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THE CAPACITY OF A COMMITTEE

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It is known that under moderate requirements of strategy - proofness, a committee is in general incapable of choosing by vote if the set of alternatives is too big. This led B. Peleg to introduce the notion of the capacity of a committee. It is shown here how this number is determined by the combinatorial structure of the committee. The general criteria are quite complex, but they are somewhat simplified by a homogeneity assumption, and they reduce to a simple arithmetical formula for symmetric committees.

Key words: Committee theory; simple game; capacity; strong representation.

1. The problem

The notion of the capacity of a committee was introduced in Peleg (1978b). Roughly speaking, it is a number that tells us among how many alternatives a given committee can choose when it is required that the outcome of voting should not be distorted by manipulations on the part of coalitions of committee members. It is not our intention in this paper to elaborate on the significance of this notion, but rather to show how the structure of a committee determines its capacity. Apart from the theoretical aspect of our result, it can be useful in two types of situations. The first is when a committee is given and it is required to determine its capacity. The second occurs when a set of alternatives is given and it is required to construct a committee that can cope with it, perhaps with additional demands on the committee or the voting procedure.

Some definitions are needed in order to render the notion of capacity precise. A *committee* or a *simple game*, is a pair $G=(N, \mathcal{W})$, where N is a (finite) society and $\mathcal{W}$ is a set of coalitions (non-empty subsets of N) satisfying

$$[S\in \mathcal{W} \text{ and } S\subset T] \Rightarrow T\in \mathcal{W}. \tag{1}$$

The elements of $\mathcal{W}$ are referred to as *winning coalitions*. Thus, the term 'committee' stands not only for the set N of members but also for an allocation of decision

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power among coalitions reflected by $\mathcal{W}$. A committee $G=(N, \mathcal{W})$ is *proper* if

$$S\in \mathcal{W} \Rightarrow N-S\notin \mathcal{W}.$$

The notion of the capacity will be defined for proper committees.

A committee has to choose one alternative out of a finite set A of $m\geq$ natives. We assume that the preferences of each member over A may take t of any linear order on A, and we denote the set of all such orders by L. . *choice function* (*SCF*) is a function $F: L^N \to A$, where L^N is the Cartesian p of copies of L corresponding to the members of N. We assume that the ch an alternative is done by means of a *SCF*.

Given a *SCF* F, we want to consider the possibility of coalitions re agreements as to the votes of their members (which do not necessarily reflec true preferences), and the impact of this on the outcome of voting. To hav quate notation for this purpose, let us agree that if $Q^N\in L^N$, S is a coalitio $P^S\in L^S$, $F(P^S, Q^{N-S})$ will denote the outcome of F when the members of S vo cording to P^S while the rest vote according to their part of Q^N. When R^N (construed as the true profile of preferences) and Q^N are elements of L^N, w that Q^N is a *strong equilibrium point of the F-voting game at* R^N unless there a coalition S and $P^S\in L^S$ such that

$$F(P^S, Q^{N-S})\neq F(Q^N) \text{ and } F(P^S, Q^{N-S})\, R^S\, F(Q^N)$$

((3) means that all members of S prefer $F(P^S, Q^{N-S})$ to $F(Q^N)$). If, moreo $F(Q^N)=F(R^N)$, we say that Q^N is *exact*. We say that F is *exactly and strongly c sistent* (*ESC*) if, for each $R^N\in L^N$, there exists an exact and strong equilibri point of the F-voting game at R^N.

The last definition was introduced in Peleg (1978a), where *SCF*s based on elimination principle were shown to be *ESC*. Another construction of *ESC SC* was exhibited in Dutta and Pattanaik (1978). On the other hand, further researc has shown that the class of *ESC SCF*s is quite restricted (see Polishchuk, 197 Oren, 1981; Ishikawa and Nakamura, 1980; Holzman, 1986). Thus, the *ESC* requir ment proved to be non-trivial, and it seems to reflect a balance between strong requirements which lead to impossibility results and weaker requirements which ar universally satisfied.

The *simple game associated with* $F: L^N \to A$ is $G^*(F)=(N, \mathcal{W}^*(F))$, where a coali tion S is an element of $\mathcal{W}^*(F)$ if for all $x\in A$

$$[R^N\in L^N \text{ and } xR^S(A-\{x\})] \Rightarrow F(R^N)=x \tag{4}$$

((4) means that whenever all members of S declare that x is their most preferred alternative, x is chosen). Thus, the winning coalitions in $G^*(F)$ are those that hav full control over choices from A done according to F.

A *SCF* $F: L^N \to A$ is a *strong representation of a committee* G *of order* m if

(i) F is *ESC*,

(ii) $G^*(F)=G$. (5)

m is the number of elements of A.

The *capacity of a proper committee* G can be defined now. First let us denote

$$OSR(G)=\{m: G \text{ has a strong representation of order } m\}. \tag{6}$$

The capacity $\mu(G)$ is defined as

$$\begin{aligned}\mu(G) &= \max(OSR(G)) \\ &= \infty, \text{ if this maximum does not exist.}\end{aligned} \tag{7}$$

Peleg (1978b) showed that $2 \in OSR(G)$ for every proper committee G. Thus, $OSR(G) \neq \emptyset$, so the maximum exists if and only if $OSR(G)$ is bounded. Peleg went on to show that the bounded case occurs if and only if $G=(N, \mathcal{W})$ satisfies

$$\bigcap\{S: S \in \mathcal{W}\}=\emptyset. \tag{8}$$

However, the bound that was found on the basis of (8) is not a sharp one. Consequently, the determination of the capacity of a committee satisfying (8) remained an open problem. The solution of this problem is the objective of the present paper.

The above definition of the capacity has the following apparent drawback. Unless we know $OSR(G)$ to be an interval (i.e., without 'holes'), knowledge of $\mu(G)$ is insufficient for answering all questions of the type: Has G a strong representation of order m? However, it turns out that $OSR(G)$ is indeed an interval. In the unbounded case, this is a corollary of Peleg's work - he showed that if (8) fails for G (such a committee is said to be *weak*) then

$$OSR(G)=\{2, 3, 4, 5, \ldots\} \quad (G \text{ weak}). \tag{9}$$

In the bounded case, the 'no holes' property of $OSR(G)$ will be a corollary of the present work. In view of this, the capacity $\mu(G)$ can be interpreted as the turning point between ability and inability on the part of the committee G to cope with a choice problem of this size, in the sense explained above. For instance, if $\mu(G)=4$ then one can provide suitable procedures for G to choose an alternative out of 2, 3 or 4 alternatives, but no such procedures exist when there are 5 or more alternatives. To avoid misunderstandings, it should be made clear that the intention is not to provide one procedure for varying sizes of choice problems (2, 3 and 4 in the example), but one for each size.

