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ARTICLE

On convex holes in *d*-dimensional point sets

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Abstract

Given a finite set $A \subseteq \mathbb{R}^d$, points $a_1, a_2, \ldots, a_\ell \in A$ form an ℓ-hole in A if they are the vertices of a convex polytope, which contains no points of A in its interior. We construct arbitrarily large point sets in general position in $\mathbb{R}^d$ having no holes of size $O(4^d d \log d)$ or more. This improves the previously known upper bound of order $d^{d+o(d)}$ due to Valtr. The basic version of our construction uses a certain type of equidistributed point sets, originating from numerical analysis, known as (t, m, s)-nets or (t, s)-sequences, yielding a bound of 2^{7d}. The better bound is obtained using a variant of (t, m, s)-nets, obeying a relaxed equidistribution condition.

Keywords: convex holes, Erdős–Szekeres theorem, (t,m,s)-nets

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1. Introduction

A finite set $A \subseteq \mathbb{R}^d$ is in *general position* if any k-dimensional affine subspace of $\mathbb{R}^d$, with $k < d$, contains at most $k + 1$ points of A. Points $a_1, a_2, \ldots, a_\ell \in A$ are in *convex position* if they are the vertices of a convex polytope. If that polytope is empty, i.e., contains no points of A in its interior, the points $a_1, a_2, \ldots, a_\ell$ are said to form an *ℓ-hole* in A.

A classic result of Erdős and Szekeres [5] asserts that for any positive integer ℓ, every sufficiently large finite set A in general position in $\mathbb{R}^2$ contains ℓ points in convex position. Erdős [4] went on to ask if one can also guarantee an ℓ-hole in a large enough $A \subseteq \mathbb{R}^2$ in general position. Harborth [7] proved that one can always find a 5-hole, while Horton [8] constructed arbitrarily large sets without any 7-hole. The remaining case $\ell = 6$ turned out to be more challenging, but was settled in the affirmative by Nicolás [10] and, independently, Gerken [6].

Another question studied is the asymptotic behaviour, as $n \to \infty$, of the number of ℓ-holes guaranteed to exist in a set A of n points in general position in $\mathbb{R}^2$. For $\ell = 3, 4$ this number was shown to be $\Theta(n^2)$ by Katchalski and Meir [9] and Bárány and Füredi [2]. The order of magnitude for $\ell = 5, 6$ is not known, but very recently Aichholzer et al. [1] proved it is superlinear for $\ell = 5$.

Turning to higher dimensions, much less is known. Valtr [14] gave a simple projection argument to extend the Erdős–Szekeres result to any dimension $d \geq 2$: for every ℓ, any sufficiently

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large finite set A in general position in $\mathbb{R}^d$ contains ℓ points in convex position. Regarding holes, he defined:

$$h(d) \stackrel{\text{def}}{=} \max\{\ell : \text{any large enough } A \subseteq \mathbb{R}^d \text{ in general position contains an } \ell\text{-hole}\}.$$

Using this notation, the two-dimensional results recalled above say that $h(2) = 6$. Valtr proved the following bounds for $d \geq 3$:

$$2d + 1 \leq h(d) \leq 2^{d-1}(P(d-1) + 1),$$

where $P(d-1)$ is the product of the smallest $d-1$ prime numbers (and thus is asymptotically $d^{d+o(d)}$). For $d = 3$ he gave the better upper bound $h(3) \leq 22$. These remained the best known bounds on $h(d)$ for almost 30 years. In this note, we improve the upper bound to become exponential in d.

Theorem 1. *For all $d \geq 3$ we have $h(d) < 2^{7d}$.*

In fact, the upper bound that we get, as explained below, is slightly better, with the exponent reduced from $7d$ to less than $7d - 8\sqrt{\frac{d-2}{3}}$. For low values of d, we get even better bounds, e.g.,

$$h(3) \leq 32, \; h(4) \leq 240, \; h(5) \leq 988, \; h(6) \leq 8000,$$

which (except for $d = 3$) improve upon those of Valtr.

