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Note

Representations of Families of Triples over $GF(2)$*

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Let **B** be any family of 3-subsets of $[n] = \{1, ..., n\}$ such that every i in $[n]$ belongs to at most three members of **B**. It is shown here that there exists a $3 \times n$ (0, 1)-matrix M such that every set of columns of M indexed by a member of **B** is linearly independent over $GF(2)$. The proof depends on finding a suitable vertex-coloring for the associated 3-uniform hypergraph. This matrix result, which is a special case of a conjecture of Griggs and Walker, implies the corresponding special case of a conjecture of Chung, Frankl, Graham, and Shearer and of Faudree, Schelp, and Sós concerning intersecting families of subsets.

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1. INTRODUCTION

Let $[n]$ denote the set $\{1, 2, ..., n\}$, and let $2^{[n]}$ denote the collection of all subsets of $[n]$. Let $\binom{[n]}{t}$ denote the collection of all t-subsets. Throughout the paper, $\mathbf{B}$ denotes a nonempty subset of $\binom{[n]}{t}$. We say that $\mathbf{F} \subseteq 2^{[n]}$ is an *intersecting family over* $\mathbf{B}$ if for every F, F' in $\mathbf{F}$ there exists B in $\mathbf{B}$ such that $B \subseteq F \cap F'$. We are interested in the maximum size of an intersecting family over $\mathbf{B}$, denoted by $v(\mathbf{B})$.

Let B be any set in $\mathbf{B}$. The collection of all subsets of $[n]$ that contain B is intersecting over $\mathbf{B}$, so it follows that $v(\mathbf{B}) \geq 2^{n-t}$. Of course, $v(\mathbf{B})$ can be larger than this, in general. Now suppose that $X = \{a_1, ..., a_t\} \in \binom{[n]}{t}$ and suppose that $\mathbf{B} = \mathbf{B}_n(X)$ consists of all cyclic translates $X + i = \{a_1 + i, ..., a_t + i \pmod n\}$ of X in $[n]$. It was conjectured in papers by Chung, Frankl, Graham, and Shearer [CFGS] and by Faudree, Schelp, and Sós [FSS] that in this case the lower bound is sharp, i.e., $v(\mathbf{B}_n(X)) = 2^{n-t}$. Graham [G] has offered \$100 for a proof of this conjecture. It was shown to be true for arbitrary X when $t = 1$ or 2. A deeper result is that it holds for arbitrary t for the particular set $X = \{1, ..., t\}$.

It was observed by Griggs and Walker [GW] that $v(\mathbf{B})$ must equal the desired value, 2^{n-t}, provided that there exists a partition of $2^{[n]}$ into collections $\mathbf{A}_1, ..., \mathbf{A}_{2^{n-t}}$ such that each $\mathbf{A}_i$ is an *anticluster for* $\mathbf{B}$: This means that for every A, A' in $\mathbf{A}$ with $A \neq A'$, $A \cap A'$ includes *no* set B in $\mathbf{B}$. Such a partition into anticlusters exists for $\mathbf{B}$ provided that there exists a $t \times n$ (0, 1)-matrix M with the property that for every B in $\mathbf{B}$ the t columns of M indexed by B are linearly independent over $GF(2)$. In this case, the matrix M is said to be *suitable for* $\mathbf{B}$. A suitable matrix yields a partition of $2^{[n]}$ into 2^{n-t} anticlusters for $\mathbf{B}$ in the following way: The rows of M generate a t-dimensional subspace $\mathbf{S}$ of $\mathbb{Z}_2^n$. The space $\mathbb{Z}_2^n$ is partitioned into 2^{n-t} affine subspaces parallel to $\mathbf{S}$. It can be shown that for each $\mathbf{v}$ in $\mathbb{Z}_2^n$, the 2^t subsets of $[n]$ whose characteristic vectors belong to the affine subspace $\mathbf{S} + \mathbf{v}$ form an anticluster for $\mathbf{B}$.

Griggs and Walker conjecture that for any X there exists a suitable matrix for $\mathbf{B}_n(X)$. They prove this for all X if $t = 1$ or 2, and for arbitrary t if $X = \{1, ..., t\}$. They obtain considerable new support for the original conjecture of Chung *et al.* by constructing, for arbitrary X in $\binom{[n]}{t}$, a matrix that is suitable for the collection of all *ordinary* translates $X + i$, which means, for $X = \{a_1 < a_2 < \cdots < a_t\}$, that $0 \leq i \leq n - a_t$. Their proof is a greedy selection of the columns of the matrix one-by-one together with a simple counting argument that there is always a feasible choice for the next column.