An important class of committees consists of the symmetric committees. $G=(N, \mathcal{W})$ is *symmetric* if there exists an integer k such that

$$S \in \mathcal{W} \Leftrightarrow |S| \geq k \tag{10}$$

(where $|S|$ denotes the number of members of S). For symmetric committees we shall use the notation $G=(n, k)$, where $n=|N|$ and k is taken from (10).

As to the capacity of a symmetric committee $G=(n, k)$, in view of (2) and (9) we can restrict our interest to the case $n/2<k<n$. Then

$$\mu(G)=[(n+1)/(n-k+1)] \tag{11}$$

(where $[x]$ is the largest integer not exceeding x). It should be noticed that for fixed n, $\mu(G)$ is an increasing function of k (in the weak sense). This can be interpreted as saying that an increase in the special majority required for decisions brings about an increase in the capacity.

That $\mu(G)$ is at least as large as indicated in (11) was proved by Peleg, but the equality follows, as a special case, from the general formula for $\mu(G)$ that we shall find here. This general formula is more complex and involves new notions about committees and coalitions in them. Therefore we shall not state it immediately, but devote the next section to its presentation.

2. The theorem

Let $G=(N, \mathcal{W})$ be a committee with $N\in\mathcal{W}$. (This mild assumption is clearly a consequence of (8) and (1); since (1) is part of the definition of a committee and (8) is the case in which we want to determine the capacity, we should not worry about making it.) For the purpose of the following definitions, it will be convenient to refer to all subsets of N, including $\emptyset$, as coalitions. A coalition S is *blocking* if $N-S\notin\mathcal{W}$. $\mathcal{B}$ will denote the collection of blocking coalitions. ($N\in\mathcal{W}$ implies $\emptyset\notin\mathcal{B}$.) A coalition S is *minimal blocking* if S is blocking but none of its proper subcoalitions is blocking. $\mathcal{B}_m$ will denote the collection of minimal blocking coalitions. By a similar definition, we let $\mathcal{W}_m$ denote the collection of *minimal winning* coalitions.

Definition 1. A coalition S is *meager* if it is either non-blocking ($S\notin\mathcal{B}$) or minimal blocking ($S\in\mathcal{B}_m$). $\mathcal{M}$ will denote the collection of meager coalitions.

A meager coalition is characterized by the property that none of its proper subcoalitions is blocking. Since $\emptyset\notin\mathcal{B}$, this implies that $\{s\}\in\mathcal{W}$ for all $s\in N$.

Definition 2. Let $p\geq r$ be positive integers. A coalition S is *of type* (p, r) if S is the union of p meager coalitions, at least r of which are non-blocking.

Since a coalition can be written in many ways as a union of subcoalitions, we do not have here a classification of coalitions into disjoint types, but rather a hierarchy of properties of coalitions that can serve in a more refined description of the power of coalitions in the committee. In particular, if p, r, $\tilde{p}$, $\tilde{r}$ are positive integers and S is a coalition then

$$[\tilde{p}\geq p\geq r\geq\tilde{r} \text{ and } S \text{ is of type } (p, r)]\Rightarrow S \text{ is of type } (\tilde{p}, \tilde{r}). \tag{12}$$

To prove (12), one should only add to the p-fold union $\tilde{p}-p$ times $\emptyset$. It should also be noticed that since the property of being non-blocking and the property of being meager are hereditary (in the sense that if they hold for a coalition they also hold for its subcoalitions), so is the property of being of type (p, r). A last remark con

ceming Definition 2 is that we could require a disjoint union and get an equivalent definition.

Definition 3. Let p be a positive integer. A coalition S is *p-blocking* if S is the disjoint union of p blocking coalitions.

Thus, a coalition is p-blocking if it can partition itself into p-pairwise disjoint blocking subcoalitions. Clearly, 1-blocking means blocking and $\tilde{p}$-blocking implies p blocking for all positive integers p less than $\tilde{p}$.

Definition 4. A committee G is *singular* if there exists a 2-blocking coalition of type (2,1). Otherwise, G is *regular*.

In order to give an example of a singular committee, we recall now the notion of a *weighted majority game*. This is a simple game $G=(N, \mathcal{W})$ for which there exist a positive quota q and nonnegative weights $w_1, \ldots, w_n$ (with the convention $N=\{1, \ldots, n\}$), such that

$$S \in \mathcal{W} \Leftrightarrow w(S) \geq q \quad \left(\text{where } w(S)=\sum_{i \in S} w_i\right). \tag{13}$$

If (13) is satisfied we write $G=[q; w_1, \ldots, w_n]$.

Consider now the committee $G=[7; 1, 1, 1, 2, 2, 2]$. Clearly, S is blocking if and only if $w(S) \geq 3$. Take the coalition $\{2, 3, 4, 5\}$. On the one hand it is $\{2, 4\} \cup \{3, 5\}$, hence 2-blocking, on the other hand it is $\{2, 3\} \cup \{4, 5\}$, hence of type (2,1). Therefore G is singular.

This example could work because $\{4, 5\} \in \mathcal{M}_m$ even though $w(\{4, 5\})>3$. In the absence of this phenomenon we can ensure regularity. To phrase this precisely, we need to recall two more notions.

The first is that of the dual game (Shapley, 1962). If $G=(N, \mathcal{W})$ is a simple game with $N \in \mathcal{W}$, then the *dual simple game* is $G^*=(N, \mathcal{M})$. That is to say, a coalition is winning in the dual game if and only if it is blocking in the original game, which in turn is the same as saying that its complement is not winning in the original game. Clearly, $(G^*)^*=G$, so G is determined by G^*.

The second is that of a *homogeneous weighted majority game*. This is a weighted majority game $G=(N, \mathcal{W})$ for which there exist $q; w_1, \ldots, w_n$ such that not only (13) is satisfied but also

$$S \in \mathcal{M}_m \Rightarrow w(S)=q. \tag{14}$$

Proposition. *If G^* is a homogeneous weighted majority game then G is regular.*

Indeed, if $G^*=[q; w_1, \ldots, w_n]$ and $w(S)=q$ for all $S \in \mathcal{M}_m$ (this is (14) applied to $G^*=(N, \mathcal{M})$), and T is any coalition, then:

$$T \text{ is 2-blocking in } G \Rightarrow w(T) \geq 2q. \tag{1}$$

$$T \text{ is of type (2,1) in } G \Rightarrow w(T) < 2q. \tag{1'}$$

This shows that G is regular. This proposition plays a reassuring role in our contex since it will be seen later that singularity has an unpleasant effect on the capacity But now we go on to define another one-parameter family of properties o coalitions.