In order to explain the source of our improvement, we recall that generalising Horton's original two-dimensional construction, Valtr constructed for any $d \geq 2$ arbitrarily large point sets in $\mathbb{R}^d$, which he called d-Horton sets, containing no hole of size greater than $2^{d-1}(P(d-1) + 1)$. The key property that he used, and the one responsible for the superexponential term $P(d-1)$ in the bound, is the following: For two relatively prime moduli q_1 and q_2 and any two residue classes $r_1 (\bmod q_1)$ and $r_2 (\bmod q_2)$, their intersection is equidistributed in the sense that it contains one of any $q_1 q_2$ consecutive integers (by the Chinese remainder theorem). We generalise Horton's construction in a different way, using another kind of equidistribution that is 'cheaper' to achieve. Instead of recruiting larger and larger prime factors as the dimension grows, we use the fixed prime 2. The relevant notion of equidistribution is captured by the following definition due to Sobol' [13].

Definition 2. Let $t \leq m$ be non-negative integers, and let s be a positive integer.

A subset $X \subseteq [0, 1)^s$ is a (t, m, s)*-net* in base 2 if $|X| = 2^m$ and every dyadic sub-box B of $[0, 1)^s$ of the form

$$B = \prod_{i=1}^{s} \left[\frac{b_i}{2^{k_i}}, \frac{b_i + 1}{2^{k_i}}\right),$$

where b_i, k_i are non-negative integers, $b_i < 2^{k_i}$, and $\sum_{i=1}^{s} k_i = m - t$, contains exactly 2^t points of X.

For a real number $y \in [0, 1]$ let $y = \sum_{j=1}^{\infty} \frac{y_j}{2^j}$ with $y_j \in \{0, 1\}$ be a binary expansion of y, and $[y]_m = \sum_{j=1}^{m} \frac{y_j}{2^j}$ its length m truncation (which may depend on the choice of expansion). For $x \in [0, 1]^s$ we write $[x]_m$ for the point in $[0, 1]^s$ obtained by applying this truncation coordinatewise.

An infinite sequence $x_0, x_1, \ldots$ of points in $[0, 1)^s$ with prescribed binary expansions of their coordinates is a (t, s)*-sequence* in base 2 if for every non-negative integer a and every integer $m > t$, the set $X_{a,m} \subseteq [0, 1]^s$ given by

$$X_{a,m} = \{[x_n]_m : a2^m \leq n < (a+1)2^m\}$$

is a (t, m, s)-net in base 2.

These notions (and their analogs in bases other than 2) have been studied intensively in discrepancy theory with applications to numerical analysis. The goal is, for a given dimension s, to construct (t, s)-sequences and hence (t, m, s)-nets with t as small as possible (t is called the quality parameter with lower values corresponding to stronger uniformity of the net/sequence). It has been observed [see, e.g. [11, Lemma 1]] that the existence of a (t, s)-sequence implies the existence of $(t, m, s+1)$-nets for all $m > t$. Various constructions have been proposed, the best amongst them using global function fields. We will use the following upper bound on the lowest possible value of t.

Theorem 3 (Xing and Niederreiter [15]). *For every positive integer s there exists a (t, s)-sequence in base 2 with $t \leq 5s - 8\sqrt{\frac{s-1}{3}} - 3$. Moreover, for infinitely many values of s, there exists a (t, s)-sequence in base 2 with $t < 3s$.*

These upper bounds are not sharp in general. In particular, for low values of s, better estimates are known [see [11, Table III]]: e.g., (t, s)-sequences in base 2 with $(t, s) = (0, 2), (1, 3), (1, 4), (2, 5), \ldots$ have been constructed (and can be used, as explained below, to get the upper bounds on $h(d)$ in dimensions $d = 3, 4, 5, 6$ stated above). However, as s grows, t must grow linearly in s. The strongest known lower bound, due to Schürer [12], is $t > s - (1 + o(1)) \log_2 s$.

Our generalisation of Horton's construction to higher dimensions uses (t, m, s)-nets and is summarised in the following proposition, which is proved in the next section.

Proposition 4. *Let $d \geq 2$ and let $t \leq m$ be non-negative integers so that a (t, m, d)-net in base 2 exists. Then there exists a set A of 2^m points in general position in $\mathbb{R}^d$, having no holes of size greater than $2^d(2^{t+d-1} - 2^t + 1)$.*

Together with Theorem 3, and the fact that a (t, s)-sequence entails $(t, m, s+1)$-nets for all $m > t$, this implies the upper bound on $h(d)$ stated in Theorem 1 (with something to spare). The second part of Theorem 3 shows that for infinitely many values of d, we get an upper bound on $h(d)$, which is exponentially better than stated in Theorem 1. The specific upper bounds on $h(d)$ for low values of d stated above follow by plugging in the parameters of the corresponding known constructions of (t, s)-sequences.

