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mathematical social sciences
Mathematical Social Sciences 46 (2003) 193–205
www.elsevier.com/locate/econbase

Network structure and strong equilibrium in route selection games

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Abstract

In a route selection game on a network, every player chooses a route from the origin to the destination, which are common to all players. Costs are assigned to road segments in the form of monotone nondecreasing functions of the number of players who use them. Each player incurs a total cost equal to the sum of the costs of the road segments in his route. It is known that such a game always has a Nash equilibrium in pure strategies. Here we obtain a structural characterization of those networks for which a strong equilibrium is guaranteed to exist regardless of the cost assignment. The route selection games based on networks in this class enjoy more stability as well as other desirable properties of equilibrium regarding uniqueness and efficiency.

Keywords: Network; Route selection game; Nash equilibrium; Strong equilibrium; Efficiency

JEL classification: C72; R41

1. Introduction

Networks are used in a large variety of areas, including transportation, communication, computation, etc.[2] The design of good networks is an important issue in all of these applications. One aspect of this design problem is concerned with the fact that the users of the network will typically act in a decentralized manner. In this context, the

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[2]Beckmann et al. (1956), Sheffi (1985), and Bell and Iida (1997) are a few of the texts devoted to this subject.

0165-4896/03/$ – see front matter
doi:10.1016/S0165-4896(03)00076-3

designer's goal is to ensure that the strategic interaction between the users will lead to stable and socially desirable outcomes.

Here we consider the following model of a two-terminal network and a game in strategic form associated with it. The network is described by these ingredients:

- a finite set of vertices V;
- a finite set of arcs A, where each arc $a \in A$ has a tail $t(a) \in V$ and a head $h(a) \in V$; we think of a as a one-way road segment from $t(a)$ to $h(a)$;
- two distinct vertices o (the origin) and d (the destination) in V.

We call $D = (V, A, o, d)$ a *network*. A *route* in D is a sequence of the form

$$v_0, a_1, v_1, a_2, v_2, \ldots, v_{l-1}, a_l, v_l$$

where $v_0, v_1, \ldots, v_l$ are distinct vertices, $v_0 = o$ and $v_l = d$, and $a_1, \ldots, a_l$ are arcs satisfying $t(a_i) = v_{i-1}$ and $h(a_i) = v_i$ for $i = 1, \ldots, l$. In order to eliminate redundancies, we always assume:

- every vertex in V and every arc in A belongs to at least one route in D.

A game associated with the network D is described by the following ingredients:

- a finite set of players (users) N; the number of players is n and we usually let $N = \{1, \ldots, n\}$;
- an assignment of costs to arcs depending on the number of players who use them; denoting by $c_a(k)$ the cost to each user of arc a if the total number of users of a is k, we assume that the array of numbers $C = (c_a(k))_{a \in A, 1 \leq k \leq n}$ satisfies

$$0 \leq c_a(1) \leq c_a(2) \leq \cdots \leq c_a(n) \quad \text{for all } a \in A. \tag{1}$$

A natural interpretation of $c_a(k)$ in the context of a traffic network is that it represents the travel time for road segment a when it is chosen by k users. We consider the game in which every player has to get from o to d. Thus, the strategy space of each player is the set of routes in D. With every n-tuple of strategies (routes) $r_1, \ldots, r_n$ we associate a congestion vector $(\sigma_a(r_1, \ldots, r_n))_{a \in A}$ where $\sigma_a(r_1, \ldots, r_n)$ is the number of users i such that a belongs to r_i. The disutility function of player i is defined by

$$\pi_i(r_1, \ldots, r_n) = \sum_{j=1}^{l} c_{a_j}(\sigma_{a_j}(r, \ldots, r_n))$$

where $r_1, \ldots, r_n$ is the n-tuple of routes chosen by the players and $a_1, \ldots, a_l$ are the arcs on r_i. This defines a game G that we call a *route selection game* based on the network D, with player set N and cost assignment C. We write $G = (D, N, C)$.[3]

