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On 2-colored graphs and partitions of boxes

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ABSTRACT

We prove that if the edges of a graph G can be colored blue or red in such a way that every vertex belongs to a monochromatic k-clique of each color, then G has at least $4(k-1)$ vertices. This confirms a conjecture of Bucic et al. (0000), and thereby solves the 2-dimensional case of their problem about partitions of discrete boxes with the k-piercing property. We also characterize the case of equality in our result.

1. Introduction

In this paper, a 2-*colored graph* will be a simple graph with edges colored blue or red. Bucic et al. [2] asked the following: Given an integer $k \geq 2$, what is the smallest possible number of vertices in a 2-colored graph having the property that every vertex belongs to a monochromatic k-clique of each color?

They gave the following construction, showing that $4(k-1)$ vertices suffice. First, for $k = 2$, take a 4-cycle with edges colored alternatingly. Now, for general k, blow up this 4-cycle, replacing each vertex by a monochromatic $(k-1)$-clique, with colors alternating along the 4-cycle (all edges between two adjacent $(k-1)$-cliques get the same color as the edge in the underlying 4-cycle). It is easy to verify that this 2-colored graph has the required property.

Bucic et al. [2] conjectured that this construction is optimal, and proved a lower bound of the form $(4-o_k(1))k$ on the number of vertices in any 2-colored graph with the required property. Here we prove exact optimality.

Theorem 1. *Let $k \geq 2$ be an integer, and let $G = (V, E)$ be a 2-colored graph so that every vertex in V belongs to a monochromatic k-clique of each color. Then $|V| \geq 4(k-1)$.*

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Our proof, given in Section 2, combines counting arguments with a linear algebraic trick similar to one used by Tverberg [7]. In Section 3 we characterize the case of equality in Theorem 1. Perhaps surprisingly, for $k \geq 3$ it turns out that the above example on $4(k-1)$ vertices is not the only extremal one. In Section 4 we discuss some generalizations and reformulations of the above question. These involve, in particular, partitions of a box into sub-boxes and decompositions of a bipartite graph into complete bipartite subgraphs. (The discussion of the latter clarifies the relation to Tverberg's above-mentioned proof.)

2. Proof of Theorem 1

Let $G = (V, E)$ be a 2-colored graph so that every vertex in V belongs to a monochromatic k-clique of each color. Instead of working directly with the graph, we store the information we have in the following form: We have a vertex set V and two families $\mathcal{B} = \{B_1, \dots, B_b\}$, $\mathcal{R} = \{R_1, \dots, R_r\}$ of subsets of V satisfying:

$$|B_i| \geq k \text{ and } |R_j| \geq k \quad \forall i \in [b], j \in [r], \tag{1}$$

$$|B_i \cap R_j| \leq 1 \quad \forall i \in [b], j \in [r], \tag{2}$$

$$\bigcup_{i=1}^{b} B_i = \bigcup_{j=1}^{r} R_j = V. \tag{3}$$

Indeed, given G we can construct $\mathcal{B}$ and $\mathcal{R}$ as the families of vertex sets of blue (resp. red) cliques witnessing that every vertex belongs to a monochromatic clique of each color.

Next, we claim that we can keep the same vertex set V and possibly make some adjustments to the families $\mathcal{B}$ and $\mathcal{R}$, so that the following will be satisfied in addition to (1)–(3):

$$B_i \setminus \bigcup_{i' \neq i} B_{i'} \neq \emptyset \text{ and } R_j \setminus \bigcup_{j' \neq j} R_{j'} \neq \emptyset \quad \forall i \in [b], j \in [r], \tag{4}$$

$$|B_i \cap R_j| = 1 \quad \forall i \in [b], j \in [r]. \tag{5}$$

Indeed, if $B_i \subseteq \bigcup_{i' \neq i} B_{i'}$, say, then we can discard B_i from $\mathcal{B}$ while retaining properties (1)–(3). Iterating this operation we end up with families satisfying (1)–(4). At this point, if $B_i \cap R_j = \emptyset$ then we can choose a vertex $v \in B_i \setminus \bigcup_{i' \neq i} B_{i'}$ and replace R_j by $R_j \cup \{v\}$ while retaining properties (1)–(3). It may happen that this change causes a violation of (4), namely when we had $R_{j^*} \setminus \bigcup_{j' \neq j^*} R_{j'} = \{v\}$ for some $j^* \neq j$ before the change; in this case, after adding v to R_j we discard R_{j^*}. Iterating this operation we end up with families satisfying (1)–(5).