Improvement. After the original version of the paper was written, we noticed that we may replace (t, m, d)-nets by sets satisfying a weaker condition. For $0 \leq \varepsilon < 1$, a non-empty set $X \subseteq [0, 1)^d$ is a (T, ε)-*almost net* in base 2 if $|X| = 2^n T$ for some natural number n and

$$(1 - \varepsilon)T \leq |X \cap B| \leq (1 + \varepsilon)T$$

for every dyadic box B of volume 2^{-n}. The following is a generalisation of Proposition 4.

Proposition 5. *Let $d \geq 2$ and suppose there exists a (T, ε)-almost net in base 2 in $[0, 1)^d$ of size $2^n T$. Then there exists a set A of $2^n T$ points in general position in $\mathbb{R}^d$ having no holes of size greater than $2^d(2^{d-1}(1 + \varepsilon)T - (1 - \varepsilon)T + 1)$.*

In [3], we construct for every natural number n a $(T, 1/3)$-almost net in base 2 in $[0, 1)^d$ of size $2^n T$, where $T \leq 900d \log(2d)$. This implies the following improvement:

Theorem 6. *For all sufficiently large d we have $h(d) = O(4^d d \log d)$.*

2. Horton-like constructions

Geometric idea. Our construction uses the same basic idea that is used in Horton's construction, and in Valtr's construction. Namely, if $U \subset \mathbb{R}^d$ is finite, $v \in \mathbb{R}^d$ is arbitrary and $e \in \mathbb{R}^d$ is a non-zero vector, then from the point of view of U, for large values of $t \in \mathbb{R}_+$ the convex hull

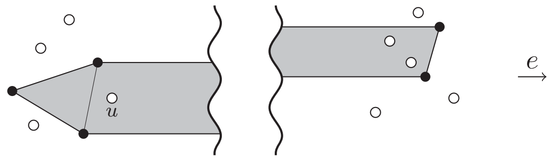

Figure 1. The set U is on the left, the set $V + te$ is on the right. The black points are the elements of U' and $V' + te$ respectively. The convex hull of $U' \cup (V' + te)$ is in grey.

conv $(U \cup \{v + te\})$ is almost equal to conv $(U) + e\mathbb{R}_+$, the Minkowski sum of the set conv (U) and the ray $e\mathbb{R}_+$. The set conv $(U) + e\mathbb{R}_+$ has two advantages: it is independent of v and it is geometrically simpler than conv $(U \cup \{v + te\})$. We extract the desirable properties into a lemma.

For $U \subset \mathbb{R}^d$, we denote by conv U its convex hull, by U^o its interior, and by $\operatorname{conv}^o U$ the interior of its convex hull. For $e \in \mathbb{R}^d$, we write $U + e$ for the translation of the set U by the vector e, and $U - e$ is defined similarly. Given a non-zero vector $e \in \mathbb{R}^d$, we denote by $\overline{p}_e$ the projection of the point $p \in \mathbb{R}^d$ on the subspace orthogonal to e. We drop the subscript e when it is clear from the context, and use the similar notation $\overline{U}$ for the projection of the set U.

Lemma 7. *Suppose $U, V \subset \mathbb{R}^d$ are finite, and $e \in \mathbb{R}^d$ is a non-zero vector. Then there exists a large $t^* = t^*(U, V)$ with the following property. For all $U' \subseteq U$, $V' \subseteq V$, with $V' \neq \emptyset$, for any point $u \in U$, and every $t \geq t^*$ we have:*

a. if $u \in (\operatorname{conv}(U') + e\mathbb{R}_+)^o$ then $u \in \operatorname{conv}^o\big(U' \cup (V' + te)\big)$ and

b. if $\overline{u} \in \operatorname{conv}^o \overline{U' \cup V'}$ then $u \in \big(\operatorname{conv}\big(U' \cup (V' + te)\big) - e\mathbb{R}_+\big)^o$.

Part (a) of the lemma is shown in Figure 1. As the lemma is intuitively plausible, we defer its proof to the end of this section.

We will use the following consequence of Lemma 7.

Lemma 8. *Suppose $U, V, W \subset \mathbb{R}^d$ are finite, and $e \in \mathbb{R}^d$ is a non-zero vector. Let $t \geq t^*(U, V)$, and $t' \geq t^*\big(U \cup (V + te), W\big)$ with t^* as in Lemma 7. Assume that $S \subseteq U \cup (V + te)$ and $u \in U$ satisfy*

- *the intersection $S \cap (V + te)$ is non-empty and*
- *$\overline{u} \in \operatorname{conv}^o \overline{S}$.*

Then $u \in \operatorname{conv}^o (S \cup \{w\})$ for every $w \in W - t'e$.