Definition 5. Let $p \geq 2$ be an integer. A coalition S is *p-critical* if

$$S \text{ is of type } (p, 2), \text{ and} \tag{17}$$

$$t \in N - S \Rightarrow S \cup \{t\} \text{ is } p\text{-blocking.} \tag{18}$$

This definition is satisfied by $S = N$ in a trivial sense for large enough p, since the emptiness of $N - S$ verifies (18) and it remains only to choose p so large that N will be of type $(p, 2)$ - $p \geq |N| + 2$ will certainly do.

For non-trivial examples of p-critical coalitions, consider for any positive integer h the simple game G_h determined by

$$G_h^* = [4; w_1, \ldots, w_{6+h}] \quad w_1 = \cdots = w_6 = 1 \quad w_7 = \cdots = w_{6+h} = 2. \tag{19}$$

(We could have given G_h directly, but the example becomes more transparent when the dual game is considered.) The coalition $\{1, \ldots, 6\}$ is 2-critical in G_h for all values of h. If one wishes to construct, for a given integer $\tilde{p} > 2$, non-trivial examples of committees with $\tilde{p}$-critical coalitions but without p-critical coalitions for any $p < \tilde{p}$, one may do so by suitably increasing the number 6 in the above example. (For instance, for $\tilde{p} = 3$, add 4 players with weight 1, keeping the other weights and the quota of G_h^* unchanged.) Observe that in these examples G^* is a homogeneous weighted majority game. This should be compared with the Proposition we had above concerning regularity.

In view of the foregoing remarks, we may define

$$\kappa(G) = \min\{p : \text{there exists a } p\text{-critical coalition in } G\}. \tag{20}$$

We are now ready for the Theorem, except for the notions of monotonicity and of faithfulness which will be explained immediately afterwards.

Theorem. *Let $G = (N, w)$ be a proper non-weak committee (i.e., (1), (2) and (8) are assumed). Then the capacity of G is given by*

$$\begin{aligned} \mu(G) &= 2 && \text{if } G \text{ is singular} \\ &= \kappa(G) - 1 && \text{if } G \text{ is regular and } N \text{ is of type } (\kappa(G), 2) \\ &= \kappa(G) && \text{if } G \text{ is regular and } N \text{ is not of type } (\kappa(G), 2). \end{aligned} \tag{21}$$

Moreover, for every integer m satisfying $2 \le m \le \mu(G)$, G has a monotonic and faithful strong representation of order m.

A *SCF* $F: L^N \to A$ is *monotonic* if whenever $F(R^N) = x$ and $R'^N \in L^N$ is obtained from R^N by an improvement of the position of x, i.e.,

$$\begin{aligned} &yR^i z \Rightarrow yR'^i z \text{ for all } y, z \in A - \{x\}, \text{ and} \\ &xR^i y \Rightarrow xR'^i y \text{ for all } y \in A, \end{aligned} \tag{22}$$

we have $F(R'^N) = x$.

Given a committee $G = (N, W)$ and a strong representation F of G, we say that F is *faithful* if whenever π is a symmetry of G, i.e., π is a permutation of N such that

$$S \in W \Leftrightarrow \pi(S) \in W, \tag{23}$$

π is also a symmetry of F, i.e.,

$$R_i^i = R^{\pi(i)} \text{ for all } i \in N \Rightarrow F(R_1^N) = F(R^N). \tag{24}$$

These two properties were added to the statement of the Theorem because they are often regarded as desirable. Another property, neutrality, cannot be achieved, but the construction that we shall give will violate it as little as possible - we shall make this clear in the course of the proof, which is deferred to the last section.

For the sake of illustration, it is suggested that the reader apply formula (21) in order to determine the capacities of the committees mentioned above as examples, or of similar committees. We shall show now how (21) reduces to (11) in the symmetric case. Assume $G = (n, k)$. Then $G^* = (n, n-k+1)$. Since G^* is symmetric, it is a homogeneous weighted majority game, so G is regular. A coalition cannot be p-critical in G except in the trivial sense discussed after Definition 5. Indeed, denote $b = n - k + 1$. (17) requires $|S| \le pb - 2$, while (18) requires $|S \cup \{t\}| \ge pb$ if $t \in N - S$, and the two are contradictory if such t exists. We conclude that $\kappa(G)$ is the minimum p such that N is of type $(p, 2)$ and $\mu(G) = \kappa(G) - 1$, by (21). This means that $\mu(G)$ is the maximum integer p such that N is not of type $(p, 2)$. Now, N is of type $(p, 2)$ if and only if $n \le pb - 2$, which in turn is equivalent to $pb > n + 1$. Hence $\mu(G)$ is the maximum integer p such that $pb \le n + 1$, so it takes the form (11).

3. Preliminaries to the proof

The proof of the Theorem relies on the results of Holzman (1986). In fact, although the notion of the capacity is defined by means of the notion of a strong representation, which is a *SCF* with certain properties, we do not construct or analyse *SCF*s in the proof. This kind of work was already done in the above mentioned paper (in a different setting, though, as we shall explain). Our program here is to present the results of that paper in this section, and give the proof of the Theorem in the next section.

As before, let N denote the society and let A denote the set of alternatives. For any set X, let $P(X)$ denote the set of all subsets of X, and let $2^X = P(X) - \{\emptyset\}$. A *monotonic characteristic function game* (*MCFG*) is a function $v: 2^N \to P(A)$ satisfying

$$S \subset T \Rightarrow v(S) \subset v(T). \tag{25}$$

$v(S)$ is construed as the set of all alternatives x such that the coalition S can force x to be the outcome of the game. If we think of a simple game, as we do in this paper, as describing an allocation of decision power among coalitions, then the following comparison is in order. While a simple game singles out those coalitions that have full decision power, a *MCFG* carries more detailed information by specifying, for each coalition, which decisions are within its power. Of course, we can derive the limited information from the detailed information; formally, we define the *simple game induced by* v as $(N, v^{-1}(A))$, i.e., S is winning if and only if $v(S) = A$.

Given a *SCF* $F: L^N \to A$, the *MCFG associated with* F, denoted $v^*(F)$, is the *MCFG* v defined by

$$x \in v(S) \Leftrightarrow (4) \text{ holds for } S \text{ and } x. \tag{26}$$

The relation between the two objects associated with a *SCF* F, the simple game $G^*(F)$ and the *MCFG* $v^*(F)$, is that $G^*(F)$ is the simple game induced by $v^*(F)$.

A *SCF* $F: L^N \to A$ is a *strong representation of a MCFG* $v: 2^N \to P(A)$ if

$$\begin{aligned} &\text{(i)} && F \text{ is } ESC, \\ &\text{(ii)} && v^*(F) = v. \end{aligned} \tag{27}$$

The relation between this notion and the similar notion for simple games (committees), is that a *SCF* $F: L^N \to A$ is a strong representation of a committee $G = (N, \mathcal{W})$ of order $m = |A|$ if and only if it is a strong representation of some *MCFG* $v: 2^N \to P(A)$ that induces G. Thus, an attempt to find conditions under which a committee G has a strong representation of order m, may be decomposed into two steps, as follows.