Thus, we may assume that the set V and the two families $\mathcal{B} = \{B_1, \dots, B_b\}$, $\mathcal{R} = \{R_1, \dots, R_r\}$ of subsets of V satisfy (1)–(5). For every $v \in V$ we write

$$I_v = \{i \in [b] : v \in B_i\}, \quad J_v = \{j \in [r] : v \in R_j\}.$$

Note that the properties of V, $\mathcal{B}$ and $\mathcal{R}$ may be expressed in terms of the subsets I_v of $[b]$ and J_v of $[r]$, for $v \in V$, as follows: (1) says that every $i \in [b]$ is covered by the subsets I_v at least k times, and similarly for $[r]$ and the subsets J_v; (3) says that $I_v, J_v \neq \emptyset$; (4) says that for every $i \in [b]$ there is $v \in V$ such that $I_v = \{i\}$, and similarly for $[r]$ and J_v; (5), which implies (2), says that the product sets $I_v \times J_v$ partition $[b] \times [r]$.

Proposition 1. *If V, $\mathcal{B}$ and $\mathcal{R}$ satisfy* (4) and (5) *then* $|V| \geq b + r - 1$.

Proof. We introduce for each $i \in [b]$ a variable x_i, and for each $j \in [r]$ a variable y_j (these variables take real values). By (5) we have the identity

$$\sum_{v \in V} \left(\sum_{i \in I_v} x_i \right) \cdot \left(\sum_{j \in J_v} y_j \right) = \left(\sum_{i=1}^{b} x_i \right) \cdot \left(\sum_{j=1}^{r} y_j \right). \tag{6}$$

Now we consider the following system of homogeneous linear equations:

$$\sum_{i\in I_v} x_i - \sum_{j\in J_v} y_j = 0, \quad v \in V, \tag{7}$$

$$\sum_{i=1}^{b} x_i = 0. \tag{8}$$

It suffices to show that the system has only the trivial solution, because this implies that the number of equations $|V|+1$ is at least as large as the number of variables $b+r$. Let $(x_i)_{i\in[b]}, (y_j)_{j\in[r]}$ satisfy (7) and (8). By (7) we know that for each $v \in V$ there is a real number α_v so that $\sum_{i\in I_v} x_i = \sum_{j\in J_v} y_j = \alpha_v$. The identity (6) implies, using (8), that $\sum_{v\in V} \alpha_v^2 = 0$ and hence $\alpha_v = 0$ for all $v \in V$. Now, given $i \in [b]$ we can find by (4) some $v \in V$ such that $x_i = \sum_{i\in I_v} x_i = \alpha_v = 0$, and a similar argument shows that $y_j = 0$ for every $j \in [r]$, as required. □

Returning to the proof of Theorem 1, we may henceforth assume that $b+r \le 4(k-1)$, otherwise $|V| \ge 4(k-1)$ follows from Proposition 1. We also know that $b \ge k$, because the sets I_v, $v \in R_1$, are k or more disjoint nonempty subsets of $[b]$; similarly $r \ge k$. Thus, the relevant domain for $b+r$ in the rest of the proof is

$$2k \le b + r \le 4(k-1). \tag{9}$$

Using (1) we have

$$\sum_{v\in V} |I_v| + |J_v| \ge k(b+r), \tag{10}$$

and using (5) we have

$$\sum_{v\in V} |I_v||J_v| = br. \tag{11}$$

Since $|I_v|$ and $|J_v|$ are nonzero by (3), their product is smallest (given their sum) when one of them is 1. Hence

$$|I_v||J_v| \ge |I_v| + |J_v| - 1 \quad \forall v \in V. \tag{12}$$

Using (10)–(12) we can write

$$\begin{aligned} |V| &= \sum_{v\in V} |I_v| + |J_v| - (|I_v| + |J_v| - 1) \\ &\ge k(b+r) - \sum_{v\in V} (|I_v| + |J_v| - 1) \\ &\ge k(b+r) - \sum_{v\in V} |I_v||J_v| \\ &= k(b+r) - br \\ &\ge k(b+r) - \frac{(b+r)^2}{4}. \end{aligned} \tag{13}$$

The latter is a decreasing function of $b+r$ in the domain (9), and is therefore bounded from below by its value at $b+r = 4(k-1)$, which is $4(k-1)$. This proves that $|V| \ge 4(k-1)$, as required. □

3. Characterization of extremal graphs

If $G = (V, E)$ is a 2-colored graph having the property that every vertex in V belongs to a monochromatic k-clique of each color, then adding any edges to G (between existing vertices) and coloring them arbitrarily results in a graph with the same property. Therefore we can restrict attention to those graphs having this property which are *edge-critical*, in the sense that removing any edge entails the loss of this property.

