



Aggregation of binary evaluations with abstentions [☆]

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Abstract

A general model of aggregation of binary evaluations over interrelated issues, introduced by Wilson and further studied by Rubinstein and Fishburn and by the authors, is extended here to allow for abstentions on some of the issues. It is shown that the same structural conditions on the set of feasible evaluations that lead to dictatorship in the model without abstentions, lead to oligarchy in the presence of abstentions. Arrow's impossibility theorem for social welfare functions, Gibbard's oligarchy theorem for quasi-transitive social decision functions, as well as some apparently new theorems on preference aggregation, are obtained as corollaries.

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[☆] The first version of this paper was circulated in December 2006. Results that are essentially the same as the main result of this paper, but are phrased in a model of logical judgment aggregation, were independently obtained by Dietrich and List (2008) [10]. The overlap between the two works was discovered just before the Cowles Workshop on "Aggregation of Opinions," September 2006, where both papers were presented. The comments of an associate editor and two referees are gratefully acknowledged.

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1. Introduction

Many aggregation problems may be described using the following framework. A society has to determine its positions on each of several issues. There are two possible positions (say, 0 or 1) on each issue, but the issues are interrelated and therefore not all combinations of 0–1 positions are feasible. Some set X of 0–1 vectors (of length equal to the number of issues) is given, representing the feasible combinations of positions. The problem is how to aggregate profiles of individual evaluations, each of which belongs to the set X , into a social evaluation that is itself in X .

This general framework for aggregation theory was first proposed by Wilson [25], and was further studied by Rubinstein and Fishburn [12,22]. They showed that this framework incorporates, as special cases, the aggregation of preferences under various rationality constraints, the aggregation of equivalence relations, and more. In an earlier work [11] we investigated this framework, and pointed out that the problem of judgment aggregation may also be cast in these terms. The latter problem has received much attention recently; see, e.g., List and Pettit [15], Pauly and van Hees [21], Dietrich [5–7], Gärdenfors [13], Nehring and Puppe [20], Dietrich and List [8–10], Mongin [16].

In this paper we enrich the framework by allowing abstentions on some of the issues. An evaluation that allows for abstentions is captured by a 0–1–* vector, where a * entry represents abstention on the corresponding issue. Similar to the original framework, not all 0–1–* vectors are allowed. Our primitive notion is still a given set X of feasible 0–1 vectors. From it, we derive the set X^* of feasible 0–1–* vectors by the following rule. A 0–1–* vector is feasible (i.e., in X^*) if there is a way to replace the * entries with 0 or 1 entries so as to obtain a vector in X , and moreover, none of the replacements has its value (0 or 1) uniquely determined by the constraint that the resulting 0–1 vector be in X . Thus, a feasible 0–1–* vector can be interpreted as a partial determination of a feasible 0–1 vector, so that none of the undetermined values may be logically deduced from the determined ones, given the set X . The latter provision assures that an abstention represents a genuine non-determination of the position on the corresponding issue.

We consider two variants of the aggregation problem. In both variants, the society is allowed to abstain, and thus the range of the aggregator is X^* . In one variant, the individuals are also allowed to abstain, and thus the domain of the aggregator is the set of all profiles of vectors in X^* . In the other variant, the individuals are not allowed to abstain, and thus the domain of the aggregator is the set of all profiles of vectors in X . In both cases, we restrict attention to aggregators that satisfy the analogues of Arrow's [1] properties for social welfare functions. The independence of irrelevant alternatives property (abbreviated IIA) requires that the society's position on any given issue depend only on the individuals' positions on that same issue. The Pareto property requires that the society adopt any unanimously held 0 or 1 position.

Working in a framework without abstentions, Nehring and Puppe [19] gave a necessary and sufficient condition on X , so that every monotone IIA and Paretian aggregator must be dictatorial; namely, such X has to be totally blocked (we recall the definition in Section 2). In [11] we characterized the domains X for which every (not necessarily monotone) IIA and Paretian aggregator must be dictatorial; the conditions are that X be totally blocked and not be an affine subspace. The problem of aggregating linear preference orders over a set of alternatives may be cast in this framework, and the corresponding set X is easily seen to satisfy the two conditions when there are at least three alternatives. Hence, our result in [11] implies Arrow's impossibility theorem for the aggregation of linear preference orders. Other derivations of this theorem in sim-

ilar frameworks were given by Wilson [25], Rubinstein and Fishburn [22], Dietrich and List [9], and, subject to a monotonicity condition, Nehring [17].