(a) Find conditions under which a *MCFG* v has a strong representation.

(b) Find conditions under which at least one of the *MCFG*s that induce G (for a fixed A with $|A| = m$) satisfies the conditions found in (a).

Step (a) was done in Holzman (1986), and we shall cite the results here. Step (b) is essentially what will be done in the next section, and the results lie behind the Theorem of this paper.

In order to formulate the answer to (a), it is convenient to introduce an alternative form of expressing a *MCFG* v. The *blocking form* of v consists of the collections $\mathcal{B}(x) \subset P(N)$ for $x \in A$, defined by

$$N \in \mathcal{B}(x) \text{ and for } S \subsetneq N,\ S \in \mathcal{B}(x) \Leftrightarrow x \notin v(N - S). \tag{28}$$

The information carried by the blocking form of v is the same as that carried by v, but the form of expression is different in two ways: it uses blocking terms (which is analogous to indicating a simple game by specifying the dual game), and it collects the information by alternatives rather than by coalitions. We shall refer to *MCFG*s in either of the two forms. If $G=(N, \mathcal{W})$ is the simple game induced by v, then the collection $\mathcal{B}$ of blocking coalitions in G and the blocking form of v are related by

$$\mathcal{B}=\bigcup_{x\in A}\mathcal{B}(x). \tag{29}$$

If $S\in\mathcal{B}(x)$ we say that S is *blocking for* x. If $S\in\mathcal{B}(x)$ and there exists no T such that $T\subsetneq S$ and $T\in\mathcal{B}(x)$, we say that S is *minimal blocking for* x. We denote the collection of all such S by $\mathcal{B}_m(x)$.

A *MCFG* is said to satisfy *condition B* if there exist no enumeration $x_1,\dots,x_m$ of A and partition $S_1,\dots,S_m$ of N so that

$$S_i\in\mathcal{B}(x_i),\quad i=1,\dots,m. \tag{30}$$

Let $m=|A|$ and let k be an integer, $0\le k\le m-2$. A *MCFG* is said to satisfy *condition D(k)* if there exist no enumeration $x_1,\dots,x_m$ of A and partition $S_1,\dots,S_m$ of N so that

$$\begin{aligned}&\text{(i)}\quad S_i\in\mathcal{B}_m(x_i),\quad i=1,\dots,k,\\ &\text{(ii)}\quad S_j\notin\mathcal{B}(x_j),\quad j=k+1,\dots,m.\end{aligned} \tag{31}$$

A *MCFG* is said to be *weak* if the simple game induced by it is weak. (Recall that this means the negation of (8).)

The following result gives the answer to (a) in the non-weak case.

Main Preliminary Result (*MPR*). Let v be a non-weak *MCFG*. Then v has a strong representation if and only if v satisfies conditions B and $\bigwedge_{k=0}^{m-2} D(k)$.

In order to establish the monotonicity and the faithfulness of strong representations of a simple game, as stated in the Theorem, we shall require similar properties of strong representations of *MCFG*s. The notion of monotonicity involves only the *SCF*, so it remains the same. The notion of faithfulness, however, involves also the game. It is adapted to *MCFG*s by replacing (23) by

$$v(S)=v(\pi(S)),\ \text{for all } S\in 2^N. \tag{32}$$

Addition to MPR. Under conditions B and $\bigwedge_{k=0}^{m-2} D(k)$, v has a monotonic and faithful strong representation.

In the course of establishing the answer to (a), it was found necessary to extend the relation $S\in\mathcal{B}(x)$ to a relation $S\in\mathcal{B}(C)$, where C ranges over all subsets of A.

This extended relation amounts to what is known as an effectivity function, bu we shall introduce it only as a means of expressing further information on s representations, which will be used in the proof. To fix the ideas, suppose I strong representation of a non-weak *MCFG* v. Then v reflects the power tl distributes to the various coalitions with respect to the various alternatives, i sense of (26), which in turn refers to (4). To take care of sets of alternatives generalize (4) to read

$$[R^N \in L^N \text{ and } CR^S(A-C)] \Rightarrow F(R^N) \in C \tag{33}$$

((33) means that whenever all members of S declare that they prefer every elem of the set C of alternatives to every element of $A-C$, the chosen alternative is in Then, generalizing (26), we let the relation $C \in E(S)$ be defined by

$$C \in E(S) \Leftrightarrow (33) \text{ holds for } S \text{ and } C. \tag{34}$$

Finally, generalizing (28), we let the relation $S \in \mathcal{B}(C)$ (S is *blocking for* C) be defi ed by

$$N \in \mathcal{B}(C) \text{ and for } S \subsetneq N,\ S \in \mathcal{B}(C) \Leftrightarrow C \notin E(N-S). \tag{35}$$

We remark that this relation satisfies the following monotonicity property:

$$[S \subset S_1,\ C \supset C_1 \text{ and } S \in \mathcal{B}(C)] \Rightarrow S_1 \in \mathcal{B}(C_1). \tag{36}$$

We have used above the relation $S \in \mathcal{B}(x)$ to state conditions B and $D(k)$. We shal use now the extended relation $S \in \mathcal{B}(C)$ to state three conditions that are similar in spirit to the above conditions. They are:

Condition B^*. *There exist no partitions $C_1, \ldots, C_q$ ($q \geq 2$) of A and $S_1, \ldots, S_q$ of N so that*

$$S_i \in \mathcal{B}(C_i), \quad C_i \neq \emptyset, \quad i = 1, \ldots, q. \tag{37}$$

Condition $D^*(0)$. *There exist no partitions $C_1, \ldots, C_q$ of A and $S_1, \ldots, S_q$ of N so that*

$$S_i \notin \mathcal{B}(C_i), \quad i = 1, \ldots, q. \tag{38}$$

Condition $D^*(1)$. *There exist no $x \in A$, $C_1, C_2 \in P(A)$ and $S, T_1, T_2 \in P(N)$ so that*

(i) $\{x\}, C_1, C_2$ is a partition of A,

(ii) S, T_1, T_2 is a partition of N,

(iii) $S \in \mathcal{B}_m(x)$, (39)

(iv) $T_i \notin \mathcal{B}(C_i), \quad i = 1, 2.$

We are now ready to state the following further result, which was established in the course of the proof of *MPR*.

MPR*. Let F be a strong representation of a non-weak *MCFG* G. Then the associated relation $S \in \mathcal{B}(C)$ satisfies conditions B^*, $D^*(0)$ and $D^*(1)$.

4. The proof

We shall prove the Theorem by establishing the following four facts referring to a proper non-weak committee $G = (N, \mathcal{W})$.

