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TO VOTE OR NOT TO VOTE: WHAT IS THE QUOTA?

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The observation of a new type of perverse behavior of voting rules—Brams and Fishburn's "no-show paradox"—led Moulin to introduce the Participation Axiom (PA) It requires that an elector's failure to vote should never result in the election of a candidate whom he/she prefers to the one elected if he/she votes sincerely The present paper examines PA in the context of Condorcet-type conditions For a given quota q, $\frac{1}{2} \leq q \leq 1$, the q-Core Condition (qCC) requires that whenever there exists a candidate such that no other candidate is preferred to him/her by a fraction of q or more of the voters, the elected candidate should have this property It is shown here that PA and qCC are consistent iff $q \geq (m-1)/m$ or $m \leq 3$, where m is the number of candidates. This essentially confirms a conjecture of Moulin and extends his original result for $q = \frac{1}{2}$

1. Introduction

It is common practice, on the eve of Election Day, to call upon the public to exercise their right and vote. The argument is that by voting one can sometimes influence the outcome and secure the election of a preferred candidate. It turns out, however, that some popular voting rules may give rise to situations where one's vote results in the election of a *less* preferred candidate (compared to the outcome in case of abstention). Brams and Fishburn [1] pointed this out for the "plurality with run-off" rule, and coined the term "no-show paradox" to describe such situations.

In this paper we continue an axiomatic treatment of this phenomenon, originated by Moulin [3]. He introduced the Participation Axiom, requiring a voting rule to never give rise to situations of the type described above. He proved that if the number of candidates is 4 or more, this axiom is inconsistent with the well-known majority principle of Condorcet. Here we replace the simple majority in Condorcet's principle by a special majority, indicated by a required quota, and determine the range of values of the quota for which this inconsistency persists.

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2. Definitions and result

We let $\mathbb{N}$, the set of natural numbers, stand for the set of potential voters, and we let $A=\{a_1,a_2,\dots,a_m\}$, a non-empty finite set, stand for the set of candidates.

We let L denote the set of all linear orders on A (i.e., $P\in L$ if $P\subset A\times A$ is irreflexive, transitive and complete; $|L|=m!$). We understand aPb as "candidate a is preferred to candidate b."

A profile $P^V=(P^i)_{i\in V}$ is an assignment of elements of L to the members of some non-empty finite subset V of $\mathbb{N}$. Given a profile P^V and $\emptyset\neq W\subset V$, we denote by P^W the profile obtained by restriction of P^V to W. We let $\mathcal{P}$ denote the set of all profiles.

A *voting rule* is a function $f:\mathcal{P}\to A$. We understand $f(P^V)=a$ as saying that when V is the set of participating voters and their preferences are expressed by $(P^i)_{i\in V}$ then a is the elected candidate.

A voting rule f satisfies the *Participation Axiom* (PA) if there do not exist $V\subset\mathbb{N}$ with $1<|V|<\infty$, $P^V\in\mathcal{P}$ and $i\in V$ such that $f(P^{V\setminus\{i\}})P^i f(P^V)$.

Let q be a real number, $\frac{1}{2}\le q\le 1$. We think of q as a quota, or more precisely as the majority size required to determine binary comparisons of candidates. Indeed, given $P^V\in\mathcal{P}$ we define a binary relation $\mathrm{Dom}(q,P^V)$ on A by

$$b\mathrm{Dom}(q,P^V)a \iff |\{i\in V\colon bP^ia\}|\ge q|V|.$$

We go on to define

$$\mathrm{Core}(q,P^V)=\{a\in A\colon \text{for no } b\in A \text{ does } b\mathrm{Dom}(q,P^V)a\}.$$

For a fixed q, a voting rule f satisfies the *q-Core Condition* (qCC) is for all $P^V\in\mathcal{P}$

$$\mathrm{Core}(q,P^V)\neq\emptyset \Rightarrow f(P^V)\in\mathrm{Core}(q,P^V).$$

Finally, we say that two properties of voting rules are consistent if there exists a voting rule satisfying both properties. We are ready to state the result (as above, q is any quota in $[\frac{1}{2},1]$ and m is the number of candidates).