Like the proof of Lemma 7, we defer the proof of the preceding lemma to the end of the section.

We apply the construction in Lemma 7 repeatedly. We start with the one-element set containing the origin. At each step, we choose a direction e and replace the previously constructed set U by $U \cup (U + te)$ for suitably large t. The directions are chosen amongst the standard basis vectors as follows: for the first m steps we choose e_1 and apply the lemma relative to $\mathbb{R}^1$, for the next m steps we choose e_2 and apply the lemma relative to $\mathbb{R}^2$, and so forth, ending with m steps when we choose e_d and work in $\mathbb{R}^d$. Each point of the resulting set is of the following form:

$$P(\mathbf{a}) \stackrel{\text{def}}{=} \sum_{\substack{i \in [d] \\ j \in [m]}} a^i_j t_{i,j} e_i,$$

where $\mathbf{a} = (a^1, a^2, \ldots, a^d) \in (\{0, 1\}^m)^d$, and

$$0 \ll t_{1,m} \ll t_{1,m-1} \ll \cdots \ll t_{1,1} \ll t_{2,m} \ll t_{2,m-1} \ll \cdots \ll t_{2,1} \ll \cdots\cdots \ll t_{d,m} \ll t_{d,m-1} \ll \cdots \ll t_{d,1}$$

with the meaning of $\ll$ being supplied iteratively by Lemma 7. Note that we chose to parameterise the points so that the last entry of a^i corresponds to the first step of the construction in direction e_i, etc. We may also assume that each next $t_{i,j}$ is at least double the preceding one. This way the order between the ith coordinate values of two points $P(\mathbf{a})$ and $P(\mathbf{b})$ is determined by the lexicographic order between a^i and b^i. Our Horton-like construction will consist of appropriately chosen points of the form $P(\,\cdot\,)$.

Good sets. We next describe a sufficient condition on a set $Y \subseteq (\{0,1\}^m)^d$ that ensures the absence of large holes in $P(Y)$.

We call $a \in \{0,1\}^k$ *a binary sequence of length* k and write $k = \operatorname{len} a$. We denote the concatenation of sequences a and b by ab. We write $a \preceq b$ if a is a prefix of b. For $a \in \{0,1\}^k$, we denote by $\hat{a}$ the sequence of length $k-1$ obtained from a by removing the last element.

Definition 9. We say that a set $Y \subseteq (\{0,1\}^m)^d$ is *q-good* if every pair of distinct points $\mathbf{x}, \mathbf{y} \in Y$ satisfies $x^i \neq y^i$ for all $i \in [d]$, and the following holds true. For every $d-1$ binary sequences $a^2, \ldots, a^d$ (possibly of different lengths) and every $(q+1)$-element set $Z \subseteq Y$ obeying the condition

(C) for each $i \in \{2, 3, \ldots, d\}$, all $\mathbf{z} \in Z$ satisfy $\hat{a}^i \preceq z^i$,

there is $\mathbf{y} \in Y$ such that $a^i \preceq y^i$ for all $i \in \{2, 3, \ldots, d\}$ and $\min\{z^1 : \mathbf{z} \in Z\} < y^1 < \max\{z^1 : \mathbf{z} \in Z\}$ in the lexicographic order.

We shall see below that any (T, ε)-almost net in base 2 can be turned into a $(2^d(1+\varepsilon)T - 2(1-\varepsilon)T + 2)$-good set. In particular, since a (t, m, d)-net in base 2 is also a $(2^t, 0)$-almost net in base 2, any (t, m, d)-net in base 2 can be turned into a $(2^{t+d} - 2^{t+1} + 2)$-good set.

Definition 10. Given a finite set of points $V \subseteq \mathbb{R}^d$, we say that V is *ℓ-hole-free* if for any ℓ points $v_1, v_2, \ldots, v_\ell \in V$, there is a point $v \in V$ in the interior of $\operatorname{conv}\{v_1, v_2, \ldots, v_\ell\}$.

If the set V is in general position, the definition agrees with the usual definition of a set without ℓ-holes. The advantage of this definition is its robustness: every sufficiently small perturbation of an ℓ-hole-free set, which is not necessarily in general position, is again ℓ-hole-free.