It follows from a more general theorem of Rosenthal (1973a) on congestion games (see Section 2) that every route selection game has a Nash equilibrium in pure strategies. That is, there always exists an n-tuple of strategies $r_1, \ldots, r_n$ such that for every player i and every route r_i' we have

$$\pi_i(r_i', r_{-i}) \geq \pi_i(r_1, \ldots, r_n).$$

(Here r_i', r_{-i} is the n-tuple obtained from $r_1, \ldots, r_n$ by changing r_i to r_i'.) This is a pleasing result, but observe that the concept of Nash equilibrium ensures stability of the outcome only with respect to unilateral deviations by players; the outcome may be unstable with respect to joint deviations by coalitions of players. In fact it may even be possible for all players to benefit by a joint deviation from a Nash equilibrium. We say that a Nash equilibrium $r_1, \ldots, r_n$ is *strictly inefficient* if there exists an n-tuple of strategies $r_1', \ldots, r_n'$ such that

$$\pi_i(r_1', \ldots, r_n') < \pi_i(r_1, \ldots, r_n) \quad \text{for all } i \in N. \tag{2}$$

The possible inefficiency of Nash equilibrium in a route selection game is an effect of congestion externalities. A simple example where this occurs is given in Fig. 1. In this example, the unique Nash equilibrium is for both players to select the route formed by x, z. Indeed, this is a dominant strategy. When this equilibrium is played, each player pays 6. However, if one player uses x, w and the other uses y, z, then each player pays 5. Thus, the unique Nash equilibrium is strictly inefficient.

An n-tuple of strategies $r_1, \ldots, r_n$ is called a *strong equilibrium* (Aumann, 1959) if there does not exist a nonempty coalition of players S (subset of N) that has a choice of strategies $(r_i')_{i \in S}$ such that

$$\pi_i(r_S', r_{-S}) < \pi_i(r_1, \ldots, r_n) \quad \text{for all } i \in S.$$

(Here r_S', r_{-S} is the n-tuple obtained from $r_1, \ldots, r_n$ by changing r_i to r_i' for each $i \in S$.) Thus, a strong equilibrium is stable with respect to deviations by any coalition. In particular, it is a Nash equilibrium and it is *weakly efficient*, in the sense that it is not strictly inefficient. The existence of a strong equilibrium is a very desirable, yet quite rare phenomenon in general.

Our purpose in this paper is to characterize those networks for which the route selection game always admits a strong equilibrium, regardless of the cost assignment. Thus we identify the network structure that inherently guarantees the stability and efficiency of the outcome of the strategic interaction among the users.

Formally, we call a network D *strong* if every route selection game $G = (D, N, C)$

[3] The model presented here is the one introduced by Rosenthal (1973b), specialized to the case in which all users have the same origin and destination. In most of the other models in the literature, the usage of the network is represented by continuous variables, either by assuming a continuum of users or by allowing users to split their demand among the various routes. See Rosenthal (1973b) for a discussion of some drawbacks of the continuous approach.

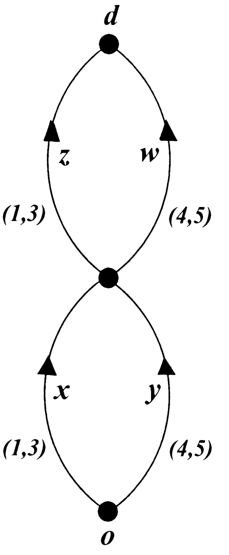

Fig. 1. The inefficiency of Nash equilibrium (in this example $N = \{1, 2\}$ and the first (second) coordinate of the vector assigned to an arc represents its cost if used by one (resp. two) players).

based on D, with an arbitrary number of users and any assignment of costs to arcs satisfying (1), has at least one strong equilibrium. We give two characterizations of strong networks, one in terms of a forbidden substructure, the other in terms of a recursive construction that produces all strong networks.

For the construction, we need to introduce three simple operations on networks. The *origin extension* of a network D is obtained by adding a new arc to D, whose tail is the new origin and whose head is the old origin. Similarly, the *destination extension* of a network D is obtained by adding a new arc to D, whose tail is the old destination and whose head is the new destination. The *parallel join* of two networks D_1 and D_2 is formed by identifying their respective origins and destinations and taking their otherwise disjoint union.

We call a network D *extension-parallel*[4] if there exists a sequence of networks D_1, $D_2, \ldots, D_m$ with $D_m = D$, in which every D_i is either a single-arc network, or the origin or destination extension of some D_j with $j < i$, or the parallel join of some D_j and D_k with $j, k < i$.

We are now ready to state our main result.