We have noticed that a slightly stronger result can be proven by this method (details omitted), which is that for arbitrary X, there exists a matrix that is suitable for all ordinary translates $X + i$ in $[n]$ together with cyclic translates in which just one element wraps around. In the notation above, there is a suitable matrix for the set of cyclic translates $X+i$, where $0 \leqslant i \leqslant n-a_{t-1}$.

Griggs and Walker apply their result on ordinary translates to prove an asymptotic result for the collection of all cyclic translates. They prove that for any fixed t-subset X of $\mathbf{N}$, there exists a suitable matrix for $\mathbf{B}_n(X)$ for infinitely many values of n. It follows that $v(\mathbf{B}_n(X))/2^{n-t} \to 1$ as $n \to \infty$, since the existence of this limit was proven by Chung *et al.* [CFGS]. Computational evidence for the matrix conjecture was obtained by Griggs and Walker in the case $t=3$ by checking for $n \leqslant 100$ and arbitrary X in $\binom{[n]}{3}$ that there is a suitable matrix for $\mathbf{B}_n(X)$. We shall prove their conjecture here for this case $t=3$.

It was suggested in [GW] that the conjectures may actually hold more generally for any $\mathbf{B}$ with the property that each element in $[n]$ belongs to at most t elements of $\mathbf{B}$. Obviously this includes the case that $\mathbf{B}=\mathbf{B}_n(X)$. Our main result here is to confirm this stronger conjecture for $t=3$.

THEOREM 1.1. *Let* $\mathbf{B} \subseteq \binom{[n]}{3}$ *such that each member i of $[n]$ belongs to at most three members of* $\mathbf{B}$. *Then there exists a suitable matrix for* $\mathbf{B}$.

COROLLARY 1.2. *The conjecture of Chung et al., holds for arbitrary X and n if $t=3$, i.e., for all X in* $\binom{[n]}{3}$, $v(\mathbf{B}_n(X))=2^{n-3}$.

In order to prove Theorem 1.1, it turns out to be worthwhile to adopt notions from the theory of hypergraphs. We introduce this terminology in Section 2 and restate the theorem in this language. The greedy coloring method is described in Section 3. The induction proof of the theorem has several main cases which are presented in Sections 4 and 5.

The conjectures described above remain open for $t \geqslant 4$. Unfortunately, the methods we use here for $t=3$ do not appear to work for larger t.

2. HYPERGRAPH INTERPRETATION

A family of subsets $\mathbf{B} \subseteq \binom{[n]}{3}$ corresponds to a hypergraph $H=(V, E)$ in which the vertex set V is $[n]$ and the edge set E is $\mathbf{B}$. Notice that H is 3-*uniform*, i.e., all edges have size 3. We assume that this is true for the rest of the paper. We abbreviate a triple $\{x, y, z\}$ in $\binom{V}{3}$ by xyz. The *degree* of a vertex x in H, denoted by $d(x)$, is the number of edges in E that contain x. Two vertices x and y are *adjacent* in H written $x \sim y$, if $xyz \in E$ for some $z \in V$. The *neighborhood* of vertex x, denoted $N(x)$, is the set of vertices adjacent to x.

Now consider what interpretation can be given in H to a suitable matrix for $\mathbf{B} \subseteq \binom{[n]}{3}$. For $1 \leqslant i \leqslant n$, column i of M is a vector in $\mathbb{Z}_2^3$. So M can be

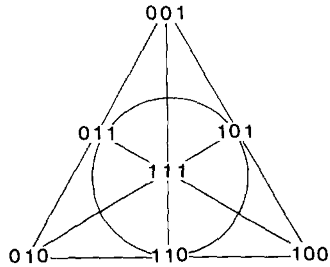

FIG. 1. The nonzero vectors represented as F_7.

viewed as a coloring of $V = [n]$, i.e., as a function $c\colon V \to \mathbb{Z}_2^3$ in which each vertex i is assigned one of eight colors (the vectors in $\mathbb{Z}_2^3$). The requirement that M be suitable is not familiar for hypergraphs. It says that for every edge $xyz \in E = \mathbf{B}$, the colors assigned to x, y, z, which we abbreviate by $c(xyz)$, must form a basis for the color set $\mathbb{Z}_2^3$. We are ready to restate Theorem 1.1 in the language of hypergraphs.