Fact 1. If G is regular, m is an integer satisfying $2 \leq m \leq \kappa(G)$ and N is not of type $(m, 2)$, then G has a monotonic and faithful strong representation of order m.

Fact 2. If G is singular and $m \geq 3$, then G has no strong representation of order m.

Fact 3. If $m \geq \kappa(G) + 1$ then G has no strong representation of order m.

Fact 4. If N is of type $(m, 2)$ then G has no strong representation of order m.

In view of Facts 1–4, the Theorem may be understood as saying that there are three specific 'reasons' for non-existence of a strong representation of G of order m. Facts 2, 3 and 4 each state that their respective hypotheses rule out strong representations, while Fact 1 states that in the absence of these reasons a strong representation does exist.

Let us see how the Theorem follows from the four facts. When G is singular, the Theorem says that G has a monotonic and faithful strong representation of order 2, which is known to be the case for every proper committee, but no strong representation of any higher order, which follows from Fact 2. When G is regular, the affirmative part of the Theorem follows from Fact 1 as soon as it is verified that N is not of type $(m, 2)$. Indeed, if $m \leq \kappa(G) - 1$, N cannot be of type $(m, 2)$, since then N would be m-critical contradicting the definition of $\kappa(G)$ as a minimum; the case $m = \kappa(G)$ results in an affirmative answer in the Theorem only when it is assumed that N is not of type $(\kappa(G), 2)$, which is just what we need to know. The negative part of the Theorem follows from Fact 3 if $m \geq \kappa(G) + 1$ and from Fact 4 if $m = \kappa(G)$ and N is of type $(\kappa(G), 2)$.

Thus, our task is to prove the four facts. We start with a few lemmata.

Lemma 1. *Let p be a positive integer. Any coalition S is either p-blocking or of type $(p, 1)$.*

Proof. By induction on p. The case $p = 1$ is obvious, as it says that S is either block-

ing or non-blocking. Assume that the Lemma is true for p but not for $p+1$, and S exemplifies its failure for $p+1$. Then surely S is not of type $(1, 1)$ - otherwise, by (12), it would be of type $(p+1, 1)$. This means that S is blocking, so we can choose some $T \subset S$ which is minimal blocking. By the induction hypothesis, $S-T$ is either p-blocking or of type $(p, 1)$. If the former is the case, S is $(p+1)$-blocking, and if the latter is the case, S is of type $(p+1, 1)$ - a contradiction. □

We shall say that G is *p-regular* (for a positive integer p) if there is no p-blocking coalition of type $(p, 1)$. Clearly, every committee is 1-regular, and in our new terminology regularity means 2-regularity.

Lemma 2. *Assume that G is p-regular, where $p \geq 2$ is an integer. Let S, T be coalitions. If S is p-blocking and T is meager, then $S-T$ is $(p-1)$-blocking.*

Proof. Suppose that $S-T$ is not $(p-1)$-blocking. By Lemma 1, $S-T$ is of type $(p-1, 1)$. Since T is meager, so is $S \cap T$, therefore, $S = (S-T) \cup (S \cap T)$ is of type $(p, 1)$. But this, together with the assumption that S is p-blocking, contradicts the p-regularity of G. □

Lemma 3. *If G is regular then G is p-regular for all positive integers p.*

Proof. By induction on p. The cases $p = 1, 2$ are obvious. Assume that G is regular and p-regular ($p \geq 2$) but not $(p+1)$-regular, and S exemplifies the latter. Then we can write

$$S = \bigcup_{i=1}^{p+1} S_i, \quad S_i \in \mathcal{B}, \quad i \neq j \Rightarrow S_i \cap S_j = \emptyset, \tag{40}$$

$$S = \bigcup_{i=1}^{p+1} T_i, \quad T_i \in \mathcal{M}, \quad T_1 \notin \mathcal{B}. \tag{41}$$

As $S_1 \cup S_2$ is 2-blocking and T_{p+1} is meager, it follows from the regularity of G, by Lemma 2, that $(S_1 \cup S_2) - T_{p+1}$ is blocking. Let us choose a coalition V such that

$$V \subset (S_1 \cup S_2) - T_{p+1}, \quad V \in \mathcal{B}_m. \tag{42}$$

A similar application of Lemma 2 shows that $(S_1 \cup S_2) - V$ is blocking. By (40) and (42),

$$S - V = ((S_1 \cup S_2) - V) \cup \left(\bigcup_{i=3}^{p+1} S_i \right), \tag{43}$$

and this coalition is p-blocking as shown by the right hand side. Applying again Lemma 2, this time using the p-regularity of G, we conclude that $(S-V) - T_{p+1}$ is $(p-1)$-blocking. By (40) and (42),

$$S - T_{p+1} = ((S - V) - T_{p+1}) \cup V, \tag{44}$$

and this coalition is p-blocking. On the other hand, by (41)

$$S - T_{p+1} = \bigcup_{i=1}^{p} (T_i - T_{p+1}), \tag{45}$$

and this coalition is of type $(p, 1)$. Therefore $S - T_{p+1}$ is a counterexample to the p-regularity of G. □

Proof of Fact 1. We assume that G is regular, m is an integer satisfying $2 \le m \le \kappa(G)$ and N is not of type $(m, 2)$. Let $A = \{x_1, \ldots, x_m\}$ be a set of m alternatives. Departing from the collection $\mathcal{B}$ of blocking coalitions in G, we define m collections of coalitions as follows:

$$\begin{aligned} &\mathcal{B}(x_1) = \cdots = \mathcal{B}(x_{m-1}) = \mathcal{B}, \\ &\mathcal{B}(x_m) = \{S \subset N : N - S \text{ is not } (m-1)\text{-blocking}\}. \end{aligned} \tag{46}$$

It is easy to verify that $\{\mathcal{B}(x_i)\}_{i=1}^{m}$ is the blocking form of a $MCFG$, which we denote by v (see (25) and (28)). We proceed now to show that v satisfies conditions B and $\bigwedge_{k=0}^{m-2} D(k)$.

Suppose first that, contrary to condition B, $S_1, \ldots, S_m$ is a partition of N so that $S_i \in \mathcal{B}(x_i)$, $i = 1, \ldots, m$. This means that for $i = 1, \ldots, m-1$, $S_i \in \mathcal{B}$, and $N - S_m$ is not $(m-1)$-blocking, but this leads to a contradiction since $N - S_m$ is the disjoint union of $S_1, \ldots, S_{m-1}$.