In investigating the framework with abstentions, our first observation is that the role of the dictatorial rule is played here by a wider class of rules, called oligarchic rules. For a non-empty subset D of the society, an aggregator is said to be D -oligarchic if for any issue, the society adopts the position 0 (resp. 1) if and only if all members of D hold the position 0 (resp. 1) on that issue; in all other situations the society abstains on the issue. The case when D is of size one corresponds to dictatorship. All D -oligarchic rules are well defined (that is, they map into X^*) and of course satisfy the IIA and Pareto properties, regardless of the structure of X . Although oligarchic rules are not as unilateral as dictatorial rules, they are still a very restricted class of rules. As D gets larger, more individuals take part in the decision process, but the process becomes less decisive (in the sense that the society tends more to abstain). When D is the entire society, the decision process is anonymous, but is deadlocked on any issue on which there is no unanimity. Therefore, if for a given set X of feasible evaluations, every IIA and Paretian aggregator into X^* must be oligarchic, then this may be viewed as a negative (if not an impossibility) result for X in the framework with abstentions.

Our main result is a characterization of those sets X for which the latter holds true, i.e., every IIA and Paretian aggregator into X^* must be oligarchic. In fact, there are two results, one for the variant in which the individuals are allowed to abstain, and one for the variant in which they are not. For both variants, the necessary and sufficient conditions on X turn out to be the same conditions from our earlier paper, namely, X should be totally blocked and should not be an affine subspace. This shows a remarkable robustness of these structural conditions on the set of feasible evaluations. The same conditions that force dictatorship when abstentions are not allowed, also force oligarchy when the society is allowed to abstain, whether or not individuals are allowed to abstain. Our result shows that when X satisfies these conditions, relaxing the assumption of completeness of evaluations and allowing abstentions does not gain us much: it only admits oligarchies instead of just dictatorship. We note that the sufficiency part of our result here (for the second variant) implies the sufficiency part of our result in [11]. Indeed, if an aggregator is known to map into X and to be oligarchic, then it must be an oligarchy of one, namely a dictatorship.

Some earlier papers on judgment aggregation, e.g., List and Pettit [15], Dietrich and List [8], recognized the potential of relaxing the completeness assumption, in particular in enabling the use of quota rules. Gärdenfors [13] criticized the completeness assumption as being strong and unnatural, and proved that if social judgment sets need not be complete, but are logically closed and consistent, then every IIA and Paretian aggregator must be oligarchic. Our work was motivated by his work, and takes care of two weaknesses in it. First, he required that the set of propositions to be evaluated have a very rich logical structure, that of an atomless Boolean algebra (in particular, it must be infinite). By contrast, we work with finite sets of issues, and identify the weakest possible structural conditions on the set X that entail oligarchy. Second, his definition of an oligarchic rule required that the society adopt a 0 or 1 position if all members of D hold that position, but did not require the ‘only if’ part, too, as we do. He used the term ‘strict oligarchy’ for our stronger notion, but his results only established the weaker version. Results that are essentially the same as our main result, but are phrased in the logical framework of judgment aggregation, were independently obtained by Dietrich and List [10].

Our result has a number of corollaries for preference aggregation. Its most direct application is to the problem of aggregating individual partial orders or linear orders into a social partial order. The results that we obtain for this problem may be known, but we are not aware of a reference.

The closest resemblance is to a result of Gibbard [14], in which the domain of the aggregators is the set of all profiles of individual weak orders. With a little extra work, we show that Gibbard’s result is also a corollary of ours. As Arrow’s [1] impossibility theorem for the aggregation of weak orders follows from Gibbard’s theorem, we also obtain it as a corollary (whereas our earlier result in [11] implied the version of Arrow’s theorem for linear orders).² Finally, we give a complete dichotomy of dictatorship and oligarchy results for aggregation problems that are intermediate between Arrow’s and Gibbard’s, in that the class of allowed social preferences lies between the class of weak orders and the class of partial orders.

In Section 2 we give the necessary definitions and formulate the main result, which we prove in Section 3. We present the corollaries for preference aggregation in Section 4.

