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Aggregation of non-binary evaluations ☆

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ABSTRACT

We study an aggregation problem in which a society has to determine its position on each of several issues, based on the positions of the members of the society on those issues. There is a prescribed set of feasible evaluations, i.e., permissible combinations of positions on the issues. The binary case of this problem, where only two positions are allowed on each issue, is by now quite well understood. We consider arbitrary sets of conceivable positions on each issue. This general framework admits the modeling of aggregation of various types of evaluations, including: assignments of candidates to jobs, choice functions from sets of alternatives, judgments in many-valued logic, probability estimates for events, etc. We require that the aggregation be performed issue-by-issue, and that the social position on each issue be supported by at least one member of the society. The set of feasible evaluations is called an impossibility domain if these requirements are satisfied for it only by dictatorial aggregation; that is to say, if it gives rise to an analogue of Arrow's impossibility theorem for preference aggregation. We obtain a two-part sufficient condition for an impossibility domain, and show that the major part is a necessary condition. For the ternary case, where three positions are allowed on each issue, we get a full characterization of impossibility domains.

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1. Introduction

There is, by now, a significant body of literature on the problem of aggregating binary evaluations. A society has to determine its position (yes/no) on each of several issues, based on the positions of the members of the society on those issues. There is a prescribed set X of feasible evaluations, i.e., permissible combinations of positions on the issues (X may be viewed as a subset of $\{0,1\}^m$, where m is the number of issues). Examples include preference aggregation (where the issues are pairwise comparisons and feasibility reflects rationality), and judgment aggregation (where the issues are logical propositions and feasibility reflects consistency). Two properties of aggregators that are suggested by the classical example of preference aggregation, may be stated in the general framework as follows. An aggregator is independent of irrelevant alternatives (IIA) if the social position on any given issue depends only on the individual positions on that same issue. An aggregator is Paretian if the society adopts any unanimously held position. A natural question is under what conditions on the set X of feasible evaluations, does the analogue of Arrow's impossibility theorem hold: any IIA and Paretian aggregator mapping profiles of evaluations in X to evaluations in X must be dictatorial (we call a set X for which this holds an impossibility domain). In [7] we gave a full answer: X is an impossibility domain if and only if it is totally blocked and is not an affine subspace of $\{0,1\}^m$.[1]

In the present paper, we extend the binary framework by allowing more than two positions on each issue. Instead of $\{0,1\}$, we have an arbitrary set P of conceivable positions on each issue. The prescribed set of feasible evaluations is now a subset X of P^m. We present a number of examples that naturally fit this framework.

Example A *(Assignments).* Suppose that there is a certain number m of available jobs, and a pool P of candidates who can fulfill any of those jobs. The natural feasibility constraint is that no candidate can be assigned to more than one job. This is reflected by the subset X of P^m consisting of all m-tuples with pairwise distinct entries. Further constraints may apply, leading to smaller sets of feasible assignments. Considering an appointments committee in this situation, the question is how to aggregate the feasible assignments suggested by the individual committee members into a feasible assignment adopted by the committee.

Example B *(Choice functions).* We refer to a situation considered in classical social choice theory, where P is a set of alternatives, and a family of m subsets $P_1,\dots,P_m$ of P is given (often this is the family of all non-empty subsets of P, but in general a subset of P appears among the P_j if it may become the set of available alternatives). The choice function of a decision maker specifies, for each P_j, his preferred element when facing a choice from P_j. The set of all choice functions may be viewed as the subset $P_1\times\cdots\times P_m$ of P^m. The set of feasible choice functions X is in general a subset of $P_1\times\cdots\times P_m$, reflecting constraints that one wishes to impose, for example rationalizability. The question that we ask here is how to aggregate the feasible choice functions of the individual decision makers into a feasible choice function for the society.

Example C *(Truth values).* Logic allows us to formulate propositions in a given language and to evaluate their truth in a given state of the world. Standard logic considers only true/false evaluations, but there are also many-valued logics, in which the truth values range over a set, say $P=\{0,1,\dots,p-1\}$. A common interpretation is that 0 means absolutely false and $p-1$ means absolutely true, with intermediate degrees of truth in-between. Given an agenda $\mathcal{A}$ consisting of m propositions in such a logic, we let X be the subset of P^m containing the logically consistent evaluations of the propositions in $\mathcal{A}$. The problem is how to aggregate the consistent evaluations held by the individual judges into a consistent evaluation adopted by the panel of judges.

[1] The concept of total blockedness was introduced by Nehring and Puppe [14] to solve a related but different characterization problem. The 'if' direction of our characterization was also proved by Dietrich and List [3].

Example D *(Probabilities).* Suppose that a panel of experts needs to evaluate the probability of each of m given events in a certain sample space. Here P is the interval $[0, 1]$, and X is the subset of P^m containing those probability evaluations of the m events that are feasible, namely, compatible with the axioms of probability. The question is how to aggregate the feasible probability evaluations submitted by the individual experts into a feasible probability evaluation adopted by the panel.

We are interested in aggregators $f : X^n \to X$ that satisfy properties analogous to those considered in the binary case. The IIA property extends naturally to the non-binary case. It requires that the social position on any given issue should depend only on the individual positions on that same issue (the acronym is better interpreted as Issue-by-Issue Aggregation). This is admittedly a strong requirement, but we observe that the non-binary framework allows us to weaken its bite when it is deemed too strong. Suppose, for example, that we face a binary evaluation problem, but we deem a certain triple of issues to be relevant to each other, so that the social position on each of them should be allowed to depend on the individual positions on all three of them. Then we can re-model the problem by combining the three issues into one composite issue admitting $2^3 = 8$ conceivable positions. Requiring the IIA property in this new model correctly captures the relevance structure.[2] This illustrates why we should be interested in the aggregation of non-binary evaluations, even if the original issues are binary in nature.

Regarding the Pareto property, there are two non-equivalent ways to extend it to the non-binary case. We say that f is Paretian if, whenever all individuals hold the same position on an issue, the society adopts that position. We say that f is supportive if, on any issue, the social position is one of the positions held by the individuals on that issue; in other words, the social position on any issue must have the support of at least one individual. Note that while the two properties coincide in the binary case, in general the latter is stronger than the former. Both require respecting unanimity, but supportiveness also requires that a unanimously rejected position be rejected by the society. The appeal of supportiveness depends on whether or not the set of positions P has additional structure that the aggregator may reasonably exploit. Thus, in Examples C and D above, where the elements form a scale, supportiveness is unattractive (in particular, in the probabilities example it rules out taking the average of the individual evaluations as the aggregate evaluation). But in examples such as A and B above, where there is no natural order on the elements of P, supportiveness makes a lot of sense. In the present paper we focus on IIA and supportive aggregators, and thus our results are interesting mainly for applications in which the set of positions P has no inner structure. We leave the study of aggregation under the weaker Pareto property for separate research.

We call the set X of feasible evaluations an impossibility domain if any IIA and supportive aggregator $f : X^n \to X$ must be dictatorial. The main question that we address in the non-binary case is what structural conditions on X make it an impossibility domain. In Theorem 1 we give two such conditions which together are sufficient for X to be an impossibility domain. The first of them is total blockedness, adapted in a non-trivial way from the binary case. The second condition is that X should be multiply constrained; this is a mild condition that requires the existence of some constraint on feasibility that involves more than two issues. Comparing this pair of conditions with the two conditions in our earlier result for the binary case, we note two differences: (a) The condition of multiple constrainedness did not appear in the binary case result (although it was used in its proof), due to the fact that it is a consequence of total blockedness in the binary case (but not in the general case). (b) On the other hand, no analogue of the non-affineness condition is needed in the non-binary case; indeed, its role in the binary case was to guarantee the monotonicity of the aggregator, which in the non-binary case follows (in a non-trivial way) from our other assumptions. As an application of Theorem 1, we obtain a general impossibility result for aggregating assignments (Example A).

[2] More generally, such re-modeling can handle in a satisfactory way any situation in which the relevance relation among the original issues is an equivalence relation. For a treatment that stays within the binary framework, but can handle any relevance relation, see Dietrich [2]. For models that require IIA only on some issues, see Mongin [13] and Dietrich and Mongin [6].

Theorem 2 is a partial converse of Theorem 1. It asserts that total blockedness is a necessary condition for X to be an impossibility domain. In other words, if X is not totally blocked then there do exist IIA and supportive aggregators $f : X^n \to X$ which are not dictatorial; in fact, we show their existence for any $n \geqslant 2$. The situation regarding the condition of multiple constrainedness is more complex: examples show that it is not a necessary condition for X to be an impossibility domain, but it cannot be entirely removed from Theorem 1 without hindering sufficiency. Still, as the condition of multiple constrainedness is mild and is satisfied in most of the interesting applications, the gap between our sufficient conditions for impossibility and our necessary condition for it may be considered to be small.