Here is a construction of an edge-critical 2-colored graph on $4(k-1)$ vertices, so that every vertex belongs to a monochromatic k-clique of each color, which generalizes the one from [2] described in the introduction. Let $k \geq 2$ be an integer, let X and Y be two disjoint sets of $2(k-1)$ vertices each, and let $B(X, Y)$ and $R(X, Y)$ be two complementary $(k-1)$-regular bipartite graphs on the bipartition (X, Y). Our graph $G = G(X, Y, B, R)$ has $X \cup Y$ as its vertex set. It has the complete bipartite graph on (X, Y) as a subgraph, with edges in $B(X, Y)$ colored blue and edges in $R(X, Y)$ colored red. We refer to $B(X, Y)$ and $R(X, Y)$ as the blue and red graphs, respectively. In addition, any two vertices in X which have a common neighbor in the blue graph are joined by a blue edge in G, and any two vertices in Y which have a common neighbor in the red graph are joined by a red edge in G. It is easy to verify that this 2-colored graph has the required property and is edge-critical.

For $k = 2$ we have $|X| = |Y| = 2$ and the blue and red graphs must be two complementary perfect matchings, resulting in the 2-colored 4-cycle described in the introduction. But for higher values of k, we have more freedom in choosing $B(X, Y)$ and $R(X, Y)$. For example, consider $k = 3$, so $|X| = |Y| = 4$. We may choose $B(X, Y)$ and $R(X, Y)$ so that each of them is the disjoint union of two 4-cycles, resulting in the blown-up 4-cycle graph from the introduction. But we can also choose $B(X, Y)$ and $R(X, Y)$ to be 8-cycles, resulting in a new example, not isomorphic to the previous one.

Note that the construction described in the introduction corresponds to the following choice of $B(X, Y)$ and $R(X, Y)$: X is equi-partitioned into X_1 and X_2, Y is equi-partitioned into Y_1 and Y_2, $B(X, Y)$ consists of all edges between X_1 and Y_1 and between X_2 and Y_2, and $R(X, Y)$ consists of all edges between X_1 and Y_2 and between X_2 and Y_1. For this choice, the resulting graph $G(X, Y, B, R)$ induces blue cliques on X_1 and X_2 and red cliques on Y_1 and Y_2, and has a total of $2(k-1)(3k-4)$ edges. Among all graphs of the form $G(X, Y, B, R)$ for a given value of k, the latter uniquely minimizes the number of edges. To see this, observe that in the graph induced on X (and similarly for Y) each vertex must have degree at least $k-2$, and the only way to have these degrees equal to $k-2$ is by using X_1, X_2, Y_1, Y_2 as above.

The next result shows that all edge-critical extremal examples for Theorem 1 are of the form $G = G(X, Y, B, R)$, thus characterizing the case of equality in that theorem.

Theorem 2. *Let $k \geq 2$ be an integer, and let $|V| = 4(k-1)$. Let $G = (V, E)$ be a 2-colored graph so that every vertex in V belongs to a monochromatic k-clique of each color, and G is edge-critical with respect to this property. Then G is isomorphic to some $G(X, Y, B, R)$, where $B(X, Y)$ and $R(X, Y)$ are complementary $(k-1)$-regular bipartite graphs on (X, Y).*

Proof. Let $G = (V, E)$ satisfy the assumptions of the theorem. In the case $k = 2$, it is easy to check directly that G must be a 4-cycle colored alternatingly, as claimed. We henceforth assume that $k \geq 3$.