THEOREM 2.1. *Let $H = (V, E)$ be a 3-uniform hypergraph of maximum degree at most three. Then there exists a coloring c of V, using elements of $\mathbb{Z}_2^3$ as the colors, such that for every $xyz \in E$, $c(xyz)$ is a basis for $\mathbb{Z}_2^3$.*

Trivially, any vertex in H of degree at least one must be assigned a nonzero color. It is convenient to notice that the nonzero elements of $\mathbb{Z}_2^3$ may be assigned to form the Fano plane F_7 in the following way. The three nonzero vectors in any two-dimensional subspace of $\mathbb{Z}_2^3$ correspond to a line in F_7, as shown in Fig. 1. Note that the three elements on the circle are considered collinear. We see that any two distinct two-dimensional subspaces in $\mathbb{Z}_2^3$ must share precisely one nonzero element. Further, any three two-dimensional subspace cover $\mathbb{Z}_2^3$ precisely when they correspond to three distinct lines in F_7 that share a single point.

We shall need to consider the connectedness of H. Two vertices, x and y, are *connected* in H if there exists a path between them, i.e., there exists a sequence $x = x_0, x_1, ..., x_r = y$ in H such that for all i, $x_i \sim x_{i+1}$. We say H is *connected* if every pair $x \neq y$ in V is connected. Given $k \in \mathbf{N}$, we say H is *k-connected* if for any subset $W \subseteq V$ with $|W| < k$, and for any vertices $x \neq y \in V - W$, there exists a path from x to y that avoids W. An important point here is that if $xyz \in E$ and $W \subseteq V$ contains z but not x nor y, we still view x and y as being adjacent in $V - W$, i.e., in the subgraph we denote by $H - W$.

3. A GREEDY COLORING FOR H

The key idea in our proof is to devise an analogue of the elegant greedy-coloring proof by Lovász [L] of Brook's Theorem. This theorem states that a connected graph G of maximum degree $\Delta \geqslant 3$ which is not the complete graph can be properly colored using just Δ colors. One can linearly order the vertices of G $(v_1, v_2, ..., v_n)$ so that for all $i<n$, vertex v_i is adjacent to at least one vertex v_j with $j>i$. It is then possible to properly color the vertices in order, one-by-one beginning with v_1, using just Δ colors, except for possibly the last vertex, v_n. The problem is to arrange the ordering and coloring so that v_n can also be colored or to use other argument. We adopt a similar strategy here, seeking to suitably order the vertices and then color them greedily. Here is a simple lemma about orderings.

LEMMA 3.1. *Suppose* $H=(V, E)$, *where* $V=[n]$ *and* $E \subseteq \binom{[n]}{3}$, *is connected. Let* $v \in V$. *Then there exists an ordering* $(v_1, ..., v_n)$ *of* V *such that* $v_n = v$ *and such that for all* $i<n$, v_i *is adjacent to some* v_j *with* $j>i$.

Proof. Starting from v_n, pick $v_{n-1} \sim v_n$, then $v_{n-2} \sim v_{n-1}$ or v_n, and so on. ∎

For the rest of the paper, assume that H satisfies the hypotheses of Theorem 2.1. Suppose for now that V has been ordered as in Lemma 3.1. One can then greedily color $V-\{v_n\}$ as follows. Any nonzero color (vector) in $\mathbb{Z}_2^3$ may be used for $c(v_1)$. Suppose that $i<n$ and that $v_1, ..., v_{i-1}$ have been suitably colored. Then a color $c(v_i)$ must be selected that is (a) linearly independent of $\{c(v_j), c(v_k)\}$ whenever j, $k<i$ and $v_i v_j v_k \in E$, and (b) linearly independent of $c(v_j)$ whenever $j<i<k$ and $v_i v_j v_k \in E$. An instance of (a) eliminates a line in F_7 from possibilities for $c(v_i)$ while an instance of (b) eliminates a point in F_7. By hypothesis, $d(v_i) \leqslant 3$ in H. By design, the ordering guarantees that (a) applies at most twice to v_i. So, at worst, nonzero vectors in F_7 are eliminated corresponding to two lines (which must intersect) and a separate point. There remains at least one nonzero vector to use for $c(v_i)$. So we can greedily color $v_1, ..., v_{n-1}$. However, we cannot color v_n in general since at worst there could be three instances of (a), corresponding to three lines in F_7. If the three lines are distinct but share one point, then they eliminate all possible colors for $c(v_n)$.