We do close this remaining gap in the ternary case, that is, when only three positions are allowed on each issue. Theorem 3 gives a full characterization in the ternary case: X is an impossibility domain if and only if it is totally blocked and it is either multiply constrained or exclusive. The latter condition requires the existence of an issue k, so that for each of the three positions on k there exists a position on another issue ℓ which is incompatible with it but is compatible with the other two positions on k. This may be understood as identifying a type of forbidden configuration, the existence of which characterizes impossibility in the case left open by the earlier conditions.

The paper is organized as follows. In Section 2 we introduce the model formally, state the general Theorems 1 and 2, and illustrate them with some examples. The two theorems are proved in Section 3. We state and prove Theorem 3 on the ternary case in Section 4.

We conclude the introduction with a brief survey of relevant literature, emphasizing contributions to the non-binary case. Arrow's [1] celebrated impossibility theorem showed that in the problem of preference aggregation, the IIA and Pareto properties force the aggregator to be dictatorial when at least three alternatives are present. Wilson [19] introduced an abstract model of aggregation of binary evaluations, and showed that Arrow's theorem applies not only to preference aggregation but also to other aggregation problems satisfying certain conditions. Rubinstein and Fishburn [16] extended Wilson's model to allow non-binary evaluations, as we do here. However, their approach was algebraic, and this led them to assume that the set of positions is a field; no algebraic structure plays a role in our treatment. Both Wilson and Rubinstein–Fishburn raised the question of characterizing impossibility domains, but their results gave only sufficient conditions for impossibility to hold, which were far from necessary.

Kalai [11] proposed a different way to generalize Arrow's theorem. In his approach, the objects to be aggregated are choice functions, in the sense explained in Example B above (note that when the subsets from which choices are made are all pairs of alternatives, and the feasible choice functions are the rationalizable ones, this is equivalent to standard preference aggregation). He made a conjecture that extends Arrow's theorem by considering choice functions from subsets of arbitrary size (thus, non-binary), and any class of feasible choice functions which is symmetric with respect to the alternatives. Shelah [17] proved a version of Kalai's conjecture, assuming that the size of the subsets from which choices are made is neither very small, nor very close to the total number of alternatives. Their model is a special case of ours, with additional combinatorial structure and symmetry.[3]

The rich recent literature on judgment aggregation, starting from the first impossibility theorem of List and Pettit [12], dealt mostly with sets of propositions in a two-valued logic (corresponding to binary evaluations). Pauly and van Hees [15] and van Hees [18] did obtain impossibility theorems for judgment aggregation in many-valued logic, in the sense explained in Example C above. But these results were confined to some specific logics and to agendas satisfying certain richness conditions, making their set-up much more special than ours. Gärdenfors [9] and Dietrich and List [5] dealt with incomplete judgment sets in two-valued logic, which is equivalent to allowing three positions (true/false/abstain) on each proposition, as modeled by us in [8]. But in these models the primitive notion of consistency or feasibility is in a binary setting, and the feasibility of three-valued evaluations is derived from it, not exogenously given as in the present paper.

[3] We have derived from our results here a generalization of Shelah's theorem, which consists in relaxing the restrictions on the subset size to the weakest possible ones. This derivation, though shorter than Shelah's original proof, is still quite tedious, and therefore it is not included in this paper.

Finally, the aggregation of probability evaluations in the sense explained in Example D above has been studied in the statistics literature, mostly under the assumption that the probabilities of all events in the sample space need to be evaluated; see a survey by Genest and Zidek [10]. Recently, inspired by the judgment aggregation model, Dietrich and List [4] offered a treatment of probability aggregation with an arbitrary family of events to be evaluated.

2. The model and the general results

We consider a finite, non-empty set of issues J. For convenience, if there are m issues in J, we identify J with the set $\{1,\dots,m\}$ of coordinates of vectors of length m. There is a non-empty, possibly infinite set P of conceivable positions on each of the issues in J. An *evaluation* is an m-tuple $x=(x_1,\dots,x_m)\in P^m$ specifying a position on each issue. There is a prescribed non-empty subset X of P^m. The evaluations in X are called *feasible*, the others are infeasible.[4]

The projection of the set X on the j-th coordinate is denoted by X_j, and its elements are referred to as the feasible positions on issue j. Note that there is no loss of generality in using the same set P of conceivable positions for all issues, as the subsets $X_j\subseteq P$ of feasible positions may differ across issues. We say that X is *binary* if $|X_j|\leqslant 2$ for every $j\in J$, and non-binary otherwise.

A *society* is a finite, non-empty set N of individuals. For convenience, if there are n individuals in N, we identify N with the set $\{1,\dots,n\}$. If we specify a feasible evaluation $x^i=(x^i_1,\dots,x^i_m)\in X$ for each individual $i\in N$, we obtain a *profile* of feasible evaluations $\mathbf{x}=(x^i_j)\in X^n$. We may view a profile as an $n\times m$ matrix all of whose rows lie in X. We use superscripts to indicate individuals (rows) and subscripts to indicate issues (columns).

An *aggregator* for N over X is a mapping $f:X^n\to X$. It assigns to every possible profile of individual feasible evaluations, a social evaluation which is also feasible. Any aggregator f may be written in the form $f=(f_1,\dots,f_m)$ where f_j is the j-th component of f. That is, $f_j:X^n\to X_j$ assigns to every profile the social position on issue j.

An aggregator f is *IIA* if for every $j\in J$ and any two profiles $\mathbf{x},\mathbf{y}\in X^n$ satisfying $x^i_j=y^i_j$ for all $i\in N$, we have $f_j(\mathbf{x})=f_j(\mathbf{y})$. This means that the social position on a given issue is determined entirely by the individual positions on that same issue. Viewing profiles as matrices, this says that the aggregation is done column-by-column. As we shall deal with IIA aggregators, we will slightly abuse notation and write also expressions of the form $f_j(x_j)$, where $x_j=(x^1_j,\dots,x^n_j)$ is the column vector of individual positions on issue j. That is, we will treat f_j also as mapping X^n_j into X_j.

An IIA aggregator f is *supportive* if we have $f_j(x_j)\in\{x^1_j,\dots,x^n_j\}$ for every $j\in J$ and every $x_j=(x^1_j,\dots,x^n_j)\in X^n_j$. This means that the social position on any issue must be one of the individual positions on that issue.[5]

An aggregator f is *dictatorial* if there exists an individual $d\in N$ such that $f(\mathbf{x})=x^d$ for every $\mathbf{x}\in X^n$. That is to say, the society always adopts the dictator's evaluation. A dictatorial aggregator is trivially IIA and supportive.

We say that X is an *impossibility domain* if for every society N, every IIA and supportive aggregator for N over X is dictatorial. Otherwise we say that X is a possibility domain. By this definition, X is a possibility domain if for some n there exists a non-dictatorial IIA and supportive aggregator $f:X^n\to X$. It will turn out, however, that in this case such aggregators exist for all $n\geqslant 3$ (and sometimes also for $n=2$).

[4] Our impossibility results extend in a straightforward way to a model with infinitely-many issues. The possibility results require compactness in the following sense: every infeasible evaluation has a restriction to a finite set of issues which is infeasible (i.e., cannot be completed to a feasible evaluation).

[5] Note that in the binary case supportiveness is equivalent to the Pareto property, which requires that $f_j(u,\dots,u)=u$ for every $j\in J$ and every $u\in X_j$. But in the general case supportiveness is stronger. We observe that both properties may be defined more generally for any aggregator, not necessarily IIA, by global conditions that refer to the entire profile. Thus f is Paretian if we have $f(\mathbf{x})=x$ whenever the profile $\mathbf{x}$ is such that $x^i=x$ for all $i\in N$; and f is supportive if we have $f_j(\mathbf{x})\in\{x^1_j,\dots,x^n_j\}$ for every $j\in J$ and every profile $\mathbf{x}$ such that for all $i,i'\in N$, $x^i_j=x^{i'}_j$ implies $x^i=x^{i'}$. These are conceptually less demanding definitions which, in the presence of IIA, yield the issue-by-issue definitions given above.