Theorem 11. *Let $d \geq 2$, m and q be positive integers, and suppose that $Y \subseteq (\{0,1\}^m)^d$ is q-good. Then the set $P(Y)$ is $(2^{d-1}q+1)$-hole-free.*

By the remark following Definition 10, we do not need to worry about general position. So, Theorem 11 gives us a purely combinatorial way to construct ℓ-hole-free sets.

Proof of Theorem 11. Let $U \subseteq Y$ be an arbitrary set of size $|U| > 2^{d-1}q$. We must show that there is a $\mathbf{y} \in Y$ such that $P(\mathbf{y}) \in \operatorname{conv}^o P(U)$.

We shall define sets $U_d \supseteq U_{d-1} \supseteq U_{d-2} \supseteq \cdots \supseteq U_1$ and binary sequences $a^d, a^{d-1}, \ldots, a^2$ inductively. We begin by setting $U_d \stackrel{\text{def}}{=} U$. Suppose $i > 1$ and U_i has been defined. Denote by U_i^i the set $\{x^i : \mathbf{x} \in U_i\}$. Let b^i be the longest binary sequence that is a prefix of all elements of U_i^i, and let α_i be an element of $\{0,1\}$ that maximises the size of

$$U_{i-1} \stackrel{\text{def}}{=} \{\mathbf{x} \in U_i : b^i\,\alpha_i \preceq x^i\};$$

in case of a tie, we pick α_i arbitrarily. Note that $|U_{i-1}| \geq |U_i|/2$. Let $\beta_i \stackrel{\text{def}}{=} 1 - \alpha_i$. We then set c^i to be the longest sequence such that $b^i\,\alpha_i\,c^i$ is a prefix of all elements of $U_{i-1}^i \stackrel{\text{def}}{=} \{x^i : \mathbf{x} \in U_{i-1}\}$ and define

$$a^i \stackrel{\text{def}}{=} b^i\,\alpha_i\,c^i\,\beta_i.$$

It is clear that a^i satisfies (C) for $Z = U_{i-1}$.

This way we obtain a nested sequence $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_d$ with $|U_1| > q$. Since Y is q-good, and $a^2, \ldots, a^d$ and U_1 satisfy condition (C) in Definition 9, there exist $\mathbf{y} \in Y$, $\mathbf{x}_{\text{small}}, \mathbf{x}_{\text{big}} \in U_1$ satisfying $a^i \preceq y^i$ for all $i \in \{2, 3, \ldots, d\}$ as well as $x^1_{\text{small}} < y^1 < x^1_{\text{big}}$ (in the lexicographic ordering). We claim that $P(\mathbf{y}) \in \operatorname{conv}^o P(U)$.

To prove this claim, we will show by induction on $i = 1, 2, \ldots, d$ that

$$\pi_i(P(\mathbf{y})) \in \operatorname{conv}^o \pi_i(P(U_i)),$$

where $\pi_i\colon \mathbb{R}^d \to \mathbb{R}^i$ is the projection map onto the first i coordinates. The base case $i = 1$ holds because of $x^1_{\text{small}} < y^1 < x^1_{\text{big}}$. Suppose that $i > 1$. There are two (similar) cases depending on the value of α_i. Suppose first that $\alpha_i = 1$. We apply Lemma 8 in $\mathbb{R}^i$ using the vector e_i, with

$$\begin{aligned}
&\{\pi_i(P(\mathbf{x})) : b^i\,1\,c^i\,1 \preceq x^i,\ \mathbf{x} \in (\{0,1\}^m)^d\} \text{ in place of } V + te,\\
&\{\pi_i(P(\mathbf{x})) : b^i\,1\,c^i\,0 \preceq x^i,\ \mathbf{x} \in (\{0,1\}^m)^d\} \text{ in place of } U,\\
&\{\pi_i(P(\mathbf{x})) : b^i\,0 \qquad\ \preceq x^i,\ \mathbf{x} \in (\{0,1\}^m)^d\} \text{ in place of } W - t'e,
\end{aligned}$$

and with $S = \pi_i(P(U_{i-1}))$, $u = \pi_i(P(\mathbf{y}))$ and $w = \pi_i(P(\mathbf{x}))$ for some $\mathbf{x} \in U_i$ such that $b^i\,0 \preceq x^i$ (such $\mathbf{x}$ exists by the maximality of b^i). Note that $S \cap (V + te)$ is non-empty by the maximality of c^i, and $\overline{u} \in \operatorname{conv}^o \overline{S}$ holds by the induction hypothesis. Therefore we deduce from Lemma 8 that $u \in \operatorname{conv}^o (S \cup \{w\}) \subseteq \operatorname{conv}^o \pi_i(P(U_i))$, as required. The case when $\alpha_i = 0$ is treated similarly by exchanging the roles of 0's and 1's, and replacing the vector e_i by $-e_i$. $\square$