[4]This term suggests a comparison with the more standard class of networks called series-parallel. They are defined by a similar recursive construction, in which networks may be joined in series as well as in parallel. Thus, the class of extension-parallel networks is more restrictive, as it allows to join two networks in series only in the special case when one of them is a single arc. For example, the network of Fig. 1 is series-parallel but not extension-parallel. See Milchtaich (2001) for a result (in a somewhat different model) relating series-parallel networks and the so-called Braess's paradox.

Theorem 1. *Let D be a network. The following conditions are equivalent*:

1. *D is strong*.
2. *There do not exist two arcs a and b in D, such that some route in D includes a but not b, another route in D includes b but not a, and yet another route in D includes both a and b*.
3. *D is extension-parallel*.

To illustrate the forbidden substructure in part (2) of the theorem,[5] note that in the network of Fig. 1 arcs x and z admit routes including each one of them without the other, as well as a route including both of them. On the other hand, it is easy to check that the network in Fig. 2, for instance, is extension-parallel. In general, the recognition problem for strong networks is rendered easy by Theorem 1. Given a network D, we can either find two arcs a and b and three corresponding routes witnessing that D is not strong, or we can exhibit a construction sequence certifying that D is strong.

It is interesting to note that the characterization given in Theorem 1 is robust with respect to several possible variations in the requirements from a strong network. Suppose that D is a network satisfying the conditions of Theorem 1, and let $G = (D, N, C)$ be any route selection game based on D. Then it is true not only that G has a strong equilibrium, but also that every Nash equilibrium of G is a strong equilibrium. Another fact is that G has a strong equilibrium $r_1, \ldots, r_n$ which is not just weakly efficient but *strictly efficient*; this means that there does not even exist an n-tuple of strategies $r'_1, \ldots, r'_n$ for which the inequalities in (2) hold as weak inequalities, and at least one of them

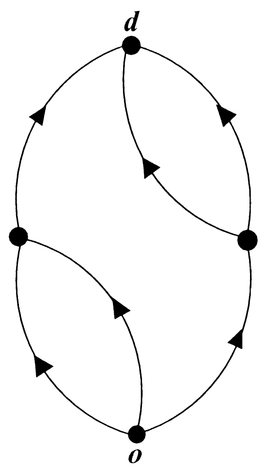

Fig. 2. A strong network.

[5]The forbidden substructure is what we called a bad configuration in Holzman and Law-Yone (1997). The relation between that paper and this one is discussed below.

holds strictly. Yet another remarkable fact is that for a generic[6] route selection game based on D, Nash equilibrium is unique up to a permutation of the players. Finally, the existence of a strong equilibrium is preserved in a variant of the definition of a route selection game based on D, which allows each player's strategy space to be a possibly different subset of the set of all routes.

In the converse direction, suppose that D is a network violating the conditions of Theorem 1. Then not only does there exist a route selection game $G=(D, N, C)$ based on D which has no strong equilibrium, but such a game can be found for any given size $n \geq 2$ of the player set. Moreover, the game can be constructed so that no Nash equilibrium of it is even weakly efficient.

The equivalence of (1) and (2) in Theorem 1, and the validity of the additional facts mentioned in the last two paragraphs, essentially follow by specializing to the network set-up the results that we obtained in earlier work (Holzman and Law-Yone, 1997) on the more general class of congestion games. We recall the necessary concepts and results in Section 2. Some extra work is needed though, because in the earlier paper the costs were not required to be nonnegative. This is taken care of in Appendix A. The main contribution of this paper is the constructive characterization given by condition (3) in Theorem 1. We prove that conditions (2) and (3) are equivalent in Section 3.

To conclude the Introduction, we point out the similarities and differences between our work and that of Milchtaich (2001).[7] Like us, Milchtaich gives characterizations of two-terminal networks[8] for which the equilibrium outcomes of the corresponding route selection games have desirable properties, regardless of the cost structure. He focuses on the efficiency of equilibrium, but also has a result implying strong equilibrium. The main difference between his model and ours is that he considers games with a continuum of players, each having a negligible effect on the congestion. He is able to extend some of his results to the case in which costs are allowed to vary across users, whereas the analogous extension would not hold true in our discrete model.[9]

[6]A property $\mathscr{P}$ is said to hold for a generic route selection game based on D, if for every player set N, the route selection game $G=(D, N, C)$ has property $\mathscr{P}$ for all cost assignments C with the possible exception of some Cs lying in the union of finitely many hyperplanes in the space of cost assignments.

[7]While our results were obtained before Milchtaich's, the presentation in this paper benefited from reading his preprint.