Our aim is to find structural conditions on the set X that can be used to classify it as a possibility or an impossibility domain. We start by introducing some terminology and tools that will be used to describe the structure of X. Note that X is a subset of the Cartesian product $\prod_{j=1}^{m} X_j$, and we may assume that $|X_j| \geqslant 2$ for all j (an issue j with $|X_j| = 1$ may be discarded without affecting the problem). A *sub-box* is a subset B of $\prod_{j=1}^{m} X_j$ of the form $B = \prod_{j=1}^{m} B_j$, where $B_j \subseteq X_j$ for each j. We call such B a 2-*sub-box* if $|B_j| = 2$ for each j. The set X induces a set of feasible evaluations in each sub-box B, namely the set $X \cap B$. When B is a 2-sub-box, this puts us in a setting that is isomorphic to the binary case of our problem. We will exploit this to lift known concepts from the binary case to the general case. These concepts are originally due, in the binary case, to Nehring and Puppe [14], but we follow and adapt the terminology we introduced in [7].

Let X be a subset of P^m, let B be a fixed sub-box, and let K be a subset of J. A *K-evaluation within B* is a vector $x = (x_j)_{j \in K} \in \prod_{j \in K} B_j$; this is a partial evaluation assigning values to issues in K only, and lying in the corresponding projection of B. Such x is said to be *feasible within B* if it can be completed (by assigning values also to issues in $J \setminus K$) to an evaluation in $X \cap B$; otherwise, it is infeasible within B. A *minimally infeasible partial evaluation within B* (abbreviated B-MIPE) is a vector $x = (x_j)_{j \in K} \in \prod_{j \in K} B_j$ that is infeasible within B, but such that every restriction of x to a proper subset of K is feasible within B. The B-MIPEs are thus the minimal obstacles to feasibility within B. We will also use the above terminology without specifying a sub-box B, when we refer to the whole box, that is, $B = \prod_{j=1}^{m} X_j$.

Our first condition on X will be expressed in terms of a directed graph G_X associated with X, that we proceed to define. The vertices of G_X are labeled by the triples uu'_j, where $j \in J$ and $u, u' \in X_j$, $u \neq u'$. Thus we have $\sum_{j=1}^{m} |X_j|(|X_j| - 1)$ vertices (note that this number is infinite if some X_j is infinite). The vertex uu'_j is to be interpreted as holding position u rather than u' on issue j, in the sense that u is the position held on j if only positions u and u' are available. There is an arc in G_X from vertex uu'_k to vertex vv'_ℓ (written $uu'_k \to vv'_\ell$) if and only if $k \neq \ell$ and there exist a 2-sub-box $B = \prod_{j=1}^{m} B_j$ with $B_k = \{u, u'\}$ and $B_\ell = \{v, v'\}$ and a B-MIPE $x = (x_j)_{j \in K}$ such that $\{k, \ell\} \subseteq K$ and $x_k = u$, $x_\ell = v'$. For such B, x, and K, we also write $uu'_k \xrightarrow[B,x,K]{} vv'_\ell$. We call the relation $uu'_k \to vv'_\ell$ relative conditional entailment, with the following interpretation. If $uu'_k \xrightarrow[B,x,K]{} vv'_\ell$ then relative to B and conditional on holding the positions prescribed in x on all issues in $K \setminus \{k, \ell\}$, holding position u rather than u' on issue k entails holding position v rather than v' on issue ℓ, because x is infeasible within B. Note that the minimality of x implies that there exists a feasible evaluation in B with entries u on k and v on ℓ that agrees with x on $K \setminus \{k, \ell\}$, and similarly for u' on k and v' on ℓ.[6] We write $uu'_k \to\to vv'_\ell$ if there exists a directed path in G_X from uu'_k to vv'_ℓ. Finally, we say that X is *totally blocked* if G_X is strongly connected, that is, for any two vertices uu'_k and vv'_ℓ we have $uu'_k \to\to vv'_\ell$.

The length of a B-MIPE $x = (x_j)_{j \in K}$ is $|K|$. We say that X is *multiply constrained* if there exists a sub-box B for which there exists a B-MIPE of length at least 3.

We are now ready to state our two general results, providing sufficient and necessary conditions, respectively, for X to be an impossibility domain.

Theorem 1. *Let X be a non-binary subset of P^m. If X is totally blocked and multiply constrained then X is an impossibility domain.*

Theorem 2. *Let X be a subset of P^m. If X is not totally blocked then X is a possibility domain*; *in fact, for every society N of* 2 *or more individuals there exists a non-dictatorial IIA and supportive aggregator for N over X.*

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[6] For a fixed 2-sub-box B, our notion of relative conditional entailment is the same as that of conditional entailment in the binary case. In the non-binary case, a MIPE in the whole box does not give rise to a conditional entailment relation, since there is more than one alternative to a given position on a given issue. That is why we had to restrict to two available positions on each of the issues k and ℓ. The restriction of positions on the other issues is not needed for the definition to make sense, but turns out to be needed for our results. Note that it renders the condition of total blockedness weaker than it would be without it.

We show next how these theorems work in a number of interesting examples. First, we apply Theorem 1 to the problem of aggregating assignments presented in Example A in the introduction.

Corollary 1. *Let $|P| = p$ and $|J| = m$, with $p \geqslant m \geqslant 4$. Let*

$$X = \left\{(x_1, \ldots, x_m) \in P^m \;\middle|\; x_1, \ldots, x_m \text{ are pairwise distinct}\right\}$$

be the set of feasible assignments (of people in P to the jobs in J). Then X is an impossibility domain.

Proof. Clearly, $X_j = P$ for all $j \in J$, so X is non-binary. For any $k, \ell \in J$, $k \neq \ell$, and any $u, v, w \in P$, $v \neq u \neq w$, we have $uv_k \to wu_\ell$. Indeed, there exist $y = (y_1, \ldots, y_m)$, $z = (z_1, \ldots, z_m)$ in X with $y_k = u$, $y_\ell = w$, $z_k = v$, $z_\ell = u$, and we can take B to be a 2-sub-box that contains y and z and $x = (x_k, x_\ell)$ to be the B-MIPE with $x_k = x_\ell = u$, to witness that $uv_k \to wu_\ell$. Using this fact repeatedly, we show that X is totally blocked. Indeed, it suffices to show that $uu'_k \to\to vv'_\ell$ for any two vertices with $k \neq \ell$. If $u = v'$ this holds in one step, and otherwise taking $j \in J \setminus \{k, \ell\}$ we have $uu'_k \to v'u_j \to vv'_\ell$. Next, we check that X is multiply constrained by taking four distinct elements t, u, v, w of P and considering the sub-box B with $B_1 = \{t, u\}$, $B_2 = \{t, v\}$, $B_3 = \{t, w\}$, $B_4 = \{u, v, w\}$, $B_j = P$ for $5 \leqslant j \leqslant m$, and the B-MIPE (x_1, x_2, x_3) with $x_1 = u$, $x_2 = v$, $x_3 = w$. It now follows from Theorem 1 that X is an impossibility domain. $\square$

Two comments are in order about Corollary 1. First, note that its proof takes advantage of the fact that the definition of multiple constrainedness does not require a MIPE of length at least 3 in the whole box, just in some sub-box. Indeed, it is easy to see that in this example all MIPEs have length 2, but we were able to find a suitable sub-box B with a B-MIPE of length 3. Secondly, although the corollary was stated here for $p \geqslant m \geqslant 4$ (to allow the construction of a sub-box B as mentioned), it does in fact hold true more generally for all $p \geqslant m$ such that $p \geqslant 3$ and $m \geqslant 2$. We omit the proof of this fact, which uses ideas that will be developed in Section 4. This fact shows, in particular, that unlike total blockedness, multiple constrainedness is not a necessary condition for an impossibility domain.

Next we apply Theorem 2 to a special case of the problem of judgment aggregation in many-valued logic presented in Example C. Recall that in the introduction we pointed out that supportiveness may be too strong a requirement in the context of this problem. But here we obtain a possibility result, which is only strengthened by our use of supportiveness rather than the plain Pareto property.