Suppose next that some condition $D(k)$ is violated, where k is an integer, $0 \le k \le m-2$. The k alternatives that appear in part (i) of (31) in the case of a counterexample are not necessarily those denoted above $x_1, \ldots, x_k$, since a different enumeration of the alternatives may be involved. To avoid restricting the generality, we assume that there is a permutation σ of A and there is a partition $S_1, \ldots, S_m$ of N so that

$$\begin{aligned} &\text{(i)} \quad S_i \in \mathcal{B}_m(\sigma(x_i)), \quad i = 1, \ldots, k, \\ &\text{(ii)} \quad S_j \notin \mathcal{B}(\sigma(x_j)), \quad j = k+1, \ldots, m. \end{aligned} \tag{47}$$

We distinguish two possible cases.

Case I: There exists i, $1 \le i \le k$, with $\sigma(x_i) = x_m$. In this case $\bigcup_{h \ne i} S_h$ is of type $(m-1, 2)$, by (46), (47) and the fact that $k \le m-2$. Since $S_i \in \mathcal{B}_m(x_m)$, for all $t \in S_i$, $S_i - \{t\} \notin \mathcal{B}(x_m)$, which means that $(\bigcup_{h \ne i} S_h) \cup \{t\}$ is $(m-1)$-blocking. This shows that $\bigcup_{h \ne i} S_h$ is $(m-1)$-critical, which contradicts the assumption that $m \le \kappa(G)$.

Case II: There exists j, $k+1 \le j \le m$, with $\sigma(x_j) = x_m$. In this case $\bigcup_{h \ne j} S_h$ is of type $(m-1, 1)$. On the other hand, since $S_j \notin \mathcal{B}(x_m)$, $\bigcup_{h \ne j} S_h$ is $(m-1)$-blocking. This shows that G is not $(m-1)$-regular, which, by Lemma 3, is incompatible with the assumption that G is regular.

This establishes the satisfaction of $\bigwedge_{k=0}^{m-2} D(k)$. We go on to show that the simple game induced by v is G. This is equivalent to (29), and in view of (46) we only have to show that $\mathcal{W}(x_m) \subset \mathcal{W}$. If this is false, let S be a non-blocking coalition in $\mathcal{W}(x_m)$. By the definition of $\mathcal{W}(x_m)$, $N-S$ is not $(m-1)$-blocking, so by Lemma 1 it is of type $(m-1, 1)$. By adding S we see that N is of type $(m, 2)$, contrary to our assumption.

By the previous Section, v has a strong representation F that is monotonic and faithful in the sense that (32) implies (24). F is a monotonic strong representation of G of order m, and by examining (46) it is seen that (23) implies (32), hence (24), so F is also faithful as a strong representation of G.

The proof of Fact 1 is now complete, but the following remark is of some interest. (46) gives a construction of a *MCFG* whose induced simple game is G. But there are in general many *MCFG*s that induce G. It could conceivably be the case that, although the construction (46) fails, there is another *MCFG* that induces G and satisfies conditions B and $\bigwedge_{k=0}^{m-2} D(k)$. If this happened, then G would have a strong representation of order m, but some hypothesis of Fact 1 would be false. By Facts 2–4 this cannot happen, so that if anything works, (46) must work. Another property of (46) as a construction of a *MCFG* from a simple game is that it introduces as little discrimination as possible among the alternatives. Since it is easy to see that one cannot generally require that all the $\mathcal{W}(x_i)$ coincide (if they are to be associated with a strong representation of G), the situation where $m-1$ of them coincide is best in a certain sense. It is in this limited sense that we said above, when presenting the Theorem, that the construction would violate neutrality as little as possible.

In proving Facts 2–4, we shall assume that G does have a strong representation of order m, and show this to be incompatible with each of the respective hypotheses. So, from now on we shall refer to a *MCFG* and, more generally, to a relation $S \in \mathcal{W}(C)$, that are associated with such a strong representation as explained in Section 3. Also, it will be assumed that $m \geq 3$, which is indeed the case under the hypotheses of each of the Facts 2, 3 and 4. (N is not of type (2,2) since G is proper.)

We shall say that a coalition S is *meager for* x (where $x \in A$) if $S \notin \mathcal{W}(x)$ or $S \in \mathcal{W}_m(x)$. $\mathcal{M}(x)$ will denote the collection of all coalitions that are meager for x. This terminology is analogous to that used in defining the collection $\mathcal{M}$ of meager coalitions in G. From (29) we obtain that

$$\mathcal{M} = \bigcap_{x \in A} \mathcal{M}(x). \tag{48}$$

Lemma 4. *Let x_1, x_2 be distinct alternatives and let T_1, T_2 be coalitions. Then*

$$[T_1 \in \mathcal{M}(x_1) \text{ and } T_2 \notin \mathcal{W}(x_2)] \Rightarrow N-(T_1 \cup T_2) \in \mathcal{W}(A-\{x_1, x_2\}). \tag{49}$$

Proof. Denote $S = N-(T_1 \cup T_2)$, $C = A-\{x_1, x_2\}$, and suppose $T_1 \in \mathcal{M}(x_1)$, $T_2 \notin \mathcal{W}(x_2)$ and $S \notin \mathcal{W}(C)$. W.l.o.g. $T_1 \cap T_2 = \emptyset$. Consider the partitions $\{x_1\}$, $\{x_2\}$, C

of A and T_1, T_2, S of N. The second and third components of the latter are not blocking for the respective components of the former. As to the first components, there are two possibilities: either $T_1 \notin \mathcal{B}(x_1)$, in which case condition $D^*(0)$ is violated, or $T_1 \in \mathcal{B}_m(x_1)$, in which case condition $D^*(1)$ is violated. (For the conditions, see the end of Section 3.) Hence (49) must hold. □

Lemma 5. *Let x_1, x_2 be distinct alternatives and let S_1, S_2 be disjoint coalitions with $S_i \in \mathcal{B}(x_i)$, $i=1, 2$. Then there do not exist coalitions T_1, T_2 that satisfy $T_1 \cup T_2 = S_1 \cup S_2$, $T_1 \in \mathcal{W}(x_1)$ and $T_2 \notin \mathcal{B}(x_2)$.*

Proof. If such T_1, T_2 existed, we would have $N-(T_1 \cup T_2) \in \mathcal{B}(A-\{x_1, x_2\})$ by (49), which together with $S_i \in \mathcal{B}(x_i)$, $i=1, 2$, would constitute a violation of condition B^*. □