As in the proof of Theorem 1, we associate with G two families $\mathcal{B} = \{B_1, \ldots, B_b\}$, $\mathcal{R} = \{R_1, \ldots, R_r\}$ of subsets of V satisfying (1)–(3). Clearly, the blue edges of G are those pairs $\{u, v\}$ contained in some B_i, and the red edges are those pairs $\{u, v\}$ contained in some R_j (by edge-criticality, there can be no other edges in G). In the main part of the proof below, we assume that V, $\mathcal{B}$, $\mathcal{R}$ satisfy (4) and (5) as well; at the end of the proof we will justify this assumption. We also use the notations I_v and J_v for $v \in V$ as introduced in the proof of Theorem 1. According to that proof, the only values of $b + r$ which may result in $|V| = 4(k-1)$ are $4(k-1)$ and $4(k-1)+1$ (if $b + r < 4(k-1)$ then (13) forces $|V|$ to be larger, and if $b + r > 4(k-1) + 1$ then Proposition 1 does that).

Case 1 $b + r = 4(k-1)$

Because $|V| = 4(k-1)$, (13) must hold as an equality. This implies that (10) and (12) hold as equalities, and $b = r = 2(k-1)$. Equality in (10) means that every B_i and every R_j is of size exactly k. Equality in (12) means that for every $v \in V$, at least one of I_v, J_v is a singleton. For $j \in [r]$, the sets I_v, $v \in R_j$, partition $[b]$ into k nonempty subsets. This implies that $|I_v| \leq k-1$, and similarly $|J_v| \leq k-1$, for every $v \in V$. Therefore $|I_v||J_v| \leq k-1$ for every $v \in V$, but since $\sum_{v\in V} |I_v||J_v| = 4(k-1)^2$ we must have equality for every $v \in V$. This means that we can partition V into two sets:

$$X = \{v \in V : |I_v| = k-1, |J_v| = 1\}, \quad Y = \{v \in V : |I_v| = 1, |J_v| = k-1\}.$$

As $\sum_{v\in V} |I_v| = kb = 2k(k-1)$, we must have $|X| = |Y| = 2(k-1)$.

Now, consider a vertex $v \in X$. There is $j \in [r]$ such that $v \in R_j$. Since the sets I_u, $u \in R_j$, partition $[b]$ into k subsets, one of which is I_v of size $k-1$, all other I_u must be singletons, so that $R_j \setminus \{v\} \subseteq Y$. This accounts for $k-1$ red edges from v into Y. As this holds for every $v \in X$, and similarly every $v \in Y$ must have at least $k-1$ blue edges into X, the complete bipartite graph on (X, Y) must appear in G and be colored so that the blue graph $B(X, Y)$ and the red graph $R(X, Y)$ are both $(k-1)$-regular. The above also implies that the neighbors of every $v \in X$ in the red graph must form a red clique in Y, and the neighbors of every $v \in Y$ in the blue graph must form a blue clique in X. This shows that $G(X, Y, B, R)$ is contained in G, and as G is edge-critical, they must coincide.

Case 2 $b + r = 4(k-1) + 1$

We will show that this case cannot occur. Consider the mapping from $\mathcal{B} \cup \mathcal{R}$ into V defined as follows. To each B_i we assign, using (4), an element u of B_i which belongs to no other $B_{i'}$; if among the possible choices of u for a given B_i there is one which belongs to more than one of the sets R_j, we assign to B_i such a u. Similarly, to each R_j we assign an element u of R_j which belongs to no other $R_{j'}$, with priority to such u which belongs to more than one of the sets B_i. As $|\mathcal{B} \cup \mathcal{R}| > |V|$, the mapping is not injective, so we can find some $u \in V$ which was assigned to some B_i and to some R_j.

Assume w.l.o.g. that $b \geq 2(k-1) + 1$. For the set R_j just found, there is no vertex v such that $|I_v| > 1$ and $J_v = \{j\}$; indeed, if there were such v it would be given priority as the vertex assigned to R_j, over the actual assignment of u which belongs to a unique B_i. It follows that we can partition R_j into two sets:

$$S = \{v \in R_j : |I_v| = 1\}, \quad T = \{v \in R_j : |I_v|, |J_v| \geq 2\}.$$

Write $|R_j| = k + \ell$, where $\ell \geq 0$, and $|S| = s$.

Due to the size of R_j, the difference between the two sides of (10) is at least ℓ. Hence the first inequality in (13) holds with a slack of at least ℓ.