Example 1. Consider the agenda $\mathcal{A} = \{\alpha, \beta, \alpha \wedge \beta\}$, and suppose that each proposition φ in $\mathcal{A}$ may be assigned any truth value $T(\varphi)$ in $P = \{0, 1, \ldots, p-1\}$, subject to the consistency requirement $T(\alpha \wedge \beta) = \min\{T(\alpha), T(\beta)\}$. This leads to the set of feasible evaluations:

$$X = \left\{(x_1, x_2, x_3) \in P^3 \;\middle|\; x_3 = \min\{x_1, x_2\}\right\}.$$

Note that any feasible evaluation with a 0 entry must have at least two 0 entries; and conversely, any evaluation with at least two 0 entries is feasible, except $(0, 0, x_3)$ with $x_3 \neq 0$. It follows that G_X does not contain any arc from a vertex of the form $0u_k$ to one of the form vw_ℓ with $v \neq 0$. Indeed, such an arc would require the existence of an infeasible evaluation with $x_k = 0$, $x_\ell = w$ that becomes feasible upon replacing w by $v \neq 0$, which is impossible by the above. Thus X is not totally blocked and hence, by Theorem 2, it is a possibility domain. This can also be verified directly, by noting that the 'least-belief' rule $x_j = \min\{x^i_j \mid i \in N\}$ for $j = 1, 2, 3$ yields an aggregator with the required properties.

Our last example in this section will show that the assumption of multiple constrainedness, although not a necessary condition for impossibility, cannot be entirely dropped from Theorem 1 without losing sufficiency. In other words, there do exist non-binary totally blocked possibility domains.

Example 2. Let $P = \{0, 1\}^2$ and let $m \geqslant 3$. Thus, positions are two-bit binary vectors, and we use the notation $u = (\underline{u}, \overline{u})$ for $u \in P$. We think of the m issues as ordered cyclically, identifying the subscript $m + 1$ with 1. Consider the set of feasible evaluations:

$$X = \{(x_1, \ldots, x_m) \in P^m \mid \bar{x}_j = \underline{x}_{j+1} \text{ for } j = 1, \ldots, m\}.$$

Clearly $|X_j| = |P| = 4$ for all $j \in J$, so X is non-binary. Any two vertices of the form uu'_j and vv'_{j+1}, where $\overline{u} \neq \overline{u}' = \underline{v}' \neq \underline{v}$, are joined in G_X by arcs going in both directions; this is because, as $m \geqslant 3$, on issues j and $j + 1$ positions u and v are compatible, and so are u' and v', but u and v' are not, and nor are u' and v. Using these arcs, it is easy to see that X is totally blocked. However, X is a possibility domain: assuming that n is odd (or replacing N in the following by an odd cardinality subset) the ‘bit-by-bit majority’ rule whereby $\underline{x}_j = \mathrm{maj}\{\underline{x}^i_j \mid i \in N\}$ and $\bar{x}_j = \mathrm{maj}\{\bar{x}^i_j \mid i \in N\}$ for all $j \in J$ yields an aggregator with the required properties.

The example just given may be understood as a binary example with unrestricted domain, embedded as a restricted quaternary domain. But it is by no means true that all totally blocked possibility domains are derived from binary underlying examples. See, for instance, Example 4 in Section 4.

3. Proof of Theorems 1 and 2

3.1. *Winning coalitions and* 2*-neutrality*

Throughout the proof of Theorem 1, we consider an IIA and supportive aggregator $f : X^n \to X$. In each step of the proof, we will establish properties of f, using some of the conditions on X assumed in the theorem and/or some of the properties of f established earlier (the conditions and properties used in each step will be stated explicitly). Eventually we will show that f is dictatorial.

We consider the components f_j of f. Given the IIA property, each f_j will be viewed as mapping columns of positions on issue j, of the form $x_j = (x^1_j, \ldots, x^n_j) \in X^n_j$, into X_j. Several of the properties of f will be expressed in terms of the behavior of the f_j on columns that consist of at most 2 different positions. This behavior is captured by the collections of winning coalitions defined as follows. For an issue j and an ordered pair of distinct positions $u, u' \in X_j$, we say that a subset S of N is a *uu'_j-winning coalition* if

$$x^i_j = \begin{cases} u & \text{if } i \in S \\ u' & \text{if } i \in N \setminus S \end{cases} \quad \Rightarrow \quad f_j(x_j) = u.$$

Thus, S is uu'_j-winning if it prevails on issue j when its members hold the position u while the others hold the position u'. We denote by $\mathcal{W}^{uu'}_j$ the collection of all uu'_j-winning coalitions. As f is supportive, we have $N \in \mathcal{W}^{uu'}_j$ and $\emptyset \notin \mathcal{W}^{uu'}_j$ for every j, u, u', and moreover $S \in \mathcal{W}^{uu'}_j \Leftrightarrow N \setminus S \notin \mathcal{W}^{u'u}_j$ (we will refer to the latter as *duality*). We say that f is 2*-neutral* if all the collections of winning coalitions coincide, that is, there exists one collection of winning coalitions, that we denote by $\mathcal{W}$, such that $\mathcal{W}^{uu'}_j = \mathcal{W}$ for every j, u, u'. This property means that when it comes to choosing between 2 positions on an issue, f treats equally all positions on all issues.

Lemma 1. *If $uu'_k \to vv'_\ell$ in the graph G_X then $\mathcal{W}^{uu'}_k \subseteq \mathcal{W}^{vv'}_\ell$.*

Proof. Suppose that $uu'_k \xrightarrow[B,x,K]{} vv'_\ell$ for a suitable 2-sub-box B and a B-MIPE $x = (x_j)_{j \in K}$. Assume, for the sake of contradiction, that the coalition S is in $\mathcal{W}^{uu'}_k$ but not in $\mathcal{W}^{vv'}_\ell$. In Table 1 we construct a profile of feasible evaluations and the resulting social evaluation, all restricted to issues in K (for ease of exposition, we assume that $K = \{1, \ldots, r\}$ and $k = 1$, $\ell = 2$).

Table 1
Construction for Lemma 1.

	1	2	3	$\cdots$	r
S	u	v	x_3	$\cdots$	x_r
$N \setminus S$	u'	v'	x_3	$\cdots$	x_r
	u	v'	x_3	$\cdots$	x_r

The fact that $uu'_1 \xrightarrow[B,x,K]{} vv'_2$ implies that each of the rows corresponding to S and $N \setminus S$ in the table is feasible within B, i.e., may be completed to an evaluation in $X \cap B$. As f is supportive, the resulting social evaluation must also lie in $X \cap B$. However, by our assumptions on S and supportiveness, the social positions on issues in K are as indicated in the table, and thus coincide with x, which is infeasible within B. $\square$

Proposition 1. *If X is totally blocked then f is 2-neutral.*

Proof. By repeated applications of Lemma 1, it follows that $uu'_k \to\to vv'_\ell$ implies $\mathcal{W}_k^{uu'} \subseteq \mathcal{W}_\ell^{vv'}$. Therefore, if X is totally blocked then all the collections of winning coalitions coincide. $\square$

3.2. *Establishing 2-monotonicity*

We say that f is 2-*monotone* if each of the collections $\mathcal{W}_j^{uu'}$ is closed with respect to supersets: $S \in \mathcal{W}_j^{uu'}$ and $S \subset T$ imply $T \in \mathcal{W}_j^{uu'}$.

Proposition 2. *If X is non-binary and totally blocked then f is 2-monotone.*

Proof. By Proposition 1, f is 2-neutral. The failure of 2-monotonicity can then be stated as the existence of $S \subset T$ such that $S \in \mathcal{W}$, $T \notin \mathcal{W}$; or equivalently, the existence of a partition S_1, S_2, S_3 of N with $S_1, S_2 \in \mathcal{W}$ (taking $S_1 = S$, $S_2 = N \setminus T$, $S_3 = T \setminus S$). We fix such a partition and work towards a contradiction. For any $j \in J$ and any $(a, b, c) \in X_j^3$, the value assumed by f_j on the column in which the members of S_1 hold position a, those of S_2 hold position b, and those of S_3 hold position c, will be denoted (by a slight abuse of notation) $f_j(a, b, c)$. We need the following lemma.

Lemma 2. *Let $y = (y_1, \dots, y_m) \in X$ be fixed. If $uu'_k \to vv'_\ell$ in the graph G_X and $f_k(u, u', y_k) = y_k$ then $f_\ell(v, v', y_\ell) = y_\ell$.*

Proof. Suppose that $uu'_k \xrightarrow[B,x,K]{} vv'_\ell$ for a suitable 2-sub-box B and a B-MIPE $x = (x_j)_{j \in K}$. For ease of exposition, we assume that $K = \{1, \dots, r\}$ and $k = 1, \ell = 2$. As x is a B-MIPE, there exists $z = (z_1, \dots, z_m) \in X \cap B$ with $z_1 = u$, $z_2 = v$, $z_j = x_j$ for $j = 3, \dots, r$, and there exists $z' = (z'_1, \dots, z'_m) \in X \cap B$ with $z'_1 = u'$, $z'_2 = v'$, $z'_j = x_j$ for $j = 3, \dots, r$. Using these two evaluations and the fixed evaluation y in the statement of the lemma, we construct in Table 2 a profile of feasible evaluations and the resulting social evaluation that we denote by $s = (s_1, \dots, s_m) \in X$.