Theorem 2.1. *PA and qCC are consistent if and only if*

$$q\ge(m-1)/m \quad\text{or}\quad m\le 3.$$

In the special case $q=\frac{1}{2}$, qCC becomes the classical Condorcet principle: if there exists a candidate who beats every other candidate by a majority (more than half) then this candidate should be elected. Moulin [3] proved the result for this case (namely, PA and $\frac{1}{2}$CC are consistent iff $m\le 3$) and conjectured that a result similar to Theorem 2.1 was true.

The proof of the "only if" statement in Theorem 2.1 will be carried out first for a particular case (in the following section) and then in its general form in Section 4. The "if" statement will be proved in Section 5. Further comments on the basic formulation, the result and a related open problem will conclude the paper in Section 6.

3. Illustration of inconsistency

For the case $m=4$, the theorem asserts that PA and qCC are inconsistent whenever $\frac{1}{2} \le q < \frac{3}{4}$. Here we shall prove that PA and $\frac{5}{8}$CC are inconsistent. Our purpose in doing this is to illustrate how an apparently sensible requirement like qCC can force a violation of PA, which may seem paradoxical. In addition, the general proof will be easier to understand with the particular case in mind.

Consider a profile P^W described by:

13	4	11	4
a_1	a_2	a_3	a_4
a_2	a_3	a_4	a_1
a_3	a_4	a_1	a_2
a_4	a_1	a_2	a_3

This notation means that there are 13 voters with preference $a_1Pa_2Pa_3Pa_4$, 4 voters with preference $a_2Pa_3Pa_4Pa_1$, etc. Observe that the four different preferences are obtained by arranging the candidates in a cyclical order and breaking the cycle at each of the four possible places. The choice of the number of voters with each preference will be understood later. For definiteness, let $W=\{1,2,\dots,32\}$ with voters $1,2,\dots,13$ having the first preference, voters 14, 15, 16, 17 having the second preference, etc.

We have

$$b\text{Dom}(\tfrac{5}{8},P^W)a \quad \text{iff} \quad |\{i \in W\colon bP^ia\}| \ge \tfrac{5}{8}\cdot 32 = 20.$$

Thus

$$a_1\text{Dom}(\tfrac{5}{8},P^W)a_2\text{Dom}(\tfrac{5}{8},P^W)a_3\text{Dom}(\tfrac{5}{8},P^W)a_4,$$

but not $a_4\text{Dom}(\frac{5}{8},P^W)a_1$, since $4+11+4<20$. We conclude that

$$\text{Core}(\tfrac{5}{8},P^W) = \{a_1\}.$$

Now consider a profile P^V obtained by adding to P^W 4 new voters with preference $a_4Pa_1Pa_3Pa_2$:

13	4	11	4	4
a_1	a_2	a_3	a_4	a_4
a_2	a_3	a_4	a_1	a_1
a_3	a_4	a_1	a_2	a_3
a_4	a_1	a_2	a_3	a_2

For definiteness, let the new voters be 33, 34, 35, 36 and let $V=\{1,2,\dots,36\}$.
We have

$$b\text{Dom}(\tfrac{5}{8},P^V)a \quad \text{iff} \quad |\{i\in V\colon bP^ia\}| \geq \tfrac{5}{8}\cdot 36 = 22.5.$$

Thus

$$a_3\text{Dom}(\tfrac{5}{8},P^V)a_4\text{Dom}(\tfrac{5}{8},P^V)a_1\text{Dom}(\tfrac{5}{8},P^V)a_2,$$

but for no b does $b\text{Dom}(\tfrac{5}{8},P^V)a_3$. We conclude that

$$\text{Core}(\tfrac{5}{8},P^V) = \{a_3\}.$$

Assume that f is a voting rule that satisfies both PA and $\tfrac{5}{8}$CC, and consider a sequence of profiles starting with P^V and deleting the voters in $V\setminus W$ one at a time, ending with P^W. By $\tfrac{5}{8}$CC, $f(P^V)=a_3$, and by PA at each step in the process the elected candidate can only move downward in the preference $a_4Pa_1Pa_3Pa_2$. Yet, by $\tfrac{5}{8}$CC, $f(P^W)=a_1$, which is a contradiction.

4. Proof of inconsistency

We shall prove here that PA and qCC are inconsistent whenever $m\geq 4$ and $\tfrac{1}{2}<q<(m-1)/m$. Together with Moulin's result for $q=\tfrac{1}{2}$, this will establish the "only if" statement in Theorem 2.1. We remark that our proof would break down for $q=\tfrac{1}{2}$; this case genuinely requires a separate proof.