2. The model and the main result

We consider a finite, non-empty set of issues J . For convenience, if there are m issues in J , we identify J with the set $\{1, \dots, m\}$ of coordinates of vectors of length m . A vector $x = (x_1, \dots, x_m) \in \{0, 1\}^m$ is an *evaluation*. We shall also speak of partial evaluations: if K is a subset of J , a vector $x = (x_j)_{j \in K} \in \{0, 1\}^K$ with entries for issues in K only is a K -evaluation.

We assume that some non-empty subset X of $\{0, 1\}^m$ is given. The evaluations in X are called *feasible*, the others are infeasible. We shall also use this terminology for partial evaluations: a K -evaluation is feasible if it lies in the projection of X on the coordinates in K , and is infeasible otherwise.

We consider also vectors $x = (x_1, \dots, x_m) \in \{0, 1, *\}^m$, which we call *pseudo-evaluations*. We construct from $X \subseteq \{0, 1\}^m$ a set $X^* \subseteq \{0, 1, *\}^m$, defined as follows. First, given $x \in \{0, 1, *\}^m$, we say that a vector $y \in \{0, 1\}^m$ *agrees* with x , if $x_j = y_j$ for all j such that $x_j \in \{0, 1\}$. We let X^* be the set of all pseudo-evaluations $x \in \{0, 1, *\}^m$ that satisfy the following two conditions:

there exists some $y \in X$ that agrees with x ; (1)

for every j such that $x_j = *$, there exist $y, y' \in X$
that agree with x , so that $y_j = 0$ and $y'_j = 1$. (2)

Note that $X^* \cap \{0, 1\}^m = X$. The pseudo-evaluations in X^* are called *feasible*, the others are infeasible.

Example 1. Suppose that there are three issues, that are associated, respectively, with the logical propositions p , q , and $p \wedge q$. If 1 denotes ‘true’ and 0 denotes ‘false,’ the logically consistent evaluations are given by the set:

$$X = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}.$$

Taking this as the set of feasible evaluations, we obtain by the above definition that the set of feasible pseudo-evaluations is:

$$X^* = X \cup \{(*, 0, 0), (0, *, 0), (1, *, *), (*, 1, *), (*, *, 0), (*, *, *)\}.$$

For example, $(*, *, 0)$ is in X^* because each of the $*$ entries may be either 0 or 1 for a suitable evaluation in X with third entry 0. Intuitively, it is possible to hold that $p \wedge q$ is false, without

² Another derivation of Arrow’s theorem for weak orders in the judgment aggregation literature was given by Dietrich [7].

committing to the truth or falsehood of either p or q . Note that the fact that $(*, *, 0)$ is in X^* does not entail that all four 0–1 combinations in the $*$ entries are compatible with third entry 0. It is easy to check that all pseudo-evaluations not listed above are indeed not in X^* . For example, $(1, *, 0)$ is not in X^* because holding p to be true and $p \wedge q$ to be false does not leave any doubt as to q .

It will be useful to introduce a binary operation on pseudo-evaluations, that we refer to (for lack of a better name) as *multiplication*. For $x, x' \in \{0, 1, *\}^m$ we define their *product* xx' $\in \{0, 1, *\}^m$ coordinate-by-coordinate, using the rule:

$$(xx')_j = \begin{cases} x_j & \text{if } x_j = x'_j, \\ * & \text{otherwise.} \end{cases} \quad (3)$$

Note that this operation is commutative and associative, which allows us to consider products of more than two vectors while disregarding order and parentheses. We will use the symbol \prod for such products. It is easy to check that using this operation we get an equivalent definition of X^* as the closure of X under multiplication. That is:

$$X^* = \left\{ \prod_{i=1}^k x^i \mid k \geq 1, x^1, \dots, x^k \in X \right\}. \quad (4)$$

This turns out to be a convenient representation of X^* .