The social position on issue 1 is determined by an assumption of the lemma. We assume, for the sake of contradiction, that $s_2 \neq y_2$, and hence by supportiveness $s_2 \in \{v, v'\}$. Using now s as the third row, we construct in Table 3 another profile of feasible evaluations and the resulting social evaluation that we denote by $t = (t_1, \dots, t_m) \in X$.

For $j = 3, \dots, r$ we have $t_j = s_j$ for the following reason: if $s_j = y_j$ then column j is the same as in Table 2, and otherwise by supportiveness $s_j = x_j$ and $t_j = x_j$. Regarding s_2 and t_2 we have two cases. In the first case, $s_2 = v$ and therefore, since $S_2 \in \mathcal{W}$, $t_2 = v'$. In this case we use t as the second row and s as the third, constructing in Table 4 yet another profile of feasible evaluations and the resulting social evaluation that we denote by $w = (w_1, \dots, w_m) \in X$.

The social positions on issues $1, \dots, r$ are implied by our assumptions that $S_1, S_2 \in \mathcal{W}$. For $j = r + 1, \dots, m$ we have $w_j \in B_j$. Indeed, if $t_j = s_j$ then, since $S_1 \in \mathcal{W}$, we have $w_j = z_j \in B_j$. If $t_j \neq s_j$,

Table 2
First construction for Lemma 2.

	1	2	3	$\cdots$	r	$r+1$	$\cdots$	m
S_1	u	v	x_3	$\cdots$	x_r	z_{r+1}	$\cdots$	z_m
S_2	u'	v'	x_3	$\cdots$	x_r	z'_{r+1}	$\cdots$	z'_m
S_3	y_1	y_2	y_3	$\cdots$	y_r	y_{r+1}	$\cdots$	y_m
	y_1	s_2	s_3	$\cdots$	s_r	s_{r+1}	$\cdots$	s_m

Table 3
Second construction for Lemma 2.

	1	2	3	$\cdots$	r	$r+1$	$\cdots$	m
S_1	u	v	x_3	$\cdots$	x_r	z_{r+1}	$\cdots$	z_m
S_2	u'	v'	x_3	$\cdots$	x_r	z'_{r+1}	$\cdots$	z'_m
S_3	y_1	s_2	s_3	$\cdots$	s_r	s_{r+1}	$\cdots$	s_m
	y_1	t_2	s_3	$\cdots$	s_r	t_{r+1}	$\cdots$	t_m

Table 4
Third construction for Lemma 2.

	1	2	3	$\cdots$	r	$r+1$	$\cdots$	m
S_1	u	v	x_3	$\cdots$	x_r	z_{r+1}	$\cdots$	z_m
S_2	y_1	v'	s_3	$\cdots$	s_r	t_{r+1}	$\cdots$	t_m
S_3	y_1	v	s_3	$\cdots$	s_r	s_{r+1}	$\cdots$	s_m
	u	v'	x_3	$\cdots$	x_r	w_{r+1}	$\cdots$	w_m

this means that the column j outcomes in Tables 2 and 3 differ, which must be because $s_j \neq y_j$. By supportiveness, this implies that $s_j \in \{z_j, z'_j\}$, which implies that $t_j \in \{z_j, z'_j\}$, which in turn implies that $w_j \in \{z_j, z'_j\} \subseteq B_j$. It follows that $w \in X \cap B$, contradicting the fact that x is a B-MIPE with $x_1 = u, x_2 = v'$.

In the remaining case, $s_2 = v'$ and therefore, since $S_1 \in \mathcal{W}$, $t_2 = v$. In this case, we redo the construction in Table 4 with the roles of s and t interchanged, and reach the same contradiction. $\square$

Returning to the proof of Proposition 2, we treat now the following case.

Case 1. There exist $j \in J$ and pairwise distinct $a, b, c \in X_j$ such that $f_j(a, b, c) = c$.

Since X is totally blocked, we have $ab_j \to\to ac_j$. We fix $y = (y_1, \ldots, y_m) \in X$ with $y_j = c$, and apply Lemma 2 repeatedly along a path in G_X from ab_j to ac_j. The initial assumption that $f_j(a, b, c) = c$ carries over along the path to the final conclusion that $f_j(a, c, c) = c$. However, the latter contradicts the fact that $S_1 \in \mathcal{W}$.

Note that if $S_3 \in \mathcal{W}$ then S_1, S_2, S_3 play symmetric roles in our assumptions. The fact that X is non-binary means that there exist $j \in J$ and pairwise distinct $a, b, c \in X_j$, and so the assumption of Case 1 holds without loss of generality. Thus we may assume that $S_3 \notin \mathcal{W}$. This implies that whenever $a, b, c \in X_j$ are not pairwise distinct, we have $f_j(a, b, c) \in \{a, b\}$. Therefore, in negating Case 1 we can refer to all, not necessarily pairwise distinct triples, getting the following.

Case 2. For all $j \in J$ and all $a, b, c \in X_j$ we have $f_j(a, b, c) \in \{a, b\}$.

Now we need the following lemma.

Lemma 3. *Assume Case 2 holds, and let $y = (y_1, \ldots, y_m) \in X$ be fixed. If $uu'_k \to vv'_\ell$ in the graph G_X and $f_k(u, u', y_k) = u$ then $f_\ell(v, v', y_\ell) = v$.*

Table 5
Construction for Proposition 3.

	1	2	3	4	$\cdots$	r
S	x'_1	x_2	x_3	x_4	$\cdots$	x_r
T	x_1	x'_2	x_3	x_4	$\cdots$	x_r
$N \setminus U$	x_1	x_2	x'_3	x_4	$\cdots$	x_r
	x_1	x_2	x_3	x_4	$\cdots$	x_r

Proof. Consider again the construction in Table 2, with resulting social evaluation $s = (s_1, \ldots, s_m) \in X$. By our current assumption, $s_1 = u$. We assume, for the sake of contradiction, that $s_2 \neq v$. By the assumption of Case 2, this implies that $s_2 = v'$. The same assumption implies that $s_j = x_j$ for $j = 3, \ldots, r$ and $s_j \in \{z_j, z'_j\} \subseteq B_j$ for $j = r+1, \ldots, m$. This contradicts the fact that x is a B-MIPE. $\square$

We can now complete the proof of Proposition 2 by treating the remaining Case 2. We fix any $j \in J$ and distinct $a, b \in X_j$, and some $y = (y_1, \ldots, y_m) \in X$ with $y_j = b$. We apply Lemma 3 repeatedly along a path in G_X from ab_j to ba_j. Initially we have $f_j(a, b, b) = a$ because $S_1 \in \mathcal{W}$, and this carries over along the path to yield finally $f_j(b, a, b) = b$. However, the latter contradicts the fact that $S_2 \in \mathcal{W}$. $\square$

3.3. *Establishing* 2*-decomposability*

Assume that f is 2-neutral. We say that f is 2-*decomposable* if for every $U \in \mathcal{W}$ and every partition S, T of U we have either $S \in \mathcal{W}$ or $T \in \mathcal{W}$.

Proposition 3. *If X is multiply constrained and f is 2-neutral then f is 2-decomposable.*

Proof. Let B be a sub-box and $x = (x_j)_{j \in K}$ be a B-MIPE with $|K| \geqslant 3$. Assume, for the sake of contradiction, that $S, T \notin \mathcal{W}$, $S \cap T = \emptyset$, and $U = S \cup T \in \mathcal{W}$. Now, consider the construction in Table 5 (where for ease of exposition $K = \{1, \ldots, r\}$).

The fact that x is a B-MIPE guarantees that for $j = 1, 2, 3$ there exists $x'_j \in B_j$ so that the j-th row in the table is feasible within B. However, by our assumptions on S, T, and U and supportiveness, the resulting social positions coincide with x, which is infeasible within B. $\square$

3.4. *Establishing* 2*-dictatorship*

We say that f is 2-*dictatorial* if there exists an individual $d \in N$ so that each of the collections $\mathcal{W}_j^{uu'}$ is equal to $\{S \subseteq N \mid d \in S\}$. This means that d prevails on any issue when the individuals hold at most 2 distinct positions on that issue.