[8]Milchtaich considers undirected networks, as opposed to our directed networks. However, all his positive results are for series-parallel networks, and in such networks an edge can be traversed in only one direction. Milchtaich's model also differs from ours in allowing more flexibility in the way route costs are determined.

[9]The following simple example illustrates the difference. The network consists of two parallel arcs from the origin to the destination. There are two players, and they both prefer being the single user of any arc to sharing an arc with the other player. The two players have opposite preferences over the two arcs. In addition to the efficient equilibrium in which each player uses his preferred arc, there is another Nash equilibrium in which each player uses the arc he likes less. Thus, in the discrete model the property that every Nash equilibrium is weakly efficient (when the network is extension-parallel) is not preserved when the costs are allowed to vary across users. In the continuum model this property does extend to the case of heterogeneous costs. See also Milchtaich (2000) and Konishi (2002) for a sufficient condition for the uniqueness of each user's equilibrium cost in a continuum model with heterogeneous costs.

2. Congestion games

In this section we present the necessary facts about congestion games, a class of games that contains the route selection games described in the Introduction.

A congestion game is specified by the following ingredients:

- a finite nonempty set of facilities M;
- a finite set of players N, enumerated as $N = \{1, \dots, n\}$;
- for each player i, a nonempty strategy space Σ_i; every strategy $S_i \in \Sigma_i$ is a subset of M;
- an assignment of costs to facilities depending on the number of players who use them, given in the form $C = (c_a(k))_{a \in M, 1 \le k \le n}$.

With every n-tuple of strategies $S_1, \dots, S_n$ we associate a congestion vector $(\sigma_a(S_1, \dots, S_n))_{a \in A}$, where $\sigma_a(S_1, \dots, S_n) = |\{i \in N\colon\ a \in S_i\}|$. The disutility function of player i is defined by

$$\pi_i(S_1, \dots, S_n) = \sum_{a \in S_i} c_a(\sigma_a(S_1, \dots, S_n)).$$

This defines the *congestion game* $G = (M, N, \Sigma_1, \dots, \Sigma_n, C)$. The cost assignment C is called *nonnegative* if $c_a(k) \ge 0$ for all $a \in M$ and every $1 \le k \le n$; it is *positive* if all these inequalities are strict. C is called *monotone* if

$$c_a(1) \le c_a(2) \le \cdots \le c_a(n) \quad \text{for all } a \in M.$$

Route selection games on networks are naturally embedded as congestion games, by considering the arcs as facilities, and the arc set of every route as a possible strategy of every player. Note that although the nonnegativity of costs is not required in the congestion game model, we do need to assume nonnegativity of arc costs in the network model in order for the embedding to make sense. If some of the arc costs were negative, restricting attention to routes (with distinct vertices and arcs) would not be justified.

Rosenthal (1973a) introduced congestion games, and proved that they always have Nash equilibria in pure strategies. In Rosenthal (1973b) he developed the application to route selection games.

The existence of strong equilibrium in congestion games[10] was studied in Holzman and Law-Yone (1997). A preliminary observation made there was that in order to obtain any positive results it is necessary to restrict attention to monotone cost assignments. The key observation in that paper was that the existence of strong equilibrium crucially depends on the absence of what we termed bad configurations in the strategy spaces. We proceed now to recall this concept and the pertinent results.

[10] The model was presented in Holzman and Law-Yone (1997) in terms of utilities $u_a(k)$ instead of costs $c_a(k)$. The two approaches are equivalent by setting $u_a(k) = -c_a(k)$.

Let Σ be a strategy space on the facility set M (that is, Σ is a nonempty family of subsets of M). A *bad configuration* in Σ is a tuple

$$(x, y; X, Y, Z)$$

where

$$x, y \in M,$$
$$X, Y, Z \in \Sigma,$$

and the following relations hold:

$$X \cap \{x, y\} = \{x\},$$
$$Y \cap \{x, y\} = \{y\},$$
$$Z \cap \{x, y\} = \{x, y\}.$$

The following theorem was proved in Holzman and Law-Yone (1997).