Let m and q be given, $m\geq 4$ and $\tfrac{1}{2}<q<(m-1)/m$. Consider a "profile" P^W of the form:

$1-q+\delta$	$\frac{2q-1}{m-2}$	$1-q-\delta$	$\frac{2q-1}{m-2}$	$\cdots$	$\frac{2q-1}{m-2}$
a_1	a_2	a_3	a_4	$\cdots$	a_m
a_2	a_3	a_4	$\vdots$		a_1
$\vdots$	$\vdots$	$\vdots$	a_1		$\vdots$
a_{m-1}	a_m	a_1	a_2		a_{m-2}
a_m	a_1	a_2	a_3	$\cdots$	a_{m-1}

We have put the word profile in quotation marks because we only indicate the relative frequencies of the various preferences; furthermore, the indicated numbers may not be rational. Nevertheless, we shall consider the Dom relation and the Core for such "profiles", as they are determined by the relative frequencies of preferences. The number δ is positive and small, to be specified later. Notice that the indicated frequencies add up to 1 and, if δ is small enough, they are all positive.

Since $q<(m-1)/m$ implies that $1-(2q-1)/(m-2)>q$ and $\delta>0$, we have

$$\text{Core}(q,P^W) = \{a_1\}.$$

Now consider a "profile" P^V obtained by adding to P^W a number of new voters with $a_mPa_1Pa_3Pa_2$ as part of their identical preference (the rest is immaterial). The relative frequencies in P^V are

$$\lambda(1-q+\delta),\quad \lambda\frac{2q-1}{m-2},\quad \lambda(1-q-\delta),\quad \lambda\frac{2q-1}{m-2},\quad \lambda\frac{2q-1}{m-2}$$

respectively for the preferences present in P^W, and $1-\lambda$ for the new preference. The number λ will be specified later ($0<\lambda<1$).

If the following four inequalities hold, we shall have $\mathrm{Core}(q,P^V)=\{a_3\}$.

$$\lambda(q+\delta)<q, \tag{1}$$

$$1-\lambda((2q-1)/(m-2)+1-q-\delta)<q, \tag{2}$$

$$1-\lambda(1-q+\delta)>q, \tag{3}$$

$$\lambda(1-(2q-1)/(m-2))>q. \tag{4}$$

Indeed, (1) and (2) guarantee that $a_3\in\mathrm{Core}(q,P^V)$, (3) implies that $a_m\mathrm{Dom}(q,P^V)a_1$ and (4) ensures that $a_{j-1}\mathrm{Dom}(q,P^V)a_j$ for all $j\neq 1,3$.

Suppose that we can choose δ and λ so that (1)–(4) are satisfied. Then we can construct actual profiles P^W and P^V approximating the indicated relative frequencies, so that their respective Cores are $\{a_1\}$ and $\{a_3\}$. (Notice that all the inequalities above are strict, so rationality can be achieved.) Using P^V and P^W we can show that PA and qCC are inconsistent.

Thus it suffices to show that δ and λ can be chosen appropriately. The conjunction of (1)–(4) can be rewritten as

$$\max\left\{\frac{1-q}{\frac{2q-1}{m-2}+1-q-\delta},\ \frac{q}{1-\frac{2q-1}{m-2}}\right\}<\lambda<\min\left\{\frac{q}{q+\delta},\ \frac{1-q}{1-q+\delta}\right\}.$$

For $\delta=0$, the maximum on the left is less than 1 which is the minimum on the right. Hence for $\delta>0$ small enough, the left-hand side is smaller than the right-hand side, so λ can be chosen in between. As this λ will also satisfy $0<\lambda<1$, we are through.

5. Proof of consistency

We shall prove here that PA and qCC are consistent if $q\geq(m-1)/m$ or $m\leq 3$.

Assume that $m\leq 3$. For $P^V\in\mathscr{P}$, define $f(P^V)$ to be the first candidate in $\bigcap_q \mathrm{Core}(q,P^V)$, where the intersection ranges over those $q\in[\frac{1}{2},1]$ for which $\mathrm{Core}(q,P^V)\neq\emptyset$, and "first" refers to the fixed order $a_1,\dots,a_m$. It can be checked that f is well-defined and satisfies qCC for all $q\in[\frac{1}{2},1]$—this is true regardless of m—and moreover satisfies PA as well (this is actually shown in [3]).