Proposition 4. *If f is 2-neutral, 2-monotone, and 2-decomposable, then it is 2-dictatorial.*

Proof. Let U be a winning coalition of minimum cardinality (this is well defined because $N \in \mathcal{W}$). If $|U| \geqslant 2$ then we get a contradiction to minimality by using 2-decomposability. Clearly $|U| = 0$ is impossible since $\emptyset \notin \mathcal{W}$. Hence there exists $d \in N$ such that $U = \{d\}$. By 2-monotonicity it follows that $\{S \subseteq N \mid d \in S\} \subseteq \mathcal{W}$, and the reverse inclusion follows by duality. $\square$

3.5. *From* 2*-dictatorship to dictatorship*

Proposition 5. *If X is totally blocked and f is 2-dictatorial then f is dictatorial.*

Proof. Let d be the 2-dictator for f. Assume, for the sake of contradiction, that d is not a dictator for f. Then there exists at least one instance of $p \in J$ and $x_p = (x_p^1, \ldots, x_p^n) \in X_p^n$ so that $f_p(x_p) \neq x_p^d$.

Table 6
First construction for Lemma 4.

	1	2	3	⋯	r	$r+1$	⋯	m
S_1	u	v	x_3	⋯	x_r	z_{r+1}	⋯	z_m
S_2	u'	v'	x_3	⋯	x_r	z'_{r+1}	⋯	z'_m
S_3	y_1^3	y_2^3	y_3^3	⋯	y_r^3	y_{r+1}^3	⋯	y_m^3
.	.	.	.		.	.		.
.	.	.	.		.	.		.
.	.	.	.		.	.		.
S_q	y_1^q	y_2^q	y_3^q	⋯	y_r^q	y_{r+1}^q	⋯	y_m^q
	u	s_2	x_3	⋯	x_r	s_{r+1}	⋯	s_m

Table 7
Second construction for Lemma 4.

	1	2	3	⋯	r	$r+1$	⋯	m
S_1	u	s_2	x_3	⋯	x_r	s_{r+1}	⋯	s_m
S_2	u'	v'	x_3	⋯	x_r	z'_{r+1}	⋯	z'_m
S_3	y_1^3	y_2^3	y_3^3	⋯	y_r^3	y_{r+1}^3	⋯	y_m^3
.	.	.	.		.	.		.
.	.	.	.		.	.		.
.	.	.	.		.	.		.
S_q	y_1^q	y_2^q	y_3^q	⋯	y_r^q	y_{r+1}^q	⋯	y_m^q
	u	v'	x_3	⋯	x_r	t_{r+1}	⋯	t_m

Choose one instance so that the number q of distinct positions appearing in x_p is the smallest possible among all such instances. Necessarily $q \geqslant 3$, since d is a 2-dictator. Enumerate the distinct positions appearing in x_p as $w^1, \dots, w^q$, with $w^1 = f_p(x_p)$ and $w^2 = x_p^d$. Let $S_h = \{i \in N \mid x_p^i = w^h\}$, $h = 1, \dots, q$, be the partition of N according to the positions held in x_p (note that $d \in S_2$). For any $k \in J$ and any $(a^1, \dots, a^q) \in X_k^q$, the value assumed by f_k on the column in which the members of S_h hold position a^h, $h = 1, \dots, q$, will be denoted (by a slight abuse of notation) $f_k(a^1, \dots, a^q)$. Using this notation, we have $f_p(w^1, \dots, w^q) = w^1$. Note that by the minimality of q, for any $k \in J$ and any $a^1, \dots, a^q \in X_k$ that are not pairwise distinct, we have $f_k(a^1, \dots, a^q) = a^2$. We need the following lemma.

Lemma 4. *Let $y^h = (y_1^h, \dots, y_m^h) \in X$, $h = 3, \dots, q$, be $q-2$ fixed evaluations. If $uu'_k \to vv'_\ell$ in the graph G_X and $f_k(u, u', y_k^3, \dots, y_k^q) = u$ then $f_\ell(v, v', y_\ell^3, \dots, y_\ell^q) = v$.*

Proof. Suppose that $uu'_k \xrightarrow[B,x,K]{} vv'_\ell$ for a suitable 2-sub-box B and a B-MIPE $x = (x_j)_{j \in K}$. For ease of exposition, we assume that $K = \{1, \dots, r\}$ and $k = 1$, $\ell = 2$. As x is a B-MIPE, there exists $z = (z_1, \dots, z_m) \in X \cap B$ with $z_1 = u$, $z_2 = v$, $z_j = x_j$ for $j = 3, \dots, r$, and there exists $z' = (z'_1, \dots, z'_m) \in X \cap B$ with $z'_1 = u'$, $z'_2 = v'$, $z'_j = x_j$ for $j = 3, \dots, r$. Using these two evaluations and the $q-2$ fixed evaluations in the statement of the lemma, we construct in Table 6 a profile of feasible evaluations and the resulting social evaluation that we denote by $s = (s_1, \dots, s_m) \in X$.

The social position on issue 1 is determined by an assumption of the lemma, and those on issues $3, \dots, r$ by the existence of repeated values in those columns (which forces the outcome to be the position of S_2). We assume, for the sake of contradiction, that $s_2 \neq v$. Using now s as the first row, we construct in Table 7 another profile of feasible evaluations and the resulting social evaluation that we denote by $t = (t_1, \dots, t_m) \in X$.

The social position on issue 2 is v' because we assume $s_2 \neq v$ and hence, by supportiveness, s_2 is one of the other values in that column. For $j = r+1, \dots, m$ we have $t_j \in B_j$. Indeed, if $s_j = z_j$ then column j is the same in both tables, and therefore $t_j = s_j = z_j \in B_j$. Otherwise, s_j is one of the other values in that column, and hence $t_j = z'_j \in B_j$. It follows that $t \in X \cap B$, contradicting the fact that x is a B-MIPE. □

Returning to the proof of Proposition 5, we recall that for a particular $p \in J$ and $(w^1, \ldots, w^q) \in X_p^q$ we have $f_p(w^1, \ldots, w^q) = w^1$. We choose the $q-2$ evaluations $y^h = (y_1^h, \ldots, y_m^h) \in X$ so that $y_p^h = w^h$, $h = 3, \ldots, q$. We apply Lemma 4 repeatedly along a path in G_X from $w^1 w_p^2$ to $w^1 w_p^3$. The initial assumption that $f_p(w^1, \ldots, w^q) = w^1$ carries over along the path to the final conclusion that $f_p(w^1, w^3, w^3, \ldots, w^q) = w^1$. Since the latter q-tuple contains repeated values, this is a contradiction. □

Taken together, Propositions 1–5 yield a proof of Theorem 1.

3.6. *Proof of Theorem 2*

We assume that X is not totally blocked. Hence there exists a partition of the vertices of G_X into two non-empty parts V_1 and V_2 so that there is no arc in G_X from a vertex in V_1 to a vertex in V_2. For $n \geqslant 2$, we define a non-dictatorial IIA and supportive aggregator $f : X^n \to X$ component-by-component as follows

$$f_j(x_j^1, \ldots, x_j^n) = \begin{cases} x_j^1 = x_j^2 & \text{if } x_j^1 = x_j^2, \\ x_j^1 & \text{if } x_j^1 \neq x_j^2 \text{ and } (x_j^1)(x_j^2)_j \in V_1, \\ x_j^2 & \text{if } x_j^1 \neq x_j^2 \text{ and } (x_j^1)(x_j^2)_j \in V_2. \end{cases}$$

We only need to show that $f = (f_1, \ldots, f_m)$ maps X^n into X (the other required properties of f are obvious). Suppose, for the sake of contradiction, that $f(\mathbf{x}) \notin X$ for some $\mathbf{x} \in X^n$. Fix a 2-sub-box B that contains x^1 and x^2. Since $f(\mathbf{x}) \in B \setminus X$, it has a restriction $y = (y_j)_{j \in K}$ which is a B-MIPE. There exist some $k \in K$ such that $y_k = x_k^1 \neq x_k^2$ and some $\ell \in K$ such that $y_\ell = x_\ell^2 \neq x_\ell^1$ (otherwise y would be contained in x^1 or x^2, which are in $X \cap B$). By the above definition, we must have $(x_k^1)(x_k^2)_k \in V_1$ and $(x_\ell^1)(x_\ell^2)_\ell \in V_2$. We also have $(x_k^1)(x_k^2)_k \xrightarrow[B,y,K]{} (x_\ell^1)(x_\ell^2)_\ell$, which contradicts our assumption about V_1 and V_2.

