Soc Choice Welfare (1984) 1 : 123–125

Social Choice and Welfare

A Note on the Redundancy of an Axiom in the Pattanaik-Peleg Characterization of the Lexicographic Maximin Extension

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Received November 4, 1983 / Accepted January 9, 1984

Abstract. P. K. Pattanaik and B. Peleg have provided an axiomatic characterization of the lexicographic maximin extension of an ordering over a finite set to an ordering over the set of all non-empty subsets of that set. This note shows that one of their axioms, namely the Union axiom, is redundant, thus obtaining a more economical characterization composed of the remaining four axioms (which are independent by virtue of the examples given by Pattanaik and Peleg).

Given an ordering R over a finite set X (i.e., R is a complete transitive binary relation on X), a canonical way to extend it to an ordering over the set $\pi(X)$ of non-empty subsets of X is the following. Let u be any real valued function on X that represents R (i.e., $u(x) \geqq u(y)$ iff $x R y$) and assumes only positive values. Let $n = |X|$ denote the cardinality of X. For each $A \in \pi(X)$, let $v_*(A)$ be the n-vector whose first $|A|$ coordinates are the u values of the elements of A arranged in non-decreasing order, and the remaining coordinates are zero.

The *lexicographic maximin extension* of R is the ordering $\succeq_*$ over $\pi(X)$ defined by: $A \succeq_* B$ iff either $x R y$ holds for all $x, y \in A \cup B$ or $v_*(A)$ is greater or equal lexicographically than $v_*(B)$.

Let us recall the axioms used by Pattanaik and Peleg [1] in their recent characterization of the lexicographic maximin extension.

The following notations are used. R is an ordering over X and P and I are its asymmetric and symmetric factors respectively ("P" for preference, "I" for indifference). $\succeq$ is an ordering over $\pi(X)$ and $\succ$ and $\sim$ are its asymmetric and symmetric factors respectively. For $A, B \in \pi(X)$, $A \mathbf{R} B$ means that $x R y$ holds for all $x \in A$, $y \in B$; $\mathbf{P}$ and $\mathbf{I}$ are used similarly; an element of X on either side of such a statement is to be interpreted as the corresponding singleton in $\pi(X)$. Finally, $A \cap\!\!\!\!\!\bigcirc\, \mathbf{I}\, B = \emptyset$ means that $x I y$ holds for no $x \in A$, $y \in B$.

The five axioms are:

Gardenfors Principle (GP)
For all $A \in \pi(X)$ and all $x \in X \setminus A$,

(i) $x\mathbf{I}A$ implies $\{x\} \cup A \sim A$;
(ii) $[x\mathbf{R}A$ and not $A\mathbf{R}x]$ implies $\{x\} \cup A \succ A$; and
(iii) $[A\mathbf{R}x$ and not $x\mathbf{R}A]$ implies $A \succ \{x\} \cup A$.

Extension (E)
For all $A, B \in \pi(X)$ and all $x \in X \backslash (A \cup B)$, $[(A \cup B)\mathbf{R}x$ and $A \succ B]$ implies $\{x\} \cup A \succ \{x\} \cup B$.

Strong Fishburn Monotonicity (SFM)
For all $A, B \in \pi(X)$ such that $A \textcircled{\mathbf{I}} B = \emptyset$ and for all $x \in X \backslash (A \cup B)$, $\{x\} \cup A \succ \{x\} \cup B$ iff $A \succ B$.

Neutrality (N)
For all $A, B \in \pi(X)$, all one-to-one $f\colon A \to X$ and all one-to-one $g\colon B \to X$, if [for all $x \in A$ and all $y \in B$, (xPy iff $f(x)\, Pg(y)$) and (xIy iff $f(x)\, Ig(y)$)] then [$A \succeq B$ iff $f(A) \succeq g(B)$].

Union (U)
For all $a \in X$ and all $B, C \in \pi(X)$, $[\{a\} \succ B$ and $\{a\} \succ C]$ implies $\{a\} \succ B \cup C$.

Pattanaik and Peleg proved that if X contains at least 4 indifference levels and $\succeq$ is an ordering over $\pi(X)$, then $\succeq\, = \,\succeq_*$ iff $\succeq$ satisfies these five axioms. They also showed the independence of each of the first four axioms. In the special case where R is linear, they showed that U is a consequence of the other axioms.

We shall show here that under the assumptions of Pattanaik and Peleg, $\succeq\, = \,\succeq_*$ follows already from GP, E, SFM and N. Hence, these four axioms form an independent axiomatic system that characterizes the lexicographic maximin extension.

The proof given by Pattanaik and Peleg that $\succeq\, = \,\succeq_*$ follows from the five axioms consists of the following steps. Let for $A \in \pi(X)$ $\min(A) = \{a \in A : A\mathbf{R}a\}$, $\max(A) = \{a \in A : a\mathbf{R}A\}$. First they show that for all $A, B \in \pi(X)$, $A \sim_* B$ implies $A \sim B$. Then they prove

\# For all $A, B \in \pi(X)$, $\min(A)\mathbf{P}\min(B)$ implies $A \succ B$.