For each $v \in T$, the difference between the two sides of (12) is at least

$$2(|I_v| + |J_v| - 2) - (|I_v| + |J_v| - 1) = |I_v| + |J_v| - 3 \geq |I_v| - 1.$$

Since $|T| = k+\ell-s$, the second inequality in (13) holds with a slack of at least $\sum_{v \in T} |I_v| - (k+\ell-s)$. The sets I_v, $v \in R_j$, partition $[b]$, and therefore $\sum_{v \in T} |I_v| = b - s \geq 2(k-1) + 1 - s$, so the slack in the second inequality in (13) is at least $k - \ell - 1$.

Adding up the two slacks, we obtain

$$\begin{aligned} |V| &\geq k(b+r) - br + k - 1 \\ &\geq k(4(k-1)+1) - 2(k-1)(2(k-1)+1) + k - 1 \\ &= 4(k-1) + 1, \end{aligned}$$

which contradicts our assumption on $|V|$.

It remains to address the possibility that the families $\mathcal{B}, \mathcal{R}$ associated with G do not satisfy (4) and (5). In this case, by performing the steps indicated in the proof of Theorem 1, we obtain modified families $\mathcal{B}'$, $\mathcal{R}'$ which do satisfy (4) and (5) as well as (1)–(3). By the foregoing proof, $\mathcal{B}'$ and $\mathcal{R}'$ must be as described in Case 1 above, and the graph corresponding to them is isomorphic to some $G(X, Y, B, R)$. In particular, all sets in $\mathcal{B}' \cup \mathcal{R}'$ are of size k exactly. It follows that in passing from $\mathcal{B}$, $\mathcal{R}$ to $\mathcal{B}'$, $\mathcal{R}'$, the step of adding a vertex to a set could never occur. Thus, the only steps performed were deletions of sets. Therefore the original graph G contains a graph of the form $G(X, Y, B, R)$, and by edge-criticality they must coincide. $\square$

4. Generalizations and reformulations

4.1. More than two colors

It is natural to generalize the question treated here to t-colored graphs, i.e., simple graphs with edges colored in one of t colors.

Question 1 (*Bucic et al. [2]*). *Given integers $k, t \geq 2$, what is the smallest possible number of vertices in a t-colored graph having the property that every vertex belongs to a monochromatic k-clique of each color?*

Bucic et al. noted that their construction for $t = 2$ on $4(k-1)$ vertices can be adapted to one for general t using $2t(k-1)$ vertices. In fact, our more general construction in Section 3 can also be adapted as follows. Let $X_1, \ldots, X_t$ be t disjoint sets of $2(k-1)$ vertices each. For any pair of colors i, j, take the complete bipartite graph on (X_i, X_j) and color its edges i or j so that both color graphs are $(k-1)$-regular. In addition, for each color i, any two vertices in X_i which have a common neighbor in the color i graph (in any X_j, $j \neq i$) are joined by an edge colored i. This yields an edge-critical graph with the required property.

It is natural to conjecture that the above construction is optimal for any fixed number of colors t and large enough k. As observed by Bucic et al. [2], their proof of an asymptotic lower bound for the case of two colors extends to general t, yielding a lower bound of $(2t - o_k(1))k$ in Question 1. Unfortunately, it seems that our proof of the exact lower bound does not extend to general t.

We note that the above construction is not optimal if k is fixed and t is suitably large. For example, 4 vertices suffice for $k = 2$, $t = 3$. More generally, for any fixed k and large enough t congruent to $1 (\mathrm{mod}\, k)$, the answer to Question 1 is $t(k-1)+1$. Indeed, by the existence result of Ray-Chaudhuri and Wilson [6] for resolvable block designs, for such k and t the complete graph on $t(k-1)+1$ vertices can be t-colored so that each color class is a disjoint union of k-cliques spanning all vertices.

While the above generalization looks interesting in its own right, the intended application of Bucic et al. [2] suggests a different generalization. This will be explained in the following subsections.

4.2. Partition of a box into sub-boxes

A set of the form $A = A_1 \times \cdots \times A_d$, where $A_1, \ldots, A_d$ are finite sets with $|A_i| \geq 2$, is called a d-dimensional discrete box. A set of the form $B = B_1 \times \cdots \times B_d$, where $B_i \subseteq A_i$ for all $i \in [d]$, is a sub-box of A; it is said to be nontrivial if $\emptyset \neq B_i \neq A_i$ for all $i \in [d]$. It is easy to partition a d-dimensional discrete box into 2^d nontrivial sub-boxes, by cutting each A_i into two parts. The following theorem answered a question of Kearnes and Kiss [4].