Theorem 2. *Let $G = (M, N, \Sigma, \ldots, \Sigma, C)$ be a congestion game with a common strategy space Σ to all players and a monotone cost assignment C. If Σ contains no bad configuration then every Nash equilibrium of G is a strong equilibrium.*

It follows immediately from this theorem (and the existence of Nash equilibrium) that condition (2) implies condition (1) in Theorem 1.[11] The converse implication follows from:

Theorem 3. *Let M be a set of facilities, and let Σ be a strategy space on M such that no set in Σ contains another. If Σ contains a bad configuration then for every integer $n \geqslant 2$ there exists a positive and monotone cost assignment C such that the congestion game $G = (M, N, \Sigma, \ldots, \Sigma, C)$ with $|N| = n$ has no weakly efficient Nash equilibrium.*

In Holzman and Law-Yone (1997) we proved a version of Theorem 3 with a weaker assumption (no condition on containments among sets in Σ) and a weaker conclusion (no positivity requirement on the cost assignment). The proof of the current version is much more involved, and is given in Appendix A. Note that in the context of positive cost assignments the condition that no set in Σ contains another entails no loss of generality, because if S contains T then the strategy S is dominated by the strategy T. In the network application the condition of no containments among strategies (routes) is automatically satisfied.[12]

[11]The additional statements made in the Introduction about properties of equilibria of route selection games for networks satisfying the conditions of Theorem 1 follow similarly from corresponding results in Holzman and Law-Yone (1997); see Theorems 6.2, 6.1 and Corollary 5.3 there.

[12]We note that there are some further properties of the strategy space that always hold in the network application. These can be used to simplify the proof of Theorem 3 in the special case of networks. We prefer, however, to state and prove the theorem for the general congestion game set-up, which admits many other applications.

3. Proof of Theorem 1

As noted above, the equivalence between conditions (1) and (2) in Theorem 1 follows from Theorems 2 and 3. Here we show that conditions (2) and (3) are equivalent, thereby completing the proof of Theorem 1.

Suppose first that the network D satisfies condition (3), with the construction sequence $D_1, D_2, \ldots, D_m = D$. Then every D_i in this sequence, and in particular D, satisfies condition (2). This can be shown by induction on i, based on the easily checked facts that a single-arc network satisfies condition (2) and the condition is preserved under the operations of origin/destination extension and parallel join.

Conversely, suppose that the network D satisfies condition (2). We prove that D is extension-parallel by induction on the number of arcs in D. It suffices to show that if D has more than one arc then it is either the origin or destination extension of some network, or the parallel join of some two networks. Once we have shown this, the network (or networks) from which D is obtained by one of these operations must also satisfy condition (2), and we can apply the induction hypothesis to them to conclude the proof.

So, we consider a network D with more than one arc and analyze its structure going through several possible cases; in each case we obtain either a decomposition of D corresponding to one of the three operations or a violation of condition (2).

If there is only one arc going out of the origin of D, then D is the result of an origin extension. (Note that there can be no arcs going into the origin, since by assumption every arc belongs to a route.) Similarly, if there is only one arc going into the destination of D, then D is the result of a destination extension.

We assume henceforth that there are at least two arcs going out of the origin o and at least two arcs going into the destination d. Let

A_o = the set of arcs going out of o,
A_d = the set of arcs going into d.

We choose arcs $x \in A_o$ and $y \in A_d$ that belong together to some route in D. We consider the following sets of arcs:

X = the arcs in A_d that belong to some route starting with x
Y = the arcs in A_o that belong to some route ending with y.

If all the routes ending in X start with x, then the vertices and arcs of these routes form one of two networks of which D is the parallel join. (We omit the detailed verification of this fact.) A similar decomposition exists if all the routes starting in Y end with y.

Thus, we assume that there exists a route that ends with some arc z in X but does not start with x, and there exists a route that starts with some arc w in Y but does not end with y. If $w = x$ and $z = y$ then x and y violate condition (2). (Recall that x and y were chosen so that some route contains both of them.) If $w \neq x$ then w and y violate condition (2), and if $z \neq y$ then x and z do so.

Acknowledgements

We are grateful to the Editor in charge and to a referee for their helpful comments. Research supported by the Fund for the Promotion of Research at the Technion.

Appendix A. Proof of Theorem 3

Let M be a set of facilities, let Σ be a strategy space on M with no containments among its members and with a bad configuration, and let $n \geq 2$ be a given integer. We have to construct a positive and monotone cost assignment $C = (c_a(k))_{a \in M,\ 1 \leq k \leq n}$ so that in the n-player congestion game $G = (M, N, \Sigma, \ldots, \Sigma, C)$ every Nash equilibrium will be strictly inefficient. For simplicity of presentation, we will assign zero costs in some cases; a slight perturbation making them positive will not affect the argument.