Finally, they use # to show, by means of an inductive process, that for all $A, B \in \pi(X)$, $A \succ_* B$ implies $A \succ B$.

In this scheme, U is only used in proving #. Thus, our task is reduced to proving # from GP, E, SFM and N.

We start by noting that it suffices to prove # for the case $A = \{\tilde{x}\}$. Indeed, if $|A| > 1$, choose $\tilde{x} \in \min(A)$. By repeated application of GP and transitivity, $A \succeq \{\tilde{x}\}$, and by # for the special case $\{\tilde{x}\} \succ B$, hence $A \succ B$.

Lemma 1: *Let $C \in \pi(X)$, let $y \in \min(C)$ and let $z \in \max(C)$. If there exists $w \in C \backslash \{y\}$ such that zPw then $\{y, z\} \succ C$.*

Proof: By repeated application of GP and transitivity, $\{z\} \succ C \backslash \{y\}$. Hence, by E $\{y, z\} \succ C$ which proves the lemma.

In view of Lemma 1, it suffices to prove # for the case when not only $|A| = 1$ but also $B = \max(B) \cup \{\tilde{y}\}$ for some $\tilde{y} \in X$. Indeed, suppose that this special case of # is true (denote it by ##). We have to prove that for arbitrary $B \in \pi(X)$, $\tilde{x}\mathbf{P}\min(B)$ implies $\{\tilde{x}\} \succ B$. To do so, choose $\tilde{y} \in \min(B)$. If $B \neq \max(B) \cup \{\tilde{y}\}$, choose $\tilde{z} \in \max(B)$

and $\tilde{w} \in B \backslash (\max(B) \cup \{\tilde{y}\})$, and let $\tilde{B} = \{\tilde{y}, \tilde{z}\}$. The assumptions of Lemma 1 hold for $B, \tilde{y}$ and $\tilde{z}$ (with $w = \tilde{w}$), so $\tilde{B} \succ B$. Now, $\tilde{B} = \max(\tilde{B}) \cup \{\tilde{y}\}$ and $\min(\tilde{B})\,\mathbf{I}\,\min(B)$, so by # # it follows that $\{\tilde{x}\} \succ \tilde{B}$. By transitivity we obtain $\{\tilde{x}\} \succ B$ as required.

Moreover, we may assume $\max(B)\,\mathbf{P}\,\tilde{x}\,P\,\tilde{y}$. Indeed, we assume $\tilde{x}\,\mathbf{P}\,\min(B)$ and therefore $\tilde{x}\,P\,\tilde{y}$. If it is not the case that $\max(B)\,\mathbf{P}\,\tilde{x}$ then $\tilde{x}\,\mathbf{R}\,\max(B)$ since $\max(B)$ is contained in one indifference level; hence by repeated application of GP and transitivity $\{\tilde{x}\} \succ B$.

Lemma 2: *Let $D \in \pi(X)$ and let $x, y \in X$ such that $D\,\mathbf{P}\,x\,P\,y$. If there exists $z \in X \backslash D$ such that $z\,\mathbf{R}\,D$ but not $D\,\mathbf{R}\,z$ then $\{x\} \succ D \cup \{y\}$.*

Proof: Let z be as assumed. By GP and transitivity, $\{x\} \succ \{y\}$, hence by SFM $\{x, z\} \succ \{y, z\}$.

Let $C = D \cup \{y, z\}$. By Lemma 1 $\{y, z\} \succ C$, so we obtain $\{x, z\} \succ C$ by transitivity. By SFM it follows that $\{x\} \succ D \cup \{y\}$, proving the lemma.

Now, if $\max(B) \cap \max(X) = \emptyset$, we can apply Lemma 2 to $D = \max(B)$, $x = \tilde{x}$ and $y = \tilde{y}$ (with any $z \in \max(X)$) to obtain $\{\tilde{x}\} \succ \max(B) \cup \{\tilde{y}\} = B$.

It remains to deal with the case $\max(B) \subset \max(X)$. Choose $z \in \max(B)$. As there are at least 3 indifference levels below $\max(X)$, there exist $f\colon \{\tilde{x}\} \to X$ and $g\colon B \to X$ such that $g \restriction \max(B) \backslash \{z\}$ is the identity mapping and $z\,P\,g(z)\,P\,f(\tilde{x})\,P\,g(\tilde{y})$. Applying Lemma 2 to $D = g(\max(B))$, $x = f(\tilde{x})$ and $y = g(\tilde{y})$ yields $\{f(\tilde{x})\} \succ g(\max(B)) \cup \{g(\tilde{y})\}$, or equivalently $f(\{\tilde{x}\}) \succ g(B)$. Hence by N $\{\tilde{x}\} \succ B$.

Reference

1. Pattanaik PK, Peleg B (1984) An axiomatic characterization of the lexicographic maximin extension of an ordering over a set to the power set. Soc Choice Welfare 1:113–122