A *society* is a finite, non-empty set N . For convenience, if there are n individuals in N , we identify N with the set $\{1, \dots, n\}$. If we specify a feasible evaluation $x^i = (x_1^i, \dots, x_m^i) \in X$ for each individual $i \in N$, we obtain a *profile* of feasible evaluations $\mathbf{x} = (x_j^i) \in X^n$. Likewise, if we specify a feasible pseudo-evaluation $x^i = (x_1^i, \dots, x_m^i) \in X^*$ for each individual $i \in N$, we obtain a profile of feasible pseudo-evaluations $\mathbf{x} = (x_j^i) \in (X^*)^n$. We may view a profile as an $n \times m$ matrix all of whose rows lie in X (or in X^* , in the case of pseudo-evaluations). We use superscripts to indicate individuals (rows) and subscripts to indicate issues (columns).

An *aggregator* for N over X is a mapping $f: X^n \rightarrow X$. This is the type of aggregator that we studied in [11]. Here we investigate two other types of aggregators. An aggregator for N over X^* is a mapping $f: (X^*)^n \rightarrow X^*$. An aggregator for N over (X, X^*) is a mapping $f: X^n \rightarrow X^*$. The following notational conventions and definitions will apply, unless stated otherwise, to all types of aggregators. An aggregator f may be written in the form $f = (f_1, \dots, f_m)$ where f_j is the j -th component of f , i.e., it assigns to every profile the social position on the j -th issue.

An aggregator f is *independent of irrelevant alternatives* (abbreviated IIA) if for every $j \in J$ and any two profiles \mathbf{x} and \mathbf{y} satisfying $x_j^i = y_j^i$ for all $i \in N$, we have $f_j(\mathbf{x}) = f_j(\mathbf{y})$. This means that the social position on a given issue is determined entirely by the individual positions on that same issue. Viewing profiles as matrices, this means that the aggregation is done column-by-column. As we shall deal with IIA aggregators, we will slightly abuse notation and write also expressions of the form $f_j(x_j)$, where $x_j = (x_j^1, \dots, x_j^n)$ is the column vector of individual positions on the j -th issue.

An aggregator f is *Paretian* if we have $f(\mathbf{x}) = x$ whenever the profile $\mathbf{x} \in X^n$ is such that $x^i = x$ for all $i \in N$. Note that this requirement applies only to profiles in X^n , even if we consider an aggregator over X^* .³ Observe also that in the presence of IIA, this is equivalent to demanding

³ Our result would not be affected if we extended the requirement to profiles in $(X^*)^n$, but we prefer the weaker version.

that whenever all individuals hold the same 0 or 1 position on any one issue, the society adopts this position on that issue.

An aggregator f is D -oligarchic, for a non-empty $D \subseteq N$, if for every profile \mathbf{x} , every issue j , and every $u \in \{0, 1\}$, we have

$$f_j(\mathbf{x}) = u \quad \text{if and only if} \quad x_j^i = u \quad \text{for all } i \in D. \tag{5}$$

In the special case when $|D| = 1$, such an aggregator is *dictatorial*. In general, f is *oligarchic* if it is D -oligarchic for some non-empty $D \subseteq N$. Note that an oligarchic aggregator is trivially IIA and Paretian. Observe also that using the multiplication operation, rule (5) may be equivalently expressed as: for every profile \mathbf{x} we have

$$f(\mathbf{x}) = \prod_{i \in D} x^i. \tag{6}$$

Using (4), this shows that such f maps X^n , and more generally $(X^*)^n$, into X^* . We conclude that for every non-empty $D \subseteq N$, there exist indeed a D -oligarchic aggregator for N over (X, X^*) , and a D -oligarchic aggregator for N over X^* . On the other hand, an oligarchic aggregator for N over X must be dictatorial.⁴

We will be interested in conditions on X under which the only IIA and Paretian aggregators are the oligarchic ones. For this purpose, we may and will assume without loss of generality that X is *non-degenerate* in the following sense: for every issue $j \in J$ and every $u \in \{0, 1\}$ there exists $x \in X$ with $x_j = u$.