Next, assume that $q\geq(m-1)/m$. In order to define our voting rule in this case, we need some preliminaries. For $P^V\in\mathscr{P}$ and $a\in A$ we let

$$w(P^V,a)=|\{(i,c)\colon i\in V,\ c\in A \text{ and } aP^ic\}|.$$

For $P^V \in \mathscr{P}$ we denote

$$B(P^V) = \{a \in A: w(P^V, a) \geq w(P^V, b) \text{ for all } b \in A\}.$$

The candidates in $B(P^V)$ are known as the Borda winners. The following definition of a voting rule g is meaningful for $m \geq 2$. For $P^V \in \mathscr{P}$, define $g(P^V)$ to be the first candidate in

$$B(P^V) \cap \text{Core}((m-1)/m, P^V)$$

if this intersection is non-empty, otherwise let $g(P^V)$ be the first candidate in $B(P^V)$.

The satisfaction of PA follows immediately from the fact that $g(P^V) \in B(P^V)$ for all $P^V \in \mathscr{P}$. As for qCC, the basic observation is that if $b\text{Dom}(q, P^V)a$ then

$$\begin{aligned} & w(P^V, b) - w(P^V, a) \\ & \quad \geq |\{i \in V: bP^i a\}| - (m-1)|\{i \in V: aP^i b\}| \\ & \quad \geq q|V| - (m-1)(1-q)|V| = (q-(m-1)/m)m|V| \geq 0, \end{aligned}$$

and the last inequality is strict if $q > (m-1)/m$. Thus if $q > (m-1)/m$ then $B(P^V) \subset \text{Core}(q, P^V)$ for all $P^V \in \mathscr{P}$ so g satisfies qCC. It remains to show that g satisfies qCC in the case $q = (m-1)/m$. Assume it does not: let $P^V \in \mathscr{P}$ be such that

$$\text{Core}((m-1)/m, P^V) \neq \emptyset$$

but

$$g(P^V) \notin \text{Core}((m-1)/m, P^V).$$

By the definition of g it must be the case that

$$B(P^V) \cap \text{Core}((m-1)/m, P^V) = \emptyset.$$

Take an arbitrary $a \in B(P^V)$. Since

$$a \notin \text{Core}((m-1)/m, P^V),$$

we can find $b \in A$ with

$$b\text{Dom}((m-1)/m, P^V)a.$$

By the above observation $w(P^V, b) \geq w(P^V, a)$, so $b \in B(P^V)$ as well. Repeating the argument we can find $c \in A$ with

$$c\text{Dom}((m-1)/m, P^V)b,$$

and so forth. Since A is finite, this process establishes the existence of a cycle in the relation $\text{Dom}((m-1)/m, P^V)$. As

$$\text{Core}((m-1)/m, P^V) \neq \emptyset,$$

this cycle consists of less than m candidates. The intersection of less than m subsets of V, each having cardinality at least $((m-1)/m)|V|$, is necessarily non-empty. Hence there is a cycle in P^i for some $i \in V$, which is absurd.

6. Three remarks and an open problem

Remark 6.1. *On strategic voting.* The classical problem in the theory of strategic voting was to find a voting rule with the following property: voting one's true preference is always in one's best interest, when the alternative available actions are voting any other preference. The Participation Axiom also requires that voting one's true preference always be in one's best interest, but here the unique alternative action is abstention. An important aspect of this comparison concerns strategic complexity: while it may be difficult for the voter to search for a profitable alternative action in the classical setup, the voter needs to check only one alternative in the PA setup. There is a significant difference in the results obtained with the two approaches. The classical problem turned out to have a negative solution: with 3 or more candidates, no "democratic" voting rule has the desired property (Gibbard [2], Satterthwaite [4]). The results on PA indicate that it is a demanding axiom, but there do exist democratic voting rules that satisfy it, notably the plurality and Borda rules. Finally, it is arguable that a more realistic analysis should incorporate both types of strategic behavior—casting an insincere vote and not casting any vote. We feel however that the effect of insincere voting would overshadow that of abstention in such a framework.