4. The ternary case

The set X of feasible evaluations is *ternary* if $\max_{j \in J} |X_j| = 3$. In the current section we deal with this case, and obtain a full characterization of impossibility domains. Theorems 1 and 2, proved above for the general case, suffice in order to classify any multiply constrained X as a possibility or an impossibility domain. So we focus here on the case when X is not multiply constrained, which means that all B-MIPEs have length at most 2. In particular, all MIPEs (with respect to the whole box $\prod_{j=1}^m X_j$) are of length 2.

Thus feasibility is determined entirely by a pairwise compatibility relation, which we represent by a graph H_X associated with X. The vertices of H_X are labeled as u_j, where $j \in J$ and $u \in X_j$ (so the number of vertices is $\sum_{j=1}^m |X_j|$). The vertex u_j is interpreted as holding position u on issue j. There is an (undirected) edge in H_X between u_k and v_ℓ if and only if $k \neq \ell$ and $x = (x_k, x_\ell)$ with $x_k = u$, $x_\ell = v$ is not a MIPE. In this case we say that u_k and v_ℓ are *compatible*, and write $u_k \sim v_\ell$; in the other case (when x defined as above is a MIPE) we say that u_k and v_ℓ are incompatible, and write $u_k \nsim v_\ell$. Note that for $x = (x_1, \ldots, x_m) \in \prod_{j=1}^m X_j$ we have $x \in X$ if and only if every two of its entries are compatible (the 'if' statement relies on our assumption that all MIPEs are of length 2).

The condition on X that we need for the characterization of impossibility domains will require the existence of a certain type of configuration in the compatibility graph H_X. Let k be an issue with $|X_k| = 3$. We say that a vertex u_k is *excluded* by another vertex v_ℓ if $k \neq \ell$ and we have $u_k \nsim v_\ell$,

$u'_k \sim v_\ell$, $u''_k \sim v_\ell$, where u' and u'' are the other two positions in X_k. A vertex u_k is said to be excluded if it is excluded by some other vertex. We say that X is *exclusive* if there exists an issue k with $|X_k| = 3$ so that u_k is excluded for every $u \in X_k$.

We are now ready to state our characterization of impossibility domains in the ternary case.

Theorem 3. *Let X be a ternary subset of P^m. Then X is an impossibility domain if and only if X is totally blocked and is either multiply constrained or exclusive.*

Before proving Theorem 3, we illustrate it with two examples of ternary sets X. Each of them is totally blocked and not multiply constrained, and is therefore not settled by Theorems 1 and 2. The condition of exclusivity distinguishes between them and classifies them as an impossibility and a possibility domain, respectively.

Example 3. Consider the case $p = m = 3$ of the problem of aggregating assignments (presented above in Example A and Corollary 1):

$$X = \{(0,1,2), (0,2,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0)\}.$$

The same argument as in the proof of Corollary 1 shows that X is totally blocked. However, X is not multiply constrained: a B-MIPE of length 3 would be some $x \in B \setminus X$, and so it would have two equal entries, which would be infeasible by themselves. The set X is exclusive because every u_k is excluded by u_ℓ for $\ell \neq k$. It follows from Theorem 3 that X is an impossibility domain.

Example 4. Let

$$X = \{(0,1,2), (1,2,0), (2,0,1), (0,0,0)\}.$$

To verify that X is totally blocked, we observe that the following is a Hamiltonian cycle in G_X:

$$\begin{aligned} &01_1 \to 02_2 \to 21_1 \to 10_3 \to 20_1 \to 12_3 \to 01_2 \to 02_3 \to 21_2 \\ &\quad \to 10_1 \to 20_2 \to 12_1 \to 01_3 \to 02_1 \to 21_3 \to 10_2 \to 20_3 \to 12_2 \to 01_1. \end{aligned}$$

To check that X is not multiply constrained, we note that an infeasible $x \in \{0,1,2\}^3$ either has two 1 entries or two 2 entries or equals (up to rotation) one of the following: $(0,0,1), (0,0,2), (2,1,0)$. In each case, it has two entries that are infeasible by themselves. The set X is not exclusive, because 0_k, for all k, is not excluded. Indeed, for 0_k to be excluded by v_ℓ we need $0_k \nsim v_\ell$, which implies that $v \neq 0$; but then also $v_k \nsim v_\ell$ and 0_k is not excluded. It follows from Theorem 3 that X is a possibility domain. This can also be verified directly, by noting that the 'majority with 0 as default' rule yields an aggregator with the required properties for odd n (or replacing N in the following by an odd cardinality subset): $x_j = \text{maj}\{x^i_j \mid i \in N\}$ if one of the 3 positions enjoys a majority, and $x_j = 0$ otherwise, $j = 1, 2, 3$.

4.1. Proof of Theorem 3: sufficiency

The only part of the proof of Theorem 1 that used multiple constrainedness is Proposition 3. The following substitute for Proposition 3 will therefore yield a proof of the sufficiency part of Theorem 3.

Proposition 6. *If X is exclusive and f is 2-neutral then f is 2-decomposable.*

Table 8
Construction for Proposition 6.

	k	ℓ
S_1	u	v'
S_2	u'	v
S_3	u''	v
	u	v

Proof. Let k be an issue with $X_k = \{u, u', u''\}$ so that u_k, u'_k, u''_k are all excluded. The failure of 2-decomposability can be stated as the existence of disjoint $S, T \notin \mathcal{W}$ so that $S \cup T \in \mathcal{W}$; or equivalently, the existence of a partition S_1, S_2, S_3 of N with $S_1, S_2, S_3 \notin \mathcal{W}$ (taking $S_1 = S$, $S_2 = T$, $S_3 = N \setminus (S \cup T)$). We fix such a partition and apply f_k to the column in which the members of S_1 hold position u, those of S_2 hold position u', and those of S_3 hold position u''. We assume that the value of f_k on this column is u (this is without loss of generality, due to the symmetry in our assumptions). Let u_k be excluded by v_ℓ. Consider the construction in Table 8.

The row corresponding to S_1 in the table is feasible for some $v' \in X_\ell$, because $u \in X_k$. The rows corresponding to S_2 and S_3 are feasible because $u'_k \sim v_\ell$ and $u''_k \sim v_\ell$. The indicated social positions follow from our assumptions and yield a contradiction to $u_k \nsim v_\ell$. □

As a side remark, we point out that Proposition 6 is not specific to the ternary case. For the general case, we define X to be exclusive if there exist an issue k and a triple $Y_k \subseteq X_k$ so that for every $u \in Y_k$ there exists v_ℓ with $u_k \nsim v_\ell$, $u'_k \sim v_\ell$, $u''_k \sim v_\ell$ (where $Y_k = \{u, u', u''\}$). Proposition 6 remains true, with the same proof, and so the sufficiency part of Theorem 3 holds true for any non-binary X.[7] It is the necessity part that depends crucially on X being ternary.

4.2. *Proof of Theorem 3: necessity*

The necessity of total blockedness was already proved (for the general case) in Theorem 2. Thus, the following proposition will complete the proof of Theorem 3.

Proposition 7. *Let X be a ternary subset of P^m. If X is neither multiply constrained nor exclusive then X is a possibility domain*; *in fact, for every society N of* 3 *or more individuals there exists a non-dictatorial IIA and supportive aggregator for N over X.*

Proof. We assume that n is odd (otherwise we can replace N in the following by an odd cardinality subset). We say that a column $(x_j^1, \ldots, x_j^n) \in X_j^n$ of positions on issue j is majoritarian if there exists $u \in X_j$ such that $|\{i \in N \mid x_j^i = u\}| > n/2$; we denote this u by $\operatorname{maj}\{x_j^i \mid i \in N\}$. For an issue j and a fixed position $w \in X_j$, we define the 'majority with w as default' rule f_j^w by

$$f_j^w\left(x_j^1, \ldots, x_j^n\right) = \begin{cases} \operatorname{maj}\{x_j^i \mid i \in N\} & \text{if } (x_j^1, \ldots, x_j^n) \text{ is majoritarian,} \\ w & \text{otherwise.} \end{cases}$$

We will show that there exists a choice $z = (z_1, \ldots, z_m) \in \prod_{j=1}^m X_j$ of defaults for the issues, so that the resulting $f^z = (f_1^{z_1}, \ldots, f_m^{z_m})$ maps X^n into X. This will suffice, because the other required properties of f^z obviously hold regardless of the choice of z (note that supportiveness requires that n be odd).

[7] This observation may be used, for example, to obtain the more general version of the impossibility result for aggregating assignments, as stated after Corollary 1.