The first condition that appears in our characterization, named total blockedness, was introduced by Nehring and Puppe [19]. We recall its definition, as rephrased by us in [11]. Let X be a non-degenerate subset of $\{0, 1\}^m$. A *minimally infeasible partial evaluation* (abbreviated MIPE) is a K -evaluation $x = (x_j)_{j \in K}$ for some $K \subseteq J$ which is infeasible, but such that every restriction of x to a proper subset of K is feasible. By non-degeneracy, the length of any MIPE (i.e., the size of K) is at least two. We use the MIPEs to construct a directed graph associated with X , denoted by G_X . It has $2m$ vertices, labeled $0_1, 1_1, 0_2, 1_2, \dots, 0_m, 1_m$. The vertex u_j is to be interpreted as holding the position u on issue j . There is an arc in G_X from vertex u_k to vertex v_ℓ (written $u_k \rightarrow v_\ell$) if and only if $k \neq \ell$ and there exists a MIPE $x = (x_j)_{j \in K}$ such that $\{k, \ell\} \subseteq K$ and $x_k = u, x_\ell = \bar{v}$ (where \bar{v} denotes $1 - v$). The interpretation of $u_k \rightarrow v_\ell$ is that u_k conditionally entails v_ℓ in the following sense: conditional on holding the positions prescribed in the MIPE x on all issues in $K \setminus \{k, \ell\}$, holding position u on issue k entails holding position v on issue ℓ (since x is infeasible). We write $u_k \rightarrow\rightarrow v_\ell$ if there exists a directed path in G_X from u_k to v_ℓ . Finally, we say that X is *totally blocked* if G_X is strongly connected, that is, for any two vertices u_k and v_ℓ we have $u_k \rightarrow\rightarrow v_\ell$.

Example 1 (continued). For the set X of the above example, there are three MIPEs. One consists of the position 0 on the first issue and 1 on the third issue. It gives rise to the arcs $0_1 \rightarrow 0_3$ and $1_3 \rightarrow 1_1$ in G_X . Another consists of the position 0 on the second issue and 1 on the third issue. It gives rise to $0_2 \rightarrow 0_3$ and $1_3 \rightarrow 1_2$. The third MIPE is the evaluation $(1, 1, 0)$. It gives rise to six arcs: $1_1 \rightarrow 0_2, 1_1 \rightarrow 1_3, 1_2 \rightarrow 0_1, 1_2 \rightarrow 1_3, 0_3 \rightarrow 0_1, 0_3 \rightarrow 0_2$. Looking at the entire set of ten

⁴ For aggregators over X , there is a different notion of D -oligarchy that does make sense for arbitrary D . We used that notion in [11], and a similar notion called oligarchy with a default was used by Nehring and Puppe [20] and Nehring [18]. In this paper we do not require that notion.

arcs forming G_X , we note that no arcs go from a 0_k to a 1_ℓ . We conclude that G_X is not strongly connected, and hence X is not totally blocked.

The second condition that appears in our characterization was introduced by us in [11]. It requires that X not be an affine subspace of $\{0, 1\}^m$. Recall that $\{0, 1\}^m$ is a vector space over the field $\{0, 1\}$. A linear subspace is a non-empty subset closed under addition, and an *affine subspace* is a subset obtained from a linear subspace by adding a fixed vector to each of its elements. Intuitively, the condition of not being an affine subspace rules out sets X having a very specific structure, based entirely on parity requirements.

We are now ready to state the main result.

Theorem 2.1. *The following are equivalent for a non-degenerate subset X of $\{0, 1\}^m$:*

1. X is totally blocked and is not an affine subspace.
2. For every n , every IIA and Paretian aggregator $f : (X^*)^n \rightarrow X^*$ is oligarchic.
3. For some $n \geq 3$, every IIA and Paretian aggregator $f : (X^*)^n \rightarrow X^*$ is oligarchic.
4. For every n , every IIA and Paretian aggregator $f : X^n \rightarrow X^*$ is oligarchic.
5. For some $n \geq 3$, every IIA and Paretian aggregator $f : X^n \rightarrow X^*$ is oligarchic.

3. Proof of the main result

Let X be a non-degenerate subset of $\{0, 1\}^m$. In preparation for the proof, we introduce the following terminology pertaining to an IIA and Paretian aggregator $f : (X^*)^n \rightarrow X^*$. As noted earlier, f may be written in the form $f = (f_1, \dots, f_m)$ where f_j maps columns of positions on the j -th issue (of the form $x_j = (x_j^1, \dots, x_j^n) \in \{0, 1, *\}^n$) into $\{0, 1, *\}$. For each issue j and each position $u \in \{0, 1\}$, we say that a subset S of N is u_j -winning against another subset T of N , with $S \cap T = \emptyset$, if

$$x_j^i = \begin{cases} u & \text{if } i \in S \\ \bar{u} & \text{if } i \in T \\ * & \text{if } i \in N \setminus (S \cup T) \end{cases} \Rightarrow f_j(x_j) = u. \tag{7}$$