Remark 6.2. *On the critical quota.* Special majority quotas were originally introduced in an attempt to avoid the cycles that occur in the simple majority comparisons when there are more than 2 candidates. This goal is obtained—namely $\mathrm{Core}(q, P^V) \neq \emptyset$ for all $P^V \in \mathscr{P}$—if and only if $q > (m-1)/m$. Our result here reveals that $(m-1)/m$ is also the critical quota for reconciling PA and qCC. This suggests the following interpretation. As long as $\mathrm{Core}(q, \cdot)$ is non-empty valued it admits a selection nice enough to satisfy PA; but when it assumes both non-empty and empty values, there is no way to select from its non-empty values and extend the definition to all of $\mathscr{P}$ without violating PA. The validity of this insight is limited however by the fact that it does not account for the discrepancy in the results when $q = (m-1)/m$ or when $m=3$ and $q \leq \frac{2}{3}$: in these cases $\mathrm{Core}(q, P^V)$ may be empty, yet PA and qCC are consistent.

Remark 6.3. *On the number of voters.* One may want to replace the infinite set of potential voters in our formulation by a finite set of cardinality n. The consistency statement in Theorem 2.1 would of course carry over, but what about the inconsistency statement? For given m and q, it is clear from our proof in Section 4 that some finite n suffices. Incidentally, Moulin's result for $q = \frac{1}{2}$ requires $n \geq 25$ and our example case with $m=4$, $q = \frac{5}{8}$ (the midpoint of the inconsistency interval) works for $n \geq 36$. Furthermore, it is not difficult to see that for a given $m \geq 4$ and a given compact $K \subset [\frac{1}{2}, (m-1)/m)$ there exists a finite n that makes PA and qCC inconsistent for all $q \in K$. Yet, no finite n will yield the result for all $q \in [\frac{1}{2}, (m-1)/m)$. To see this, observe that given n there exists $q < (m-1)/m$ such that $\mathrm{Dom}(q, P^V)$ coincides with $\mathrm{Dom}((m-1)/m, P^V)$ whenever $|V| \leq n$, so the consistency of PA and $((m-1)/m)$CC can be invoked to obtain a contradiction.

Open Problem. There is an alternative way to apply Condorcet's principle to quotas higher than $\frac{1}{2}$. Namely, for a given $q \in (\frac{1}{2}, 1]$, a voting rule f satisfies the *q-Dominance Condition* (qDC) if for all $P^V \in \mathscr{P}$ and all $a \in A$

$$a\mathrm{Dom}(q, P^V)b \text{ for all } b \in A \setminus \{a\} \;\Rightarrow\; f(P^V) = a.$$

Clearly qDC is weaker than qCC, and qDC implies q'DC for $q' > q$ (which is not the case for qCC). The conjunction $\bigwedge_{q>1/2} q$DC amounts to the classical Condorcet principle, which is $\frac{1}{2}$CC.

The open problem is, given $m \geq 4$, to determine the range of quotas q for which PA and qDC are consistent. In particular, we do not know whether this range is strictly larger than the corresponding one for qCC, i.e., $q \geq (m-1)/m$. In our attempts at this problem we succeeded only to show that PA and qDC are inconsistent for values of q which are relatively close to $\frac{1}{2}$. The reader may observe that for our proof of inconsistency in Section 4 it suffices to assume that $f(P^V) \in \mathrm{Core}(q, P^V)$ when the latter is a singleton, which is weaker than qCC but still stronger than qDC.

An interesting way to look at this problem, as well as other related problems, is in terms of colorings of a simplex. Let

$$\Delta = \left\{ x = (x_1, x_2, \ldots, x_{m!}) \in \mathbb{R}^{m!} : x_k \geq 0, k = 1, \ldots, m!, \sum_{k=1}^{m!} x_k = 1 \right\}$$

be the standard $(m!-1)$-dimensional simplex, with its vertices $v_1, v_2, \ldots, v_{m!}$ corresponding in a one-to-one fixed manner to the orders $P_1, P_2, \ldots, P_{m!}$ of the set A of m elements. A coloring is a function $f: \Delta \to A$. It is acceptable if for all $x \in \Delta$ and all $k = 1, \ldots, m!$ the whole interval $[x, v_k]$ is colored with colors ranked at least as high as $f(x)$ in the order P_k. (This corresponds to PA.) Now suppose that a partial coloring is given: $D_a \subset \Delta$ is already colored with a, for each $a \in A$. The question is when can the partial coloring be extended to an acceptable coloring of the entire simplex. If

$$D_a = \left\{ x \in \Delta : \sum_{k \; aP_k b} x_k \geq q \text{ for all } b \in A, b \neq a \right\}, \quad a \in A,$$

then this quest[illegible] corresponds to the problem of consistency of PA and qDC described above.

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