Good sets from (T, ε)-almost nets. Here we show how to transform a (T, ε)-almost net $X \subseteq [0,1)^d$ of size $2^n T$ into a good set $Y \subseteq (\{0,1\}^m)^d$ for $m = n + \lceil \log_2 T \rceil + 1$. Fix $i \in [d]$. For $x = (x_1, \ldots, x_d) \in X$, let y_i be the unique non-negative integer such that $y_i \le x_i 2^m < y_i + 1$. Let $\widetilde{y}_i \in \{0,1\}^m$ be the m-digit binary representation of y_i. Applying the definition of a (T, ε)-almost net to the sub-boxes of the form $B = [0,1)^{i-1} \times [\frac{b}{2^n}, \frac{b+1}{2^n}) \times [0,1)^{d-i}$, we know that there are between $(1-\varepsilon)T$ and $(1+\varepsilon)T$ points x in X for which the corresponding $\widetilde{y}_i$ has any given prefix of length n. By suitably changing, if necessary, the last $\lceil \log_2 T \rceil + 1$ entries of $\widetilde{y}_i$ we obtain $y^i \in \{0,1\}^m$ so that the mapping $x \mapsto y^i$ is injective. Doing this for each $i \in [d]$, we transform every x in X into a $\mathbf{y} = (y^1, y^2, \ldots, y^d)$ in $(\{0,1\}^m)^d$, so that the resulting set $Y \subseteq (\{0,1\}^m)^d$ satisfies the requirement in Definition 9 that its elements should differ for all $i \in [d]$. Moreover, the definition of a (T, ε)-almost net implies that for any d binary sequences $a^1, a^2, \ldots, a^d$ with $\sum_{i=1}^d \operatorname{len} a^i = k \le n$, the set

$$I(a^1, a^2, \ldots, a^d) \stackrel{\text{def}}{=} \{\mathbf{y} \in Y : a^i \preceq y^i \text{ for all } i \in [d]\}$$

has size between $2^{n-k}(1-\varepsilon)T$ and $2^{n-k}(1+\varepsilon)T$. We call such a set Y a *binary* (T, ε)*-almost net* of size $2^n T$.

The next result, together with Theorem 11, implies Proposition 5, which was announced in the introduction, and hence Theorrem 6. The case $\varepsilon = 0$ yields Proposition 4, and hence Theorem 1.

Proposition 12. *If $Y \subseteq (\{0,1\}^m)^d$ is a binary (T, ε)-almost net then Y is $\lfloor 2^d(1+\varepsilon)T - 2(1-\varepsilon)T + 2 \rfloor$-good.*

Proof. Suppose that the binary sequences $a^2, \ldots, a^d$ and the set $Z \subseteq Y$ with size $|Z| > 2^d(1+\varepsilon)T - 2(1-\varepsilon)T + 2$ satisfy condition (C) in Definition 9. By condition (C) we have, using the notation introduced above, $Z \subseteq I(\emptyset, \hat{a}^2, \ldots, \hat{a}^d)$. As $|Z| > 2^{d-1}(1+\varepsilon)T$ and

$|I(\emptyset, \hat{a}^2, \ldots, \hat{a}^d)| \le (1+\varepsilon)T \max(1, 2^{n-\sum_{i=2}^d \operatorname{len} \hat{a}^i})$, we conclude that $d-1 < n - \sum_{i=2}^d \operatorname{len} \hat{a}^i$ and hence $\sum_{i=2}^d \operatorname{len} a^i = \sum_{i=2}^d \operatorname{len} \hat{a}^i + d - 1 < n$. Thus the quantity $r \overset{\text{def}}{=} n - \sum_{i=2}^d \operatorname{len} a^i$ is positive. Given a sequence $a \in \{0,1\}^r$, consider the sets $B(a) \overset{\text{def}}{=} I(a, a^2, \ldots, a^d)$ and $\hat{B}(a) \overset{\text{def}}{=} I(a, \hat{a}^2, \ldots, \hat{a}^d)$. From the discussion above we know that $|B(a)| \ge (1-\varepsilon)T$ and $|\hat{B}(a)| \le 2^{d-1}(1+\varepsilon)T$ for every $a \in \{0,1\}^r$. From condition (C) we know also that $Z \subseteq \bigcup_{a\in\{0,1\}^r} \hat{B}(a)$.

