = v(A) - \xi(A) \geqslant \bar{v}(g) - \xi(g) = \eta(g) - \xi(g)$, and therefore $\eta(A) - \eta(g) \geqslant \xi(A) - \xi(g)$. Since $g = \chi_A - f$, we have $\eta(f) \geqslant \xi(f)$.

Step 5. We prove that the order of the quantifiers in Step 4 can be reversed, that is, that there exists a single $\eta \in \text{Core}(v)$ with $\eta(A) = v(A)$ that satisfies $\eta(f) \geqslant \xi(f)$ for all $f \in \mathbf{B}_+$ with $f \leqslant \chi_A$.

Denote $\mathbf{B}_+(A)=\{f\in\mathbf{B}_+ \mid f\leqslant\chi_A\}$ and $C(A)=\{\eta\in\text{Core}(v)\mid\eta(A)=v(A)\}$. Then the sets $\mathbf{B}_+(A)$ and $C(A)$ are weak*-compact and convex in $L_\infty(\mu)$ and $ba(\mu)$ respectively.

Define a real-valued function H on $C(A)\times\mathbf{B}_+(A)$ by $H(\eta, f)=\eta(f)-\xi(f)$. Then H is affine and continuous in each of its variables separately. Thus the sets $C(A)$, $\mathbf{B}_+(A)$ and the function H satisfy the assumptions of Sion's [27] minmax theorem, and therefore

$$\min_{f\in\mathbf{B}_+(A)}\ \max_{\eta\in C(A)} H(\eta, f)=\max_{\eta\in C(A)}\ \min_{f\in\mathbf{B}_+(A)} H(\eta, f). \tag{3.3}$$

Now by Step 4, $\min_f\max_\eta H(\eta, f)\geqslant 0$ and thus by (3.3), $\max_\eta\min_f H(\eta, f)\geqslant 0$. Therefore there exists $\eta\in C(A)$ such that $H(\eta, f)\geqslant 0$ for all $f\in\mathbf{B}_+(A)$. It is clear that η satisfies (3.2).

Step 6. We now show that if ξ is in ca_+ but is not absolutely continuous with respect to μ, then there exist A and η such that (3.1) and (3.2) are satisfied.

By the Lebesgue decomposition theorem, there exist two measures, ξ_a and ξ_s, in ca_+ such that $\xi=\xi_a+\xi_s$, where ξ_a is absolutely continuous with respect to μ and the measures ξ_s and μ are mutually singular. As $\xi(S)<v(S)$, $\xi_a(S)<v(S)$. As $\xi_a\in ca_+(\mu)$, by what we have already shown, there exist $A_0\in\Sigma$ and $\eta\in\text{Core}(v)$ such that $v(A_0)-\xi_a(A_0)=\max\{v(C)-\xi_a(C)\mid C\in\Sigma\}$, $\eta(A_0)=v(A_0)>\xi_a(A_0)$, and $\eta(B)\geqslant\xi_a(B)$ for all $B\in\Sigma$ with $B\subset A_0$. Let C_0 be a carrier of μ such that $\xi_s(C_0)=0$, and let $A=A_0\cap C_0$. Since C_0 is a carrier of v, for each $C\in\Sigma$ we have

$$\begin{aligned} v(A)-\xi(A)&=v(A)-\xi_a(A)=v(A_0)-\xi_a(A)\\ &\geqslant v(A_0)-\xi_a(A_0)\geqslant v(C)-\xi_a(C)\geqslant v(C)-\xi(C). \end{aligned}$$

Hence $v(A)-\xi(A)=\max\{v(C)-\xi(C)\mid C\in\Sigma\}$. So (3.1) is satisfied by A.

Also, $\eta(A)\leqslant\eta(A_0)=v(A_0)=v(A)$, and as $\eta\in\text{Core}(v)$, we have $\eta(A)=v(A)$. Finally, if $B\in\Sigma$ and $B\subset A$ then $\eta(B)\geqslant\xi_a(B)=\xi(B)$. Thus (3.2) is satisfied by A and η.

Step 7. We now assume that $\xi\in ca$ is a signed measure, and show that there exist A and η such that (3.1) and (3.2) are satisfied.

By the Jordan decomposition theorem $\xi=\xi_++\xi_-$, where ξ_+ and ξ_- are the positive and the negative parts of ξ respectively. Let $w=v-\xi_-$. Then w is convex, continuous and $\xi_+(S)<w(S)$. Therefore we can apply what we have already shown for the game w and the measure ξ_+, and this yields the existence of $A\in\Sigma$ and $\eta\in\text{Core}(v)$ such that (3.1) and (3.2) are satisfied. Q.E.D.