Lemma 6. *Let x be an alternative and let S_1, S_2 be disjoint coalitions so that $S_1 \in \mathcal{B}(x)$, $S_2 \in \mathcal{B}$. Assume that there exists $y \in A$, $y \neq x$, so that $S_2 \in \mathcal{B}(y)$. Let T_1, T_2 satisfy $T_1 \cup T_2 = S_1 \cup S_2$. Then it is not the case that*

$$T_1 \in \mathcal{W}(x) \text{ and } T_2 \notin \mathcal{L}, \tag{50}$$

nor that

$$T_1 \in \mathcal{W} \text{ and } T_2 \notin \mathcal{B}(x). \tag{51}$$

Proof. Assuming all the hypotheses and (50), put in Lemma 5 $x_1 = x$, $x_2 = y$. $T_2 \notin \mathcal{L}$ implies, by (29), that $T_2 \notin \mathcal{B}(y)$, so T_1, T_2 are a counterexample to the conclusion of Lemma 5. Take now (51) instead of (50), put in Lemma 5 $x_1 = y$, $x_2 = x$, and interchange the roles of S_1, S_2 so that the hypothesis of Lemma 5 will be satisfied. $T_1 \in \mathcal{W}$ implies, by (48), that $T_1 \in \mathcal{W}(y)$, so T_1, T_2 are a counterexample to the conclusion of Lemma 5. This completes the proof of Lemma 6. □

Let $x \in A$, $S \subset N$. We shall say that a coalition V, disjoint from S, is an $(x; S)$-*transfer*, if for every subcoalition T of S

$$T \in \mathcal{B} \Rightarrow T \cup V \in \mathcal{B}(x), \tag{52}$$

$$T \notin \mathcal{B} \Rightarrow T \cup V \notin \mathcal{B}(x), \tag{53}$$

$$T \in \mathcal{W} \Rightarrow T \cup V \in \mathcal{W}(x). \tag{54}$$

Lemma 7. *For every alternative x and coalition S such that $N-S \in \mathcal{B}(x)$, there exists an $(x; S)$-transfer.*

Proof. Let x, S be such that $N-S \in \mathcal{B}(x)$. Choose V_1 so that

$$V_1 \subset N-S, \quad V_1 \in \mathcal{B}_m(x). \tag{55}$$

By (29) we have also $V_1 \in \mathcal{H}$. Choose further V_0 so that

$$V_0 \subset V_1, \quad V_0 \in \mathcal{H}_m.$$

Let $V = V_1 - V_0$. Clearly V is disjoint from S, and we shall prove that V is an $(x; S)$-transfer. Indeed, let $T \subset S$.

To prove (52), assume $T \in \mathcal{H}$. If $T \in \mathcal{H}(x)$ then surely $T \cup V \in \mathcal{H}(x)$. Otherwise, by (29), there exists $y \in A$, $y \neq x$, so that $T \in \mathcal{H}(y)$. Put in Lemma 6 $S_1 = V_1$, $S_2 = V_0$, $T_1 = V_0$, $T_2 = T \cup V$. All the hypotheses are satisfied, so (51) is false. But the first part of it is true, by (56), hence the second part must be false, which means that $T \cup V \in \mathcal{H}(x)$.

To prove (53), assume $T \notin \mathcal{H}$, but $T \cup V \in \mathcal{H}(x)$. Then $V_0 \notin \mathcal{H}(x)$, since $V_0 \in \mathcal{H}$ is compatible with (55) and (56) only if $V_0 = V_1$, which implies that $V = \emptyset$ and therefore $T = T \cup V$ is blocking for x but non-blocking, contradicting (29). So there exists $y \in A$, $y \neq x$, so that $V_0 \in \mathcal{H}(y)$. Put in Lemma 6 $S_1 = T \cup V$, $S_2 = V_0$, $T_1 = T$, $T_2 = T$. All the hypotheses as well as (50) are satisfied, which is a contradiction.

To prove (54), assume $T \in \mathcal{H}$, but $T \cup V \notin \mathcal{H}(x)$. As above it is seen that $V_0 \notin \mathcal{H}(x)$, since the opposite implies $V = \emptyset$ which makes our assumptions impossible in view of (48). So there exists $y \in A$, $y \neq x$, so that $V_0 \in \mathcal{H}(y)$. The assumption that $T \cup V \notin \mathcal{H}(x)$ means that $T \cup V$ has a proper subcoalition that is blocking for x. A subcoalition of the form $T^* \cup V$, $T^* \subsetneq T$, cannot be blocking for x (apply (51) to T^*). Hence there must be a subcoalition of the form $T \cup V^*$, $V^* \subsetneq V$, so that $T \cup V^* \in \mathcal{H}(x)$. Put in Lemma 6 $S_1 = T \cup V^*$, $S_2 = V_0$, $T_1 = T$, $T_2 = V_0 \cup V^*$. All the hypotheses are satisfied, so (51) is false. But the first part of it is true, hence the second part must be false, which means that $V_0 \cup V^* \in \mathcal{H}(x)$. However $V_0 \cup V^* \subsetneq V_1$ and $V_1 \in \mathcal{H}_m(x)$ – a contradiction.

Proof of Fact 2. We assume that G is singular. Let S be a 2-blocking coalition of type (2,1). Then we can write

$$S = S^* \cup S^{**}, \quad S^*, S^{**} \in \mathcal{H}, \quad S^* \cap S^{**} = \emptyset,$$

$$S = T^* \cup T^{**}, \quad T^* \in \mathcal{H}, \quad T^{**} \notin \mathcal{H}.$$

Since $S^{**} \in \mathcal{H}$, we can find by (29) $y \in A$ such that $S^{**} \in \mathcal{H}(y)$. We choose also z such that $z \neq y$. Put in Lemma 4 $x_1 = y$, $x_2 = z$, $T_1 = T^*$, $T_2 = T^{**}$. By (58), (48), (29), $T^* \in \mathcal{H}(y)$ and $T^{**} \notin \mathcal{H}(z)$, hence by (49), $N - S \in \mathcal{H}(A - \{y, z\})$. Let us call now $x \in A - \{y, z\}$. By (36) we have $N - S \in \mathcal{H}(x)$. Let V be an $(x; S)$-transfer. It exists by Lemma 7. Put in Lemma 6 $S_1 = S^* \cup V$, $S_2 = S^{**}$, $T_1 = T^* \cup V$, $T_2 = S^*$. Using (52) for S^* it is seen that all the hypotheses are satisfied. Using (54) for T^* it is seen that (50) is also satisfied, which is a contradiction.

Lemma 8. *Let x be an alternative and let S_1, S_2 be disjoint coalitions. Then*

$$[S_1 \in \mathcal{H}_m \cap \mathcal{H}(x) \text{ and } S_2 \in \mathcal{H}] \Rightarrow S_2 \in \mathcal{H}(x).$$

Proof. Suppose that the hypothesis of (59) is true but $S_2 \notin \mathcal{H}(x)$. By (29) there exists $y \neq x$, so that $S_2 \in \mathcal{H}(y)$. All the hypotheses of Lemma 6 are satisfied for $T_i = S_i$, $i = 1, 2$, and (51) is also satisfied, which is a contradiction.

Lemma 9. *There exists an alternative x so that every blocking coalition is blocking for x.*

Proof. Choose a minimal blocking coalition S. Choose further, by (29), an alternative x such that $S \in \mathcal{H}(x)$. We shall show that x satisfies the requirement of the lemma.