Thus, S is u_j -winning against T if it prevails on issue j when its members hold the position u , the members of T hold the opposite position, and the remaining individuals abstain. For each j and u this defines a binary relation on subsets of N . We denote it by \mathcal{W}_j^u and write $S\mathcal{W}_j^u T$ when S is u_j -winning against T . The system of binary relations \mathcal{W}_j^u has the following properties (that follow from the definition and the Pareto property):

$$S\mathcal{W}_j^u T \Rightarrow S \cap T = \emptyset, \tag{8}$$

$$N\mathcal{W}_j^u \emptyset, \tag{9}$$

$$S\mathcal{W}_j^u T \Rightarrow \text{not } T\mathcal{W}_j^{\bar{u}} S. \tag{10}$$

Note that, conversely, if we arbitrarily specify a system of binary relations \mathcal{W}_j^u satisfying (8)–(10), then we have implicitly defined the components f_1, \dots, f_m . Namely, given $x_j = (x_j^1, \dots, x_j^n) \in \{0, 1, *\}^n$ we let

$$S = \{i \in N \mid x_j^i = 0\} \quad \text{and} \quad T = \{i \in N \mid x_j^i = 1\}$$

Table 1
Construction for Claim 3.1.

	1	2	3	...	r
S	u	v	x_3	...	x_r
T	\bar{u}	\bar{v}	x_3	...	x_r
$N \setminus (S \cup T)$	*	*	x_3	...	x_r
	u		x_3	...	x_r

and define:

$$f_j(x_j) = \begin{cases} 0 & \text{if } S\mathcal{W}_j^0 T, \\ 1 & \text{if } T\mathcal{W}_j^1 S, \\ * & \text{otherwise.} \end{cases}$$

The resulting function $f = (f_1, \dots, f_m)$ may not map $(X^*)^n$ into X^* , but if it does then it is an IIA and Paretian aggregator. Clearly, f is oligarchic if and only if there exists a non-empty $D \subseteq N$ so that for every j and u and any two disjoint subsets S and T of N we have $S\mathcal{W}_j^u T \Leftrightarrow S \supseteq D$.

Note that when we discuss aggregators $f : X^n \rightarrow X^*$, we can use the same terminology and observations as above, but restrict attention to complementary pairs of subsets of N (i.e., $T = N \setminus S$). This restriction makes things simpler. In the proofs below, in order to avoid undue repetition, we will explicitly treat only the more difficult case of $f : (X^*)^n \rightarrow X^*$. The corresponding proofs for $f : X^n \rightarrow X^*$ can be obtained by omitting those parts of the proofs that become irrelevant. The structure of the proof is similar to that of our proof for $f : X^n \rightarrow X$ in [11], but some modifications and additional steps are needed.

In the first part of the proof of Theorem 2.1 we show the implication $1 \Rightarrow 2$. So we consider some IIA and Paretian aggregator $f : (X^*)^n \rightarrow X^*$ and, using the conditions on X in clause 1 of the theorem, we gradually establish properties of the associated system of binary relations \mathcal{W}_j^u .

Claim 3.1. *If $u_k \rightarrow v_\ell$ in the graph G_X then $\mathcal{W}_k^u \subseteq \mathcal{W}_\ell^v$.*

Proof. By the definition of G_X , there exists a MIPE $x = (x_j)_{j \in K}$ such that $\{k, \ell\} \subseteq K$ and $x_k = u$, $x_\ell = \bar{v}$. Let S and T be disjoint subsets of N so that $S\mathcal{W}_k^u T$. We have to show that $S\mathcal{W}_\ell^v T$. In Table 1 we construct a profile of feasible pseudo-evaluations, restricted to issues in K , and indicate the resulting social positions on some of the issues (for ease of exposition, we assume that $K = \{1, \dots, r\}$ and $k = 1, \ell = 2$).