Proof of Theorem A. It suffices to show that Core(v) is externally stable. Let $\mu \in ca_+$ be a measure as guaranteed in Proposition 3.1. Let $\xi \in I(v) \setminus \text{Core}(v)$. Let $S \in \Sigma$ be such that $\xi(S) < v(S)$.

We first assume that $\xi \in ca_+$. Let $\varepsilon > 0$ be such that $\xi(S) + \varepsilon\mu(S) < v(S)$. Then by Lemma 3.2 applied to $\xi + \varepsilon\mu \in ca$, there exist $A \in \Sigma$ and $\eta \in \text{Core}(v)$ such that $\eta(A) = v(A) > \xi(A) + \varepsilon\mu(A)$ and $\eta(B) \geqslant \xi(B) + \varepsilon\mu(B)$ for all $B \in \Sigma$ with $B \subset A$. Since $\sigma_v(A) \leqslant \xi(A) < v(A)$, A is essential in v. Now if $B \subseteq A$ is an essential coalition in v then $\mu(B) > 0$, and therefore $\eta(B) > \xi(B)$. Hence, $\eta \succ_A \xi$.

We now assume that ξ is in ba_+ but is not countably additive. By [30, Theorem 1.23], ξ can be decomposed uniquely into a sum of a nonnegative countably additive measure ξ^c and a nonnegative purely finitely additive measure ξ^p. As $\mu \in ca_+$, by [30, Theorem 1.22], there exists an increasing sequence of sets $C_n \in \Sigma$ such that $\xi^p(C_n) = 0$ for all n, and $\lim_{n \to \infty} \mu(T \setminus C_n) = 0$. Let $C = \bigcup_{n=1}^{\infty} C_n$. Then $\mu(T \setminus C) = 0$. Therefore C is a carrier of v. For each n let $S_n = S \cap C_n$. As v is continuous, $\lim_{n \to \infty} v(S_n) = v(S \cap C) = v(S)$. Since $\xi(S) < v(S)$, there exists a natural number k such that $\xi(S_k) < v(S_k)$. Let $D = S_k$ and $\Sigma_D = \{A \in \Sigma \mid A \subseteq D\}$. Let v_D be the restriction of v to Σ_D. It is clear that v_D is a continuous convex game on (D, Σ_D). As ξ^p vanishes on Σ_D, ξ coincides with ξ^c on Σ_D. Let ξ^c_D be the restriction of ξ^c to Σ_D. Then $\xi^c_D \in ca_+(D, \Sigma_D)$ and $\xi^c_D(D) = \xi(D) < v(D)$. Now we can apply what we have already shown to the game v_D and the measure ξ^c_D, in order to obtain the existence of an essential coalition $A \in \Sigma_D$ and a measure $\zeta \in \text{Core}(v_D)$ such that $\zeta(A) = v(A)$ and $\zeta(B) > \xi(B)$ for all $B \in \Sigma$ with $B \subseteq A$ and $\mu(B) > 0$. Since v is convex and $\zeta \in \text{Core}(v_D)$, by [10, Proposition 3.8], ζ can be extended to a measure η on (T, Σ) such that $\eta \in \text{Core}(v)$. Then $\eta \succ_A \xi$ in v and the proof is completed. Q.E.D.

Proof of Theorem B. From the definition of $MB(v)$ it is clear that Core(v) $\subseteq MB(v)$. As v is continuous Core(v) $\subseteq ca$ (see Section 2). Therefore Core(v) $\subseteq MB(v) \cap ca$. We will show that $MB(v) \cap ca \subseteq \text{Core}(v)$. Assume, on the contrary, that there is $\xi \in MB(v) \cap ca$ such that $\xi \notin \text{Core}(v)$. As $\xi(T) \leqslant v(T)$, there is $S \in \Sigma$ such that $\xi(S) < v(S)$. Therefore by Lemma 3.2, there exist $A \in \Sigma$ and $\eta \in \text{Core}(v)$ such that (3.1) and (3.2) are satisfied. Clearly, (A, η) is an objection to ξ in the game v. We show that (A, η) is a justified objection and this will contradict the fact that $\xi \in MB(v)$. Let C be any coalition in Σ. Then by the convexity of v we have

$$v(C) - \xi(C) \leqslant v(A \cup C) - \xi(A \cup C) + v(A \cap C) - \xi(A \cap C) - v(A) + \xi(A).$$

By (3.1), $v(A \cup C) - \xi(A \cup C) \leqslant v(A) - \xi(A)$. Therefore $v(C) - \xi(C) \leqslant v(A \cap C) - \xi(A \cap C)$. As $\eta \in \text{Core}(v)$, we have $v(A \cap C) \leqslant \eta(A \cap C)$, and thus $v(C) \leqslant \eta(A \cap C) + \xi(C \setminus A)$. Hence there is no counter objection of C to (A, η), and as C was arbitrary, (A, η) is a justified objection to ξ. Q.E.D.

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