We conclude the section by disposing of one particularly easy case.

CLAIM 3.2. *If* H *is connected and contains some vertex* v *with* $d(v)<3$, *then* H *can be colored greedily.*

Proof. Order the vertices as in Lemma 3.1, taking $v_n = v$. Greedily color $v_1, ..., v_{n-1}$. At most two lines in F_7 are eliminated as possible colors for v_n. Thus v_n can also be colored. ∎

4. THE PROOF FOR EASILY DISCONNECTED H

The remainder of the paper consists of the proof of Theorem 2.1. The proof is by induction on $n = |V|$. The case $n \leqslant 3$ is trivial, so we assume $n \geqslant 4$. By Claim 3.2, we may assume H is regular of degree 3.

CLAIM 4.1. *If there exists $W \subseteq V$ with $|W| \leqslant 3$ such that $H - W$ is not connected, then H can be colored by induction.*

Proof. If H itself is disconnected, separately color each component by induction. Hence we may assume H is connected. Suppose that there is a vertex x such that $H - \{x\}$ is disconnected. Let $C_1, ..., C_r$ be the vertex sets of components of $H - \{x\}$. Each C_i will contain all vertices besides x for some edge in E that contains x. Hence in each induced subgraph of H on $C_i \cup \{x\}$, x has degree at most two, so that $C_i \cup \{x\}$ can be colored greedily by Claim 3.2 (or, alternately, by induction). This can be done with x receiving the same color in each $C_i \cup \{x\}$, so that the colorings can be combined to properly color H itself.

Next suppose H is 2-connected, but there exist $x \neq y \in V$ such that $H - \{x, y\}$ is disconnected into components $C_1, ..., C_r$. For any i consider the induced subgraph on $C_i \cup \{x, y\}$ (it contains the edges $uvw \in E$ such that $u, v, w \in C_i \cup \{x, y\}$). Each component C_i is adjacent to both x and y so that in each induced subgraph $C_i \cup \{x, y\}$, x and y each have degree at most 2, and all other vertices have degree at most 3. To avoid coloring x and y the same in one component and differently in another, we shall ensure they receive different colors by adding a new "dummy' vertex d and a new edge xyd to each subgraph $C_i \cup \{x, y\}$. This new graph on $C_i \cup \{x, y, d\}$ can be colored greedily by Claim 3.2 since d has degree 1. The colors for x and y differ for all i, so by relabelling if necessary we may assume x and y receive the same pair of colors for all i. The component colorings (ignoring d) combine to give a coloring for H.

Finally, suppose that H is 3-connected but $H - \{x, y, z\}$ is disconnected into components $C_1, ..., C_r$. Each C_i is adjacent to each of x, y, and z, so the induced subgraphs $C_i \cup \{x, y, z\}$ each have maximum degree at most 3 with degree at most 2 for x, y, and z. So unless the edge xyz is already in the induced subgraph $C_i \cup \{x, y, z\}$, it can be added while keeping the maximum degree at most 3. This graph on $C_i \cup \{x, y, z\}$ can be colored by induction on $|V|$, and the colors for x, y, z receive the same colors for all i, so that the colorings can be combined to properly color H itself. ∎

5. The Proof for 4-Connected H

This continues the induction proof begun in Section 4. We assume for the remainder that H is 3-regular and 4-connected. First we treat the case that some vertex has less than 6 neighbors.

Claim 5.1. *If H contains a vertex x with $|N(x)|<6$, then H can be properly colored.*

Proof. Let x be a vertex in H with the minimum value of $|N(x)|$. Since x has degree 3, $|N(x)|\geqslant 3$. First suppose that $|N(x)|=3$. Since removing $N(x)$ disconnects x from any other vertices, while H is 4-connected, the only possibility is that $n=4$ and $E=\binom{[4]}{3}$, which is easily colored greedily.