First we introduce some terminology pertaining to a bad configuration $(x, y; X, Y, Z)$. The set $X \cup Y \cup Z$ is called the *domain* of $(x, y; X, Y, Z)$. We call $(x, y; X, Y, Z)$ *separated* if the sets $X \backslash (Y \cup Z)$ and $Y \backslash (X \cup Z)$ are both nonempty. We call it *weakly separated* if at least one of these two sets is nonempty.

Among the bad configurations in Σ (there may be more than one) we choose one $(x, y; X, Y, Z)$ according to the following rules. Firstly, we restrict attention to bad configurations with the smallest possible domain. Among these, we choose a separated one if such exists. Failing that, we choose a weakly separated one if such exists, and otherwise we choose an arbitrary one.

We will explicitly assign costs only to facilities in the domain $X \cup Y \cup Z$ of the chosen bad configuration. The costs of the other facilities are to be set sufficiently high so that all strategies that are not contained in the domain become dominated. This allows us, when looking for Nash equilibria, to restrict attention to strategies contained in the domain, i.e. to the set of strategies

$$\Sigma^* = \{S \in \Sigma: S \subseteq X \cup Y \cup Z\}.$$

For a subset $I \subseteq \{x, y\}$ let

$$\Sigma_I = \{S \in \Sigma^*: S \cap \{x, y\} = I\}.$$

Thus, the set of strategies Σ^* is partitioned into four subsets: $\Sigma_\emptyset$, Σ_x, Σ_y, Σ_{xy}. (We omit brackets and commas.)

We distinguish two possible cases.

Case 1. $(x, y; X, Y, Z)$ is separated.

In this case, we choose elements $\hat{x} \in X \backslash (Y \cup Z)$ and $\hat{y} \in Y \backslash (X \cup Z)$. We establish a few facts about strategies in Σ^*.

Claim 1. (a) Every S in Σ_x contains $\hat{x}$.

(b) Every S in Σ_y contains $\hat{y}$.
(c) Every S in $\Sigma_\emptyset$ contains both $\hat{x}$ and $\hat{y}$.

Proof. (a) Suppose $S \in \Sigma_x$ and $\hat{x} \notin S$. Then $(x, y; S, Y, Z)$ is a bad configuration with a smaller domain than $(x, y; X, Y, Z)$, contradicting our choice of $(x, y; X, Y, Z)$.

(b) Similar to (a).

(c) Let $S \in \Sigma_\emptyset$. As there are no containments among strategies in Σ, we can choose an element $s \in S \backslash Z$. Because S is contained in $X \cup Y \cup Z$, we have $s \in X \cup Y$, and we may assume without loss of generality that $s \in X$. Then we must have $\hat{y} \in S$, for otherwise $(x, s; Z, S, X)$ would be a bad configuration with a smaller domain than $(x, y; X, Y, Z)$. Now it follows also that $\hat{x} \in S$, for otherwise $(y, \hat{y}; Z, S, Y)$ would be a bad configuration with a smaller domain than $(x, y; X, Y, Z)$.

Now, consider the cost assignment given in Table A.1. Based on Claim 1, we check that every strategy in $\Sigma^* \backslash \Sigma_{xy}$ is dominated by the strategy Z. Indeed, a player who switches from some strategy S in $\Sigma_x \cup \Sigma_y$ to Z increases his cost on $\{x, y\}$ by at most 2 but saves at least 3 on $\{\hat{x}, \hat{y}\}$. Similarly, a player who switches from a strategy S in $\Sigma_\emptyset$ to Z reduces his cost from 6 to at most 4.

It follows that in every Nash equilibrium all players will use strategies in Σ_{xy}, incurring costs of at least 4 each. But this is strictly inefficient, because if some players switch to X and the other players switch to Y they will only pay 3 each.

Case 2. $(x, y; X, Y, Z)$ is not separated.

In this case, the following is true.

Claim 2. There is no strategy in $\Sigma_\emptyset$.