Theorem 3 (*Alon et al. [1]*). *Let A be a d-dimensional discrete box, and let $\{B^1, \ldots, B^m\}$ be a partition of A into m nontrivial sub-boxes. Then $m \geq 2^d$.*

Instead of requiring the sub-boxes $B^1, \ldots, B^m$ to be nontrivial, one may equivalently require that every axis-parallel line (i.e., set of the form $\{(x_1, \ldots, x_d) \in A : x_j = a_j \ \forall j \in [d] \setminus \{i\}\}$) intersects at least two of them. This observation led Bucic et al. [2] to consider families of sub-boxes $\{B^1, \ldots, B^m\}$ with the k-piercing property, namely: every axis-parallel line intersects at least k sub-boxes in the family. Generalizing the question of Kearnes and Kiss, they asked the following.

Question 2 (*Bucic et al. [2]*). *Let $d \geq 1$ and $k \geq 2$ be integers, and let $A = A_1 \times \cdots \times A_d$ be a d-dimensional discrete box with all $|A_i|$ sufficiently large. What is the smallest possible number m of sub-boxes in a partition $\{B^1, \ldots, B^m\}$ of A having the k-piercing property?*

They denoted the answer to Question 2 by $p_{\mathrm{box}}(d, k)$. The case $k = 2$ is solved by Theorem 3: $p_{\mathrm{box}}(d, 2) = 2^d$. For larger k, it is natural to consider first the 2-dimensional case ($d = 1$ is trivial). Here, cutting each A_i into k parts gives a construction with $m = k^2$ sub-boxes. But Bucic et al. [2] showed that in fact $m = 4(k-1)$ is enough. Their construction is illustrated in Fig. 1.

Bucic et al. conjectured that this construction is optimal, that is, $p_{\mathrm{box}}(2, k) = 4(k-1)$. They observed that this is the case if one restricts attention to sub-boxes which are bricks, i.e., products of intervals. In an attempt to prove optimality among partitions into general sub-boxes, they associated with any such partition of a 2-dimensional box a 2-colored graph as follows: the vertices are the sub-boxes in the partition, and two sub-boxes are joined by a blue (resp. red) edge if there is

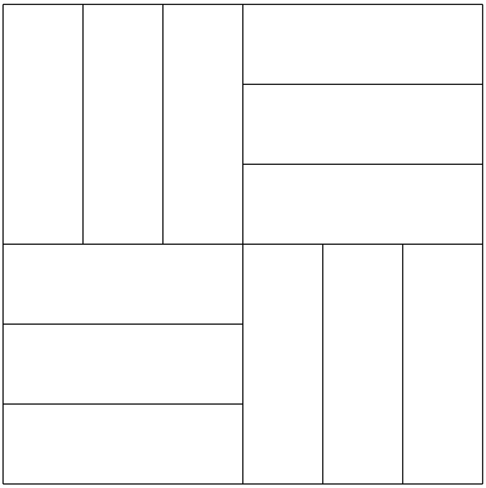

Fig. 1. A k-piercing partition of a 2-dimensional box, showing that $p_{\mathrm{box}}(2, k) \leq 4(k-1)$. Each quarter of the box consists of $k-1$ parallel sub-boxes. The illustration corresponds to $k = 4$.

a horizontal (resp. vertical) line which intersects both of them. The k-piercing property implies that every vertex belongs to a monochromatic k-clique of each color. This led them to ask for the minimum number of vertices in such a graph. Note that the 2-colored graph with $4(k-1)$ vertices constructed by them (and presented in the introduction) corresponds to the partition shown in Fig. 1.

The asymptotic lower bound that Bucic et al. [2] obtained for the question about 2-colored graphs enabled them to deduce that $p_{\mathrm{box}}(2, k) \geq (4 - o_k(1))k$. Our full solution of the question (Theorem 1) allows us to confirm their conjecture: $p_{\mathrm{box}}(2, k) = 4(k-1)$. In fact, since the reduction described in the previous paragraph does not depend on the sub-boxes being a covering of the given box, but only on their disjointness, we have the following more general statement, which is also tight.

