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On partitions of discrete boxes

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Abstract

We prove that any partition of an n -dimensional discrete box into nontrivial sub-boxes must consist of at least 2^n sub-boxes, and consider some extensions of this theorem.

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1. The theorem

A set of the form

$$A = A_1 \times A_2 \times \cdots \times A_n,$$

where A_1, A_2, \dots, A_n are finite sets with $|A_i| \geq 2$, will be called here an n -dimensional discrete box. A set of the form $B = B_1 \times B_2 \times \cdots \times B_n$, where $B_i \subseteq A_i$, $i = 1, \dots, n$, is a sub-box of A . Such a set B is said to be nontrivial if $\emptyset \neq B_i \neq A_i$ for every i .

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The following theorem answers a question posed by Kearnes and Kiss [1, Problem 5.5].

Theorem 1. *Let A be an n -dimensional discrete box, and let $\{B^1, B^2, \dots, B^m\}$ be a partition of A into nontrivial sub-boxes. Then $m \geq 2^n$.*

Proof. Let

$$B^j = B_1^j \times B_2^j \times \dots \times B_n^j, \quad j = 1, \dots, m.$$

Let us call a sub-box C of A odd if its cardinality is odd. Let $\mathcal{O}(A)$ denote the collection of all odd sub-boxes of A . For $j = 1, \dots, m$, define

$$\mathcal{O}_j(A) = \{C \in \mathcal{O}(A) \mid C \cap B^j \text{ is odd}\}.$$

A sub-box is odd if and only if each of its n factors has odd cardinality, and the nontriviality of the B^j implies that half of the odd cardinality subsets of A_i intersect B_i^j in an odd number of elements. This implies

$$\frac{|\mathcal{O}_j(A)|}{|\mathcal{O}(A)|} = \frac{1}{2^n}, \quad j = 1, \dots, m. \quad (1)$$

For each $C \in \mathcal{O}(A)$ the partition $\{B^1, B^2, \dots, B^m\}$ induces a partition of C in which at least one of the parts must have odd cardinality, which implies

$$\bigcup_{j=1}^m \mathcal{O}_j(A) = \mathcal{O}(A). \quad (2)$$

It follows from (1) and (2) that $m \geq 2^n$. \square

2. Extensions and non-extensions

2.1. Infinite boxes

The theorem remains true if in the definition of an n -dimensional discrete box we allow the sets A_1, A_2, \dots, A_n to be infinite. This follows by considering the finitely many atoms induced by the partition at hand.

2.2. Partitions mod 2

The theorem remains true, with the same proof, if $\{B^1, B^2, \dots, B^m\}$ is only assumed to be a partition mod 2, that is, $\{B^1, B^2, \dots, B^m\}$ is a multi-family of nontrivial sub-boxes of A such that every point of A is covered an odd number of times.

2.3. Conditions for equality

An obvious example of equality in the theorem is obtained by splitting each A_i into two nonempty parts, and taking B^1, B^2, \dots, B^{2^n} to be the corresponding cells. One can

derive from the above proof some conditions that any example of equality must satisfy, and one might hope that these will lead to a characterization of all such examples. In particular, one might naively conjecture that every n -dimensional example of equality may be obtained by splitting one factor into two parts, and further partitioning each of the two resulting boxes according to some $(n - 1)$ -dimensional examples of equality. However, the following partition of a $3 \times 3 \times 3$ box into 8 nontrivial sub-boxes, in which none of the factors is split into just two parts, seems to indicate that examples of equality do not obey a simple construction rule:

$$\begin{aligned}
 A &= \{1, 2, 3\} \times \{a, b, c\} \times \{\alpha, \beta, \gamma\} \\
 B^1 &= \{1\} \times \{a\} \times \{\alpha\} \\
 B^2 &= \{1\} \times \{a\} \times \{\beta, \gamma\} \\
 B^3 &= \{1\} \times \{b, c\} \times \{\alpha, \beta\} \\
 B^4 &= \{1, 2\} \times \{b, c\} \times \{\gamma\} \\
 B^5 &= \{2, 3\} \times \{a, b\} \times \{\alpha, \beta\} \\
 B^6 &= \{2, 3\} \times \{a\} \times \{\gamma\} \\
 B^7 &= \{2, 3\} \times \{c\} \times \{\alpha, \beta\} \\
 B^8 &= \{3\} \times \{b, c\} \times \{\gamma\}
 \end{aligned}$$

2.4. Partition numbers of hypergraphs

If $\mathcal{H} = (V, E)$ is a hypergraph (i.e., E is a family of subsets of V), let us define the *partition number* $\pi(\mathcal{H})$ as the least p such that E contains a partition $\{B^1, B^2, \dots, B^p\}$ of V (letting $\pi(\mathcal{H}) = \infty$ if there is no such p). If $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ are two hypergraphs, let us define their *product* $\mathcal{H}_1 \times \mathcal{H}_2$ to be the hypergraph with vertex-set $V_1 \times V_2$ and edge-set consisting of all sets of the form $B_1 \times B_2$, $B_1 \in E_1$, $B_2 \in E_2$.

Clearly, if E consists of all the proper subsets of V and $|V| \geq 2$, then the partition number of $\mathcal{H} = (V, E)$ is 2. Our theorem asserts that the product of n such hypergraphs has partition number 2^n . This raises the question whether the partition number is multiplicative with respect to hypergraph product. It is easy to see that $\pi(\mathcal{H}_1 \times \mathcal{H}_2) \leq \pi(\mathcal{H}_1) \cdot \pi(\mathcal{H}_2)$, but the following example shows that in general equality need not hold.

Let $k > 4$ be an integer, and let V_1 and V_2 be two sets of cardinality $3k$. Let E_1 consist of all subsets of V_1 of cardinality 1 or $k + 1$, and let E_2 consist of all subsets of V_2 of cardinality 1 or $2k - 1$. Then $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ satisfy $\pi(\mathcal{H}_1) = k$ and $\pi(\mathcal{H}_2) = k + 2$. However, $\pi(\mathcal{H}_1 \times \mathcal{H}_2) \leq 6k$. In order to see this, identify the vertex-set of $\mathcal{H}_1 \times \mathcal{H}_2$ with the edge-set $E(K_{3k,3k})$ of a complete bipartite graph with $3k$ vertices on each side. Find a $(k + 1)$ -regular spanning subgraph G of $K_{3k,3k}$, and partition the edge-sets of G and its bipartite complement into $3k$ stars each, centered on opposite sides. As $6k < k(k + 2)$ for $k > 4$, this is a counterexample to the multiplicativity of the partition number with respect to hypergraph product.

One may define the *mod 2 partition number* $\bar{\pi}(\mathcal{H})$ in a similar way, by considering partitions mod 2 (as in Section 2.2) instead of partitions. Here, too, multiplicativity fails in general. Let $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ be two copies of a Fano plane (vertices are points, edges are lines). Then $\bar{\pi}(\mathcal{H}_i) = 3$ for $i = 1, 2$, but $\bar{\pi}(\mathcal{H}_1 \times \mathcal{H}_2) \leq 7$, as shown by the mod 2 partition of $V_1 \times V_2$ formed by taking the product of each line with itself.

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References

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