Proof. By the assumption of Case 2, one of the sets $X \backslash Z$, $Y \backslash Z$ contains the other. We assume without loss of generality that

$$X \backslash Z \subseteq Y \backslash Z. \tag{A.1}$$

Suppose, for the sake of contradiction, that the strategy S is in $\Sigma_\emptyset$. We choose an element $s \in S \backslash Z$. Because S is contained in $X \cup Y \cup Z$, we have $s \in (X \cup Y) \backslash Z$, and therefore by (A.1) necessarily $s \in Y \backslash Z$. Now, $(y, s; Z, S, Y)$ is a bad configuration with domain contained in (and hence equal to) $X \cup Y \cup Z$. Moreover, this bad configuration is weakly separated, as witnessed by the element x. It follows then, by our rules for

Table A.1
Cost assignment in Case 1 (the cost of every other facility in $X \cup Y \cup Z$ is zero, and the cost of every facility outside $X \cup Y \cup Z$ is prohibitively high)

	x	y	$\hat{x}$	$\hat{y}$
Fewer than n users	0	0	3	3
n users	2	2	3	3

choosing $(x, y; X, Y, Z)$, that it, too, is weakly separated. Thus the containment in (A.1) is strict.

Now, if $S\backslash Z \subseteq X$ then $(x, s; Z, S, X)$ is a bad configuration with domain $X \cup Z$, which is smaller than $X \cup Y \cup Z$ by strict (A.1). It follows that $S\backslash Z \not\subseteq X$, and we assume without loss of generality that $s \notin X$ (otherwise we replace s by another element of $S\backslash Z$).

Next, we choose an element $t \in S\backslash Y$. If $t \in X$, then $(t, s; X, Y, S)$ is a bad configuration with domain contained in (and hence equal to) $X \cup Y \cup Z$. Moreover, this bad configuration is separated, as witnessed by the elements x and y. By our rules for choosing $(x, y; X, Y, Z)$, this contradicts the assumption of Case 2. Thus, we assume that $t \notin X$.

We further choose an element $u \in X\backslash Z$. Now $(x, u; Z, Y, X)$ is a bad configuration (since $u \in Y\backslash Z$ by (A.1)) with domain $X \cup Y \cup Z$. This bad configuration is separated, as witnessed by the elements t and s. Again, this contradicts the assumption of Case 2, and completes the proof of Claim 2.

We now choose strategies $\tilde{X}$ and $\tilde{Y}$ so that

$$\tilde{X} \in \arg\min_{S \in \Sigma_x} |S\backslash Z| \quad \text{and} \quad \tilde{Y} \in \arg\min_{S \in \Sigma_y} |S\backslash Z|$$

We observe that $(x, y; \tilde{X}, \tilde{Y}, Z)$ is a bad configuration with domain contained in (and hence equal to) $X \cup Y \cup Z$. By the assumption of Case 2, it cannot be separated. We assume without loss of generality that $\tilde{X}\backslash Z \subseteq \tilde{Y}\backslash Z$. We have

$$|\tilde{X}\backslash Z| = p \quad \text{and} \quad |\tilde{Y}\backslash Z| = q$$

for some positive integers p and q with $p \leq q$. Let

$$\tilde{Y}\backslash Z = \{w_1, w_2, \ldots, w_q\}.$$

Note that $Z \cup \{w_1, \ldots, w_q\}$ is the entire domain, every strategy in Σ_x contains at least p of the w_i's, and every strategy in Σ_y contains all q of them.

Now, consider the cost assignment given in Table A.2. We check that every strategy in $\Sigma^*\backslash\Sigma_{xy}$ is dominated by the strategy Z. Indeed, a player who switches from some strategy S in Σ_x to Z incurs a cost of at most $3p - 1$ for y but saves at least $3p$ on $\{w_1, \ldots, w_q\}$. Similarly, a player who switches from a strategy S in Σ_y to Z incurs a cost of at most $3q - 1$ for x but saves $3q$ on $\{w_1, \ldots, w_q\}$. By Claim 2, this is all we need to check.

It follows that in every Nash equilibrium all players will use strategies in Σ_{xy} and pay at least $3(p + q) - 2$ each. But this is strictly inefficient, because if some players switch

Table A.2
Cost assignment in Case 2 (the cost of every facility in $Z\backslash\{x, y\}$ is zero, and the cost of every facility outside $Z \cup \{w_1, \ldots, w_q\}$ is prohibitively high)

	x	y	w_1	w_2	$\cdots$	w_q
Fewer than n users	0	0	3	3	$\cdots$	3
n users	$3q - 1$	$3p - 1$	3	3	$\cdots$	3

to $\tilde{X}$ and the other players switch to $\tilde{Y}$, the former will pay $3p$ each and the latter $3q$ each.

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