Our aim is to find $\mathbf{y} \in Y$ that is contained in some $B(a)$ and whose first coordinate is sandwiched between the first coordinates of two elements in Z.

Suppose first that $Z \cap \hat{B}(a)$ is non-empty for three (or more) distinct sequences $a \in \{0,1\}^r$, say for $a^{(1)}, a^{(2)}, a^{(3)}$. We may assume that, of the three, $a^{(1)}$ is the lexicographically smallest and $a^{(3)}$ is the lexicographically largest. Then we may pick $\mathbf{y}$ to be any element of $B(a^{(2)})$, for its first coordinate is between those of elements in $Z \cap \hat{B}(a^{(1)})$ and in $Z \cap \hat{B}(a^{(3)})$.

So, we may assume that Z is entirely contained in $\hat{B}(a) \cup \hat{B}(a')$ for some pair $a, a' \in \{0,1\}^r$. Then either $\hat{B}(a)$ or $\hat{B}(a')$ contains more than $2^{d-1}(1+\varepsilon)T - (1-\varepsilon)T + 1$ elements of Z. By size considerations, at least two of them must be in the respective $B(\cdot)$-set, and at most one of the two is extremal in Z, so choosing the other one as our $\mathbf{y}$ works. □

Proofs of the geometrical lemmas. It remains to prove Lemma 7 and 8.

Proof of Lemma 7. Because there are only finitely many subset pairs (U', V') and points $u \in U$, it suffices to prove the assertion for any one such choice. We may then pick the largest t^* over all choices of (U', V') and u.

Proof of part (a). Pick $v \in V'$ arbitrarily. Let $u \in (\operatorname{conv}(U') + e\mathbb{R}_+)^o$ be arbitrary. Let $B(u, \varepsilon)$, with $\varepsilon > 0$, be a closed ball around u that is contained in $\operatorname{conv}(U') + e\mathbb{R}_+$.

Assume, for contradiction's sake, that $u \notin \operatorname{conv}^o(U' \cup \{v + te\})$. Then there is a hyperplane through u such that the convex set $\operatorname{conv}(U' \cup \{v+te\})$ lies entirely on one of its sides. Pick a unit normal vector w to this hyperplane, such that the halfspace $H \overset{\text{def}}{=} \{x : \langle w, x-u\rangle > 0\}$ does not meet $\operatorname{conv}(U' \cup \{v+te\})$. Consider the point $\widetilde{u} \overset{\text{def}}{=} u + \varepsilon w$, and note that $\widetilde{u} \in H$.

Since $\operatorname{dist}(u, \widetilde{u}) = \varepsilon$, it follows that $\widetilde{u} \in \operatorname{conv}(U') + e\mathbb{R}_+$, and so we may write $\widetilde{u} = u_0 + et_0$ with $u_0 \in \operatorname{conv} U'$ and $t_0 \in \mathbb{R}_+$. Define points $p \overset{\text{def}}{=} \frac{t_0}{t_0 - t} v + \frac{t}{t-t_0} u_0$ and $u' \overset{\text{def}}{=} (t_0/t)(v+te) + (1-t_0/t)u_0$. We may pick t^* large enough so that $\operatorname{dist}(p, u_0) < \varepsilon$ for $t \ge t^*$. Since $\widetilde{u} = (t_0/t)(v+te) + (1 - t_0/t)p$, it then follows that $\operatorname{dist}(\widetilde{u}, u') < \varepsilon$, and hence $u' \in H$. Since $u' \in \operatorname{conv}(U' \cup \{v+te\})$, this contradicts the definition of H.

Proof of part (b). We first note that it suffices to show that $u \in \operatorname{conv}\big(U' \cup (V' + te)\big) - e\mathbb{R}_+$, for we may then apply this to all points in a sufficiently small neighbourhood of u to conclude that in fact $u \in \big(\operatorname{conv}\big(U' \cup (V'+te)\big) - e\mathbb{R}_+\big)^o$.