Next suppose that $|N(x)|=4$. There can be no vertex $y\neq x$ that belongs to all three edges through x, or else the three vertices in $N(x)-\{y\}$ would disconnect $\{x, y\}$ from the rest of H (which must be nonempty), a contradiction. Hence we may assume that the edges through x are xpq, xqr, and xrs. Order V as in Lemma 3.1 with x being last and greedily color the vertices in the list. Then some color is available for x as well: The nonzero colors for p, q determine a line L_1 in F_7. The colors for q, r determine a line L_2 that either meets L_1 only at $c(q)$ or else $L_2=L_1$. The colors for r, s determine a line L_3 that either equals L_2 or else meets it only at $c(r)$ and not at $c(q)$. The three lines share a point only if some two of them coincide, so that they do not together cover F_7. A color must remain for x to compute the coloring of H.

Finally suppose that $|N(x)|=5$. We may assume that the edges through x are xpq, xpr, xst. Ordering V with x last and coloring greedily will not work in general. Instead, we replace the three edges above by just two: pqr and pst. This preserves the degrees of p, q, r, s, and t, but there is one fewer vertex. By induction this smaller graph, call it H', can be colored. We claim that this can be extended to a coloring of H. Notice that since $pq, pr\subseteq pqr$ and $st\subseteq pst$, the colors for each pair pq, pr, st determine a line in F_7. It remains to show that some point in F_7 is not covered by these three lines. Suppose not, i.e., suppose the lines are distinct and some point in F_7 lies on all three. The lines for pq and pr meet at $c(p)$, so this must be the point of i intersection. Then the line for st must contain $c(p)$. But this is a contradiction since by design the edge pst belongs to H', so that $c(pst)$ must be linearly independent. Hence there remains an available point in F_7 to use for $c(x)$ in order to complete the coloring of H. ∎

Owing to Claim 5.1, it remains to consider graphs in which every vertex has six neighbors. Let $x\in V$ and label its neighbors so that it belongs to edges xpq, xrs, stu.

CLAIM 5.2. *Let H satisfy all conditions stated above. If every pair of neighbors of x is adjacent in H, then H can be colored.*

Proof. Consider any neighbor of x, say p. This vertex p lies on two edges besides xpq, and it must be adjacent to each of r, s, t, and u. So it must be adjacent to each precisely once and never again adjacent to x or q. It follows that, in fact, H must be the Fano plane, F_7 where V gives the point and E gives the lines. This is isomorphic to the hypergraph $H' = (V', E')$ in which $V' = [7]$ and $E' = \mathbf{B}_7(\{1, 2, 4\})$. Here is a coloring for this hypergraph, presented as a suitable matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}. \quad \blacksquare$$

Henceforth we consider H such that some pair of neighbors of x, say p and r, is not adjacent. First we treat the case that the removal of $\{p, q, r, s\}$ does not disconnect H.

CLAIM 5.3. *Let H satisfy the conditions stated above. Suppose $H - \{p, q, r, s\}$ is connected. Then H can be colored.*

Proof. Assume H satisfies the hypotheses of the claim. Order the vertices of $H - \{p, q, r, s\}$ with x last as in Lemma 3.1. Then put $\{p, q, r, s\}$ at the beginning of the list. Select three colors from a line in F_7, say $\{100, 010, 110\}$. Assign color 100 to p and r, 010 to q, and 110 to s, so that each pair on $\{p, q, r, s\}$ has independent colors (except for p and r, which are not adjacent anyway). With this start, continue to greedily color the vertices in the ordered list. It suffices to show that some color will be available for x. This is clearly the case, since p, q, r, s together exhaust only one line in F_7, while t, u at worst exhaust another line, leaving at least two choices for $c(x)$ to complete the coloring of H. $\blacksquare$

We shall continue to further restrict the remaining possibilities for H until all are exhausted. Claim 5.3 implies that we may assume hereafter that whenever two neighbors of x are not adjacent in H, then removing those two neighbors as well as the other two neighbors of x that appear with them on edges through x must disconnect H. In particular, since p is not adjacent to r, $H - \{p, q, r, s\}$ is disconnected. Let A denote the set of vertices not connected to x in $H - \{p, q, r, s\}$. Notice that each of $\{p, q, r, s\}$ is adjacent to each component of A since H is 4-connected.

Suppose for contradiction that $p \sim t$ and $p \sim u$. It cannot be that $ptu \in E$, since already $xtu \in E$ and we are assuming $|N(t)| = 6$. Thus the edges containing p must be xpq, pta, pub, where a and b are not yet specified.

However, at least one of a and b, say a, must belong to A since $p \sim A$. This implies there is a path from x to A in $H-\{p, q, r, s\}$, contradicting the definition of A. Therefore it is not possible that $p \sim t$ and $p \sim u$.