If not, then let T be a blocking coalition that is not blocking for x. Let y be an alternative, $y \neq x$. Put in Lemma 4 $x_1 = y$, $x_2 = x$, $T_1 = S$, $T_2 = T$. Since $S \in \mathcal{H}_m$, by (48) $S \in \mathcal{H}(y)$. We also assume $T \notin \mathcal{H}(x)$, so (49) gives $N - (S \cup T) \in \mathcal{H}(A - \{x, y\})$. By (36) and (29) this implies that $N - (S \cup T) \in \mathcal{H}$. Choose a coalition V such that $V \subseteq N - (S \cup T)$, $V \in \mathcal{H}_m$. Putting in Lemma 8 $S_1 = S$, $S_2 = V$ yields $V \in \mathcal{H}(x)$. Putting again in Lemma 8 $S_1 = V$, $S_2 = T$ yields $T \in \mathcal{H}(x)$, a contradiction.

Lemma 10. *For every integer $p \geq \kappa(G)$ there exists a p-critical coalition.*

Proof. By induction on p. The case $p = \kappa(G)$ is obvious. Assume that S is a p-critical coalition. We shall show that there exists a $(p+1)$-critical coalition. If $N - S$ is meager then, since S is of type $(p, 2)$, $N = S \cup (N - S)$ is of type $(p+1, 2)$, therefore N is $(p+1)$-critical. Otherwise, we can find a coalition V so that $V \subseteq N - S$ and $V \in \mathcal{H}_m$. Consider the coalition $S \cup V$. It is of type $(p+1, 2)$, and if $t \in N - (S \cup V)$ then, as S is p-critical, $S \cup \{t\}$ is p-blocking, hence $S \cup V \cup \{t\}$ is $(p+1)$-blocking, so $S \cup V$ is $(p+1)$-critical. □

Proof of Fact 3. We assume that $m \geq \kappa(G) + 1$. By Lemma 10 there exists an $(m-1)$-critical coalition. Let $S^{(1)}$ be such a coalition. Let $x_1, \ldots, x_m$ be an enumeration of the alternatives so that x_1 satisfies the requirement of Lemma 9.

Our intention now is to show that it is possible to find coalitions $V_2, \ldots, V_{m-1}$ so that, denoting

$$S^{(i)} = S^{(1)} \cup \left(\bigcup_{j=2}^{i} V_j \right), \quad i = 1, \ldots, m-1, \tag{60}$$

V_i will be an $(x_i; S^{(i-1)})$-transfer, $i = 2, \ldots, m-1$. The following Claim will be used both in finding such coalitions and in using them in our proof.

Claim. *Let q be an integer, $2 \leq q \leq m$, and suppose that we found coalitions $V_2, \ldots, V_{q-1}$ (the case $q = 2$ is for the initial step when nothing was found yet), so that V_i is an $(x_i; S^{(i-1)})$-transfer, $i = 2, \ldots, q-1$. Then:*
(a) *V_i are pairwise disjoint and disjoint from $S^{(1)}$.*
(b) *$N - S^{(q-1)} \notin \mathcal{H}(x_q)$.*

Proof. (a) Since V_i is an $(x_i; S^{(i-1)})$-transfer, it is disjoint from $S^{(i-1)}$, hence by (6 it is disjoint from $S^{(1)}$ and all V_j, $j \le i-1$. (b) Let $S_1, \ldots, S_{m-1}$ be a partition of S into meager coalitions, at least two of which are non-blocking. (Such a partition ca be found since $S^{(1)}$ is of type $(m-1, 2)$.) Let σ be a permutation of the set of in dices $\{1, \ldots, m\}$ satisfying $\sigma(q) = m$.

We define a partition $T^1, \ldots, T^m$ of N by

$$\begin{aligned} T^i &= N - S^{(q-1)} && \text{if } i = q \\ &= S_{\sigma(i)} \cup V_i && \text{if } 2 \le i \le q-1 \\ &= S_{\sigma(i)} && \text{otherwise.} \end{aligned} \tag{61}$$

Assuming that $N - S^{(q-1)} \in \mathcal{W}(x_q)$, it can be seen that $T^i \in \mathcal{W}(x_i)$ for all i. Indeed, if $i = q$ this is our assumption; if $2 \le i \le q-1$ this follows by an application of (54); otherwise by (48). Moreover, if $i \ne q$ and $S_{\sigma(i)} \notin \mathcal{B}$ then $T^i \notin \mathcal{B}(x_i)$. Indeed, if $2 \le i \le q-1$ this follows by an application of (53), otherwise by (29). Hence $T^i \notin \mathcal{B}(x_i)$ occurs for at least two values of i. We are witnessing a violation of one of the conditions $D(k)$, $0 \le k \le m-2$. This completes the proof of (b). □

Using the Claim, we can find coalitions $V_2, \ldots, V_{m-1}$ as intended, going step by step. Indeed, (b) implies that $N - S^{(q-1)} \in \mathcal{B}(x_q)$, so by Lemma 7 we can find V_q that is an $(x_q; S^{(q-1)})$-transfer, and proceed to the next step. Finally we have $V_2, \ldots, V_{m-1}$ and $N - S^{(m-1)} \notin \mathcal{W}(x_m)$.

Hence we can find $t \in N - S^{(m-1)}$ so that $N - (S^{(m-1)} \cup \{t\}) \in \mathcal{B}(x_m)$. As $t \in N - S^{(1)}$ and $S^{(1)}$ is $(m-1)$-critical, $S^{(1)} \cup \{t\}$ is $(m-1)$-blocking. So we can write

$$S^{(1)} \cup \{t\} = \bigcup_{i=1}^{m-1} T_i, \quad t \in T_1, \quad T_i \in \mathcal{B}, \quad i \ne j \Rightarrow T_i \cap T_j = \emptyset. \tag{62}$$

Consider now the partition $T_1, T_2 \cup V_2, \ldots, T_{m-1} \cup V_{m-1}, N - (S^{(m-1)} \cup \{t\})$ of N. The ith component of this partition is blocking for x_i. Indeed, for $i = 1$ this is a consequence of the choice of x_1; for $i = 2, \ldots, m-1$ this follows by an application of (52); for $i = m$ by the choice of t. Thus we have a counterexample to condition B.

Proof of Fact 4. We assume that N is of type $(m, 2)$. Hence we can find a partition $S_1, \ldots, S_m$ of N into meager coalitions, at least two of which are non-blocking. Let $x_1, \ldots, x_m$ be an enumeration of the alternatives. By (29) and (48), $S_i \in \mathcal{W}(x_i)$ for all i and $S_i \notin \mathcal{B}(x_i)$ for at least two values of i. Thus, one of the conditions $D(k)$, $0 \le k \le m-2$, is violated.

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