As $\overline{u}$ is in the interior of $\operatorname{conv} \overline{U' \cup V'}$, we may write it as a convex combination, in which the coefficients of every point in $\overline{U'}$ and of every point in $\overline{V'}$ are non-zero. Indeed, for sufficiently small $\varepsilon > 0$, the point $\overline{u}_\varepsilon \overset{\text{def}}{=} (1+\varepsilon)\overline{u} - \frac{\varepsilon}{|U'|+|V'|} \sum_{u'\in U'} \overline{u'} - \frac{\varepsilon}{|U'|+|V'|} \sum_{v'\in V'} \overline{v'}$ is in $\operatorname{conv} \overline{U' \cup V'}$. Writing $\overline{u}_\varepsilon$ as a convex combination of the points in $\overline{U' \cup V'}$, and rearranging, we obtain an expression for $\overline{u}$ as a convex combination with strictly positive coefficients.

Fix such a convex combination, say $\overline{u} = \sum_{u'\in U'} \alpha_{u'} \overline{u'} + \sum_{v'\in V'} \beta_{v'} \overline{v'}$. Since the β's are positive, we may choose t^* large enough so that for $t \ge t^*$, the convex combination $\sum_{u'\in U'} \alpha_{u'} u' + \sum_{v'\in V'} \beta_{v'}(v' + te)$ is above u in the direction e, and so $u \in \operatorname{conv}\big(U' \cup (V'+te)\big) - e\mathbb{R}_+$. □

Proof of Lemma 8. From Lemma 7(b) applied to the sets $U' = S \cap U$ and $V' = (S - te) \cap V$, using $\overline{u} \in \operatorname{conv}^o \overline{S}$ we deduce that $u \in (\operatorname{conv} S - e\mathbb{R}_+)^o$ (note that $V' \ne \emptyset$ because $S \cap (V + te)$ is non-empty). Then, from Lemma 7(a) applied to the sets $U \cup (V + te)$ and W in place of U and V, the direction $-e$ in place of e, with $U' = S$ and $V' = \{w + t'e\}$, we obtain the desired conclusion. □

3. Problems and remarks

- We suspect that every large enough set in general position in $\mathbb{R}^d$ contains an exponentially large hole. However, we were unable to improve Valtr's bound $h(d) \geq 2d + 1$. We do have an argument (details omitted) showing that constructions along the lines of Horton's, Valtr's and ours cannot avoid exponentially large holes: if a set in $\mathbb{R}^d$ consists of points of the form $P(\mathbf{a})$ and is large enough, then it contains a hole of size 2^d.
- Let $f_{d,\ell}(n)$ be the least number of ℓ-holes in an n-point set in general position in $\mathbb{R}^d$. It is possible to give lower bounds on $f_{d,\ell}(n)$. First, for $\ell \leq h(d)$, we may cut the n-point set into linearly many equally large pieces by parallel hyperplanes. If each piece is large enough, then it contains an ℓ-hole, and so $f_{d,\ell}(n) = \Omega(n)$ in this case.

  Second, $f_{d+1,\ell+1}(n+1) \geq \frac{n+1}{\ell+1} \cdot f_{d,\ell}(\lceil n/2 \rceil)$ holds. Indeed, suppose $P \subset \mathbb{R}^{d+1}$ is in general position and $p \in P$ is arbitrary. Pick any hyperplane that passes only through p, and push it slightly towards the side containing more points of P. Consider the central projection towards p to the hyperplane of points on this larger side; we may think of it as a set in $\mathbb{R}^d$. Every ℓ-hole in this set entails an $(\ell + 1)$-hole in P. As an $(\ell + 1)$-hole arises in this manner at most $\ell + 1$ times, the bound follows.

  Taken together with the known lower bounds on $f_{2,\ell}(n)$ and with the aforementioned bound of Valtr, these two observations yield $f_{d,\ell}(n) = \Omega(n^d)$ for $\ell = d + 1, d + 2$, $f_{d,d+3}(n) = \Omega(n^{d-1} \log^{4/5} n)$, $f_{d,d+4}(n) = \Omega(n^{d-1})$, and $f_{d,d+k}(n) = \Omega(n^{d-k+2})$ for $k = 5, \ldots, d + 1$.
- It would be interesting to characterise large sets that contain no holes of some fixed size. In this connection we conjecture that, for each $n, \ell \in \mathbb{N}$, every sufficiently large ℓ-hole-free set in general position in $\mathbb{R}^2$ contains an n-point subset whose order type is the same as that of an n-point Horton set.

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