Corollary 1. *Let $k \geq 2$ be an integer, let A be a 2-dimensional discrete box, and let $\{B^1, \ldots, B^m\}$ be a family of m disjoint sub-boxes of A having the k-piercing property. Then $m \geq 4(k-1)$.*

The question of determining $p_{\mathrm{box}}(d, k)$ when both d and k are greater than 2 remains wide open. Bucic et al. [2] attempted a reduction to colored graphs similar to the above, but it led to a less natural and less tractable question than in the case $d = 2$. Their best bounds for general d and k are of the form $e^{\Omega(\sqrt{d})}k \leq p_{\mathrm{box}}(d, k) \leq 15^{d/2}k$ (of course, when k is small relative to d, the bounds $2^d = p_{\mathrm{box}}(d, 2) \leq p_{\mathrm{box}}(d, k) \leq k^d$ may be better).

4.3. Decomposition of a bipartite graph into complete bipartite subgraphs

A well-studied parameter of a graph $G = (V, E)$ is the minimum number of edge-disjoint complete bipartite subgraphs of G which cover the edge set E. The best known result is that of Graham and Pollak [3], saying that any such decomposition of the complete graph $G = K_n$ must consist of at least $n-1$ complete bipartite subgraphs. For more general results about decomposition of an arbitrary graph G, see e.g. Kratzke et al. [5]. The case when G itself is complete bipartite is of course uninteresting, because there is a decomposition into one subgraph. But it becomes interesting under some constraints on the decomposition, as we will see below.

A 2-dimensional discrete box $A = A_1 \times A_2$ (discussed in the previous subsection) may be viewed as the edge set of a complete bipartite graph on (A_1, A_2). A partition of A into sub-boxes is then a decomposition of a complete bipartite graph into complete bipartite subgraphs. We can restate

Corollary 1 from this point of view, as follows. (The notation $G = (A_1, A_2; E)$ refers to a bipartite graph with vertex bipartition (A_1, A_2) and edge set E.)

Corollary 2. *Let $k \geq 2$ be an integer. Let $G = (A_1, A_2; E)$ be a bipartite graph, and let $\{G^i = (B^i_1, B^i_2; E^i)\}_{i\in[m]}$ be a decomposition of G into m complete bipartite subgraphs. Assume that every vertex in A_1 (resp. A_2) belongs to B^i_1 (resp. B^i_2) for at least k values of $i \in [m]$. Then $m \geq 4(k-1)$.*

Proposition 1 may also be reformulated in this terminology, as follows.

Corollary 3. *Let $G = (A_1, A_2; E)$ be a complete bipartite graph, and let $\{G^i = (B^i_1, B^i_2; E^i)\}_{i\in[m]}$ be a decomposition of G into m complete bipartite subgraphs. Assume that for every vertex x in A_1 (resp. A_2) there is $i \in [m]$ such that $B^i_1 = \{x\}$ (resp. $B^i_2 = \{x\}$). Then $m \geq |A_1| + |A_2| - 1$.*

Indeed, Tverberg's [7] proof of Graham and Pollak's theorem inspired the proof of Proposition 1.

This point of view on partition problems for 2-dimensional discrete boxes suggests a generalization to higher dimensions expressed in terms of d-partite hypergraphs. (Recall that in such a hypergraph H there is a partition $(A_1, \ldots, A_d)$ of the vertex set, so that every edge contains exactly one vertex from each A_j. We use the notation $H = (A_1, \ldots, A_d; E)$. We say that H is complete d-partite if every d-tuple meeting each A_j is an edge.) In particular, the following question asks for a d-partite version of Corollary 2.

Question 3. *Let $d, k \geq 2$ be integers. Let $H = (A_1, \ldots, A_d; E)$ be a complete d-partite hypergraph, and let $\{H^i = (B^i_1, \ldots, B^i_d; E^i)\}_{i\in[m]}$ be a decomposition of H into m complete d-partite subhypergraphs. Assume that for every $\ell \in [d]$ and for every $(d-1)$-tuple of vertices $x_j \in A_j$, $j \in [d] \setminus \{\ell\}$, there are at least k values of $i \in [m]$ such that $x_j \in B^i_j$ for all $j \in [d] \setminus \{\ell\}$. If all $|A_j|$ are sufficiently large, what is the smallest possible number m of subhypergraphs in such a decomposition?*

This is a reformulation of Question 2, so the answer is the same $p_{\mathrm{box}}(d, k)$ investigated by Bucic et al. [2]. Hopefully, this interpretation of the question may suggest a useful approach, but we were unable to extend the methods of this paper to handle it.

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