Observe that each of the rows corresponding to S and to T in the table differs from the MIPE x in exactly one place, and therefore by the minimality of a MIPE these rows are feasible (i.e., can be extended to a feasible evaluation on J). The row corresponding to $N \setminus (S \cup T)$ is the product, in the sense of (3), of the two rows above it. By (4), this row can be extended to a feasible pseudo-evaluation on J . The resulting social positions are determined in the first column by our assumption that $S\mathcal{W}_1^u T$, and in columns $3, \dots, r$ by the Pareto property. Because the social pseudo-evaluation must be feasible, this has consequences for the entry in the second column. Since x is infeasible, (1) implies that this entry cannot be \bar{v} , and (2) implies that it cannot be $*$, either. Hence it must be v , which shows that $S\mathcal{W}_2^v T$. \square

By repeated applications of Claim 3.1, it follows that $u_k \rightarrow v_\ell$ implies $\mathcal{W}_k^u \subseteq \mathcal{W}_\ell^v$. Therefore, if X is totally blocked then $\mathcal{W}_k^u = \mathcal{W}_\ell^v$ for any u_k and v_ℓ . Thus, there exists one common

Table 2
Construction for Claim 3.2.

	1	2	3	4	...	r
S	\bar{x}_1	x_2	x_3	x_4	...	x_r
T	x_1	\bar{x}_2	x_3	x_4	...	x_r
$N \setminus U$	x_1	x_2	\bar{x}_3	x_4	...	x_r
	x_1	x_2		x_4	...	x_r

binary relation, that we denote by \mathcal{W} , with which all \mathcal{W}_k^u coincide. In this case we say that f is *neutral*, that is, it treats equally all issues and their negations. We define the collection \mathcal{B} of *blocking* coalitions as follows: for $S \subseteq N$ we have

$$S \in \mathcal{B} \iff \text{not } (N \setminus S)\mathcal{W}S. \tag{11}$$

We recall also that Nehring and Puppe [19] showed (see also [11]) that if X is totally blocked then there exists a MIPE of length at least three.

Claim 3.2. *If there exists a MIPE of length at least three, and f is neutral, then \mathcal{B} is decomposable in the following sense: if $U \in \mathcal{B}$ and (S, T) is any partition of U then either $S \in \mathcal{B}$ or $T \in \mathcal{B}$.*

Proof. Suppose, for the sake of contradiction, that $S, T \notin \mathcal{B}$, $S \cap T = \emptyset$, and $U = S \cup T \in \mathcal{B}$. Let $x = (x_j)_{j \in K}$ be a MIPE with $|K| \geq 3$. Now, consider the construction in Table 2 (where for ease of exposition $K = \{1, \dots, r\}$).

The validity of the construction follows from the minimality of the MIPE x . The resulting social positions are determined in columns 1 and 2 by our assumption that $S, T \notin \mathcal{B}$, and in columns 4, ..., r by the Pareto property. Now, because x is infeasible, the social position on issue 3 cannot be x_3 , and by (2) it cannot be $*$, either. So it must be \bar{x}_3 , which shows that $(N \setminus U)\mathcal{W}U$, and thus $U \notin \mathcal{B}$. This contradicts our assumption. \square

Claim 3.3. *Under the assumptions of Claim 3.2, there exists a non-empty $D \subseteq N$ so that $\{i\} \in \mathcal{B}$ for every $i \in D$, and $N \setminus D \notin \mathcal{B}$.*

Proof. Let D be the set of all $i \in N$ such that $\{i\} \in \mathcal{B}$. To prove that D is non-empty, we observe that $N \in \mathcal{B}$ by the Pareto property, and use Claim 3.2 repeatedly, until we find a singleton in \mathcal{B} . To show that $N \setminus D \notin \mathcal{B}$, we note that if $N \setminus D \in \mathcal{B}$ we can start the repeated applications of Claim 3.2 with $N \setminus D$ and thus find some $i \in N \setminus D$ such that $\{i\} \in \mathcal{B}$, contradicting the definition of D . \square

We will show that f is D -oligarchic, for the set D produced in Claim 3.3. In order to do that, we need to establish some further properties of \mathcal{W} . The first is monotonicity, which requires that a social position of 0 or 1 be preserved if the individual positions are maintained or changed in its direction. Formally, we say that \mathcal{W} is *monotone* if for any $S, T, S', T' \subseteq N$ so that $S \cap T = \emptyset$ and $S' \cap T' = \emptyset$ we have

$$S\mathcal{W}T \text{ and } S \subseteq S' \text{ and } T' \subseteq T \implies S'\mathcal{W}T'. \tag{12}$$

Table 3
Construction for Claim 3.4.