It follows by earlier statements that $H-\{p, q, t, u\}$ is disconnected, and we let B denote the set of vertices in this graph that are not connected to x. Similar arguments applied to r instead of p show that $H-\{r, s, t, u\}$ is disconnected, and we let C denote the set of vertices disconnected from x in this graph. Let F denote the set of vertices not belonging to $\{x\} \cup N(x) \cup A \cup B \cup C$. The set F contains those "extra" vertices that remain connected to x when all vertices in any two of the three pairs pq, rs, and tu are removed. We observe that by their definitions, the sets A, B, C, and F are pairwise disjoint.

We next record a general observation.

PROPOSITION 5.4. *Let $G=(V, E)$ be a 3-uniform hypergraph that is 4-connected and regular of degree 3. Suppose V is partitioned into two sets, X and Y, with $|X|$, $|Y| \geqslant 2$. Then at least four edges meet both X and Y.*

Proof. Suppose at most three edges meet both X and Y. Then for each of these edges e, precisely one of $X \cap e$ and $Y \cap e$ consists of a single vertex. Removing this vertex for each such edge clearly disconnects G if $|X|$, $|Y| \geqslant 4$. If $\min\{|X|, |Y|\}=2$ or 3, it can be verified by checking the few cases that arise that removing these vertices disconnects G. However, we have a contradiction, since G is 4-connected. ∎

Returning to the proof of the theorem, we consider any vertex $a \in A$. Since all vertices have 6 neighbors in H, by hypothesis, and since a is adjacent only to vertices in $\{p, q, r, s\} \cup A$, it follows that $|A| \geqslant 3$. By Proposition 5.4, at least 4 edges meet both A and $V-A$, i.e., both A and $\{p, q, r, s\}$, since no other vertices in $V-A$ are adjacent to A. Similar remarks apply to B and C. Altogether, at least 12 edges meet both $A \cup B \cup C$ and $N(x)$. Since H is 3-regular, each vertex in $N(x)$ must belong to 2 of these 12 edges, and none of these 12 edges can meet $N(x)$ twice. It follows that no two vertices in $N(x)$ are adjacent except for the three pairs pq, rs, and tu. Since $p \sim A$ and $p \sim B$, it follows that the three edges containing p are xpq, paa', and pbb', where $a, a' \in A$ and $b, b' \in B$. Similar remarks apply to every element of $N(x)$. No extra vertices are adjacent to $N(x)$, so that the set F is empty. Similarly, applying Proposition 5.4 to each component of A, B, and C, we find that each of A, B, and C is connected.

The entire proof has now been reduced to this one final case.

CLAIM 5.5. *Assume H satisfies the conditions stated above. Among these is that no two neighbors of x are adjacent, except that $p \sim q$, $r \sim s$, and $t \sim u$. Then H can be colored greedily.*

Proof. Since $p \sim A$, B, it follows that the edges containing p are xpq, paa' (where a, $a' \in A$), and pbb' (where b, $b' \in B$). Suppose for contradiction that $H - \{a, a', b, b'\}$ is not connected. Consider a component D that remains such that $x \notin D$. Since x is connected in H to $N(x)$ and C without passing through A or B, and since A and B are not adjacent in H, it must be that D is properly contained in one of A and B, say $D \subset A$. It follows that every path from D to x in H passes through a or a'. Then $H - \{a, a'\}$ is disconnected, a contradiction.

Therefore it must be that $H - \{a, a', b, b'\}$ is actually connected. We may perform a greedy coloring as follows: First order $H - \{a, a', b, b'\}$ with p last (instead of the usual x) and then put a, a', b, b' first in the list. We can color a, a', b, b' with $c(a) = c(b) \neq c(a') = c(b')$. Then color the rest of the list greedily, as usual. It remains to observe that no matter how it proceeds, there remains a color (indeed, at least two) to use at p in order to complete the coloring of all of H. ∎

Note added in proof. Aharoni and Holzman have recently constructed families $\mathbf{B} \subseteq \binom{[n]}{t}$ of maximum degree t such that $v(\mathbf{B}) > 2^{n-t}$. The cyclic translate conjectures remain open. The stronger conjecture that there exists a suitable matrix for any $\mathbf{B}$ of maximum degree t remains open provided $n \geqslant t/2$.

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