The plan of the proof is the following. First, we will introduce two conditions on $z = (z_1, \ldots, z_m)$. Next, we will prove that if the choice of defaults z satisfies the two conditions then f^z maps X^n into X. Finally, we will show how to choose the defaults so that the two conditions are satisfied.

Let $J_3 = \{j \in J \mid |X_j| = 3\}$. We note that we need to discuss only the choices of z_j for $j \in J_3$; if $|X_j| < 3$ then every column of positions on issue j is majoritarian, and z_j is immaterial. Our conditions are expressed in terms of the compatibility graph H_X.

Condition 1. *For every $k \in J_3$, the vertex z_k of H_X is not excluded.*

Let $k, \ell \in J_3$, $k \neq \ell$. We denote by $H_X^{k\ell}$ the bipartite subgraph that H_X induces on the 6 vertices corresponding to X_k and X_ℓ. We say that the pair k, ℓ is matchable if $H_X^{k\ell}$ has a perfect matching, that is, it contains 3 pairwise disjoint edges (and possibly other edges).

Condition 2. *For every $k, \ell \in J_3$, $k \neq \ell$, if k, ℓ is matchable then $z_k \sim z_\ell$.*

Assume now that $z = (z_1, \ldots, z_m)$ satisfies Conditions 1 and 2. Suppose, for the sake of contradiction, that $f^z(\mathbf{x}) \notin X$ for some $\mathbf{x} \in X^n$. For $j \in J$ and $w \in X_j$ we will use the notation $N_j^w = \{i \in N \mid x_j^i = w\}$ and will refer to this as the support set corresponding to w_j. Observe that since the individual evaluations are feasible, any two support sets that correspond to incompatible vertices must be disjoint. As $f^z(\mathbf{x})$ is infeasible, it contains a MIPE which, by our assumption that X is not multiply constrained, is necessarily of length 2. So there exist $k, \ell \in J$, $k \neq \ell$, $u \in X_k$, $v \in X_\ell$, so that $f_k^{z_k}(x_k) = u$, $f_\ell^{z_\ell}(x_\ell) = v$, and $u_k \nsim v_\ell$. Hence we have $N_k^u \cap N_\ell^v = \emptyset$.

There are three cases to consider depending on whether both, one, or none of the two columns x_k and x_ℓ is majoritarian.

Case 1. x_k and x_ℓ are majoritarian.

Then both N_k^u and N_ℓ^v are majorities, and therefore must intersect, contradicting the above.

Case 2. x_k is not majoritarian, x_ℓ is majoritarian.

Then $k \in J_3$ and $u = z_k$. Let u', u'' be the other two elements of X_k. By Condition 1, the vertex u_k is not excluded, in particular not by v_ℓ. Therefore one of u'_k, u''_k must be incompatible with v_ℓ, say $u'_k \nsim v_\ell$. This implies that $N_k^{u'} \cap N_\ell^v = \emptyset$ which, together with $N_k^u \cap N_\ell^v = \emptyset$, implies that $N_\ell^v \subseteq N_k^{u''}$. But this contradicts the assumption of Case 2, by which N_ℓ^v is a majority and $N_k^{u''}$ is not.

Case 3. x_k and x_ℓ are not majoritarian.

Then $k, \ell \in J_3$ and $u = z_k$, $v = z_\ell$. If k, ℓ is matchable then, by Condition 2, we must have $u_k \sim v_\ell$, a contradiction. On the other hand, if $H_X^{k\ell}$ has no perfect matching then it must have two vertices so that every edge is incident to at least one of them (this follows from König's duality theorem, and may also be verified directly for a 3-by-3 bipartite graph). This implies that the union of the support sets corresponding to these two vertices is N. Hence at least one of these two support sets must be a majority, contradicting the assumption of Case 3.

It remains to show that $z = (z_1, \ldots, z_m)$ may be chosen so that Conditions 1 and 2 are satisfied. It will be convenient to use the following notation for $k \in J_3$:

$$E_k = \{u_k \mid u \in X_k,\ u_k \text{ is excluded}\},$$

$$F_k = \{u_k \mid u \in X_k,\ u_k \text{ is not excluded}\}.$$

We will need the following lemma.

Lemma 5. *Let $k, \ell \in J_3$, $k \neq \ell$. Suppose that k, ℓ is matchable and there exist a vertex in F_k and a vertex in F_ℓ which are incompatible. Then every edge of $H_X^{k\ell}$ that has one end in F_k has its other end in F_ℓ, and vice versa.*

Proof. By a suitable choice of notation for the vertices, we may assume that $u_k \sim v_\ell$, $u'_k \sim v'_\ell$, $u''_k \sim v''_\ell$, and $u_k \nsim v'_\ell$, $u_k \in F_k$, $v'_\ell \in F_\ell$. Since u_k is not excluded, in particular not by v'_ℓ, we must have $u''_k \nsim v'_\ell$. Similarly, since v'_ℓ is not excluded, in particular not by u_k, we must have $u_k \nsim v''_\ell$. These two derived incompatibilities in turn imply further ones: as u_k is not excluded by v''_ℓ, we must have $u'_k \nsim v''_\ell$; and as v'_ℓ is not excluded by u''_k, we must have $u''_k \nsim v_\ell$. Taking stock, we see that the edges of $H_X^{k\ell}$ include $u_k \sim v_\ell$, $u'_k \sim v'_\ell$, $u''_k \sim v''_\ell$, and possibly also $u'_k \sim v_\ell$, but no other edge.

We claim that $v_\ell \in F_\ell$. Indeed, it is clearly not excluded by any vertex in $H_X^{k\ell}$. Suppose that v_ℓ is excluded by w_j, $j \neq k, \ell$. Then we have $v_\ell \nsim w_j$, $v'_\ell \sim w_j$, $v''_\ell \sim w_j$. Note that if w_j is compatible with a vertex in $H_X^{k\ell}$, and that vertex is incident to only one edge in $H_X^{k\ell}$, then w_j must be compatible also with the other end of that edge (to see this, consider a feasible evaluation that includes w_j and the vertex it is known to be compatible with). Using this, we conclude that $u_k \nsim w_j$, $u'_k \sim w_j$, $u''_k \sim w_j$, so u_k is also excluded by w_j. This contradicts our assumption that $u_k \in F_k$. A similar argument shows that $u'_k \in F_k$. So the four vertices $u_k, u'_k, v_\ell, v'_\ell$ are in F_k and F_ℓ respectively.

Regarding u''_k and v''_ℓ we distinguish two cases. If $u'_k \sim v_\ell$ then $u''_k \in E_k$ and $v''_\ell \in E_\ell$ because they are excluded by v_ℓ and u'_k, respectively. If $u'_k \nsim v_\ell$ then arguments similar to the above show that either $u''_k \in E_k$ and $v''_\ell \in E_\ell$ or $u''_k \in F_k$ and $v''_\ell \in F_\ell$. In any case, we see that every edge of $H_X^{k\ell}$ either has its two ends in F_k and F_ℓ, or it has them in E_k and E_ℓ. □

Returning to the proof of Proposition 7, we introduce an auxiliary graph Γ. The vertex set of Γ is J_3, and vertices $k, \ell \in J_3$, $k \neq \ell$, are joined by an (undirected) edge in Γ if k, ℓ satisfy the premises of Lemma 5. For each connected component C of Γ we proceed as follows. We choose an arbitrary vertex p of C and some $u \in X_p$ so that $u_p \in F_p$ (this is possible because we assume that X is not exclusive). Then we choose some $x = (x_1, \ldots, x_m) \in X$ so that $x_p = u$. For each vertex j of C we assign the corresponding default to be $z_j = x_j$. Doing this separately for each connected component of Γ, we determine all entries of $z = (z_1, \ldots, z_m)$ corresponding to issues in J_3.

We check that Condition 1 is satisfied by working within each connected component C of Γ. Given any vertex j of C, we apply Lemma 5 repeatedly along a path in Γ from p to j. According to the initial choice we have $x_p \in F_p$, and this carries over along the path to yield $x_j \in F_j$. Hence $z_j = x_j$ is not excluded.

Now suppose that Condition 2 is violated for $k, \ell \in J_3$, $k \neq \ell$. This means that k, ℓ is matchable but $z_k \nsim z_\ell$. By Condition 1 we have $z_k \in F_k$ and $z_\ell \in F_\ell$. Thus k, ℓ satisfy the premises of Lemma 5, and are therefore joined by an edge in Γ. Hence $z_k = x_k$ and $z_\ell = x_\ell$ are compatible, contradicting our assumption that they are not. □

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