	1	2	3	...	r
S	x_1	\bar{x}_2	x_3	...	x_r
T'	\bar{x}_1	x_2	x_3	...	x_r
$T \cap S'$	\bar{x}_1	\bar{x}_2	x_3	...	x_r
$N \setminus (T \cup S')$	*	*	x_3	...	x_r
$S' \setminus (S \cup T)$	*	\bar{x}_2	x_3	...	x_r
$T \setminus (S' \cup T')$	\bar{x}_1	*	x_3	...	x_r
	x_1		x_3	...	x_r

We recall also that in [11] we proved that if X is not an affine subspace then there exist two K -evaluations x and y for the same $K \subseteq J$ that differ in exactly two places, such that x is a MIPE and y is feasible.

Claim 3.4. *If there exist two K -evaluations x and y for the same $K \subseteq J$ that differ in exactly two places, such that x is a MIPE and y is feasible, and f is neutral, then \mathcal{W} is monotone.*

Proof. Let $x = (x_j)_{j \in K}$ and $y = (y_j)_{j \in K}$ be as in the assumption, and let S, T, S', T' satisfy the premises of (12). Consider the construction in Table 3 (again, for ease of exposition, $K = \{1, \dots, r\}$) and the two issues where y differs from x are 1 and 2).

Let us justify this construction. First, it is easy to check using our assumptions on S, T, S', T' that the six sets listed in the table form a partition of N . Next, we justify each of the six corresponding rows (referring to them as rows 1, ..., 6 in the order that they appear in the table). Rows 1 and 2 each differ from the MIPE x in exactly one place, and hence can be extended to feasible evaluations on J . Row 3 coincides with y , which we assume to be feasible. Row 4 is the product of rows 1 and 2, and hence by (4) can be extended to a feasible pseudo-evaluation on J . Similarly, row 5 is the product of rows 1 and 3, and row 6 is the product of rows 2 and 3. Now, the social position on issue 1 follows from our assumption that $S \mathcal{W} T$, and those on issues 3, ..., r follow from the Pareto property. Because x is infeasible, the social position on issue 2 cannot be x_2 , and by (2) it cannot be *, either. Hence it must be \bar{x}_2 , which shows that $S' \mathcal{W} T'$, as required. □

The next property of \mathcal{W} that we establish requires that a social position of 0 or 1 be preserved if some of the abstaining individuals turn into opposers. Formally, we say that \mathcal{W} is *robust to opposition* if for any $S, T, T' \subseteq N$ so that $S \cap T = \emptyset$ and $S \cap T' = \emptyset$ we have

$$S \mathcal{W} T \quad \text{and} \quad T \subseteq T' \quad \Rightarrow \quad S \mathcal{W} T'. \tag{13}$$

Note that this property is not supported by the same intuition as monotonicity, since here the change in the individual positions goes in the opposite direction. Nevertheless, we show that under our conditions this property holds. Note also that taken together, monotonicity and robustness to opposition imply that $S \mathcal{W} T$ depends only on S .

Claim 3.5. *If there exists a MIPE of length at least three, f is neutral, and \mathcal{W} is monotone, then \mathcal{W} is robust to opposition.*

Proof. Let $x = (x_j)_{j \in K}$ be a MIPE with $|K| \geq 3$, and let S, T, T' satisfy the premises of (13). Consider the construction in Table 4 (where for ease of exposition $K = \{1, \dots, r\}$).

Table 4
Construction for Claim 3.5.

	1	2	3	4	...	r
S	x_1	\bar{x}_2	x_3	x_4	...	x_r
T	\bar{x}_1	x_2	x_3	x_4	...	x_r
$N \setminus (S \cup T')$	*	*	x_3	x_4	...	x_r
$T' \setminus T$	*	x_2	*	x_4	...	x_r
	x_1		x_3	x_4	...	x_r

The four sets listed in the table form a partition of N . The corresponding rows 1, ..., 4 are justified as follows. Rows 1 and 2 each differ from the MIPE x in exactly one place, and hence can be extended to feasible evaluations on J . Row 3 is their product, and hence by (4) can be extended to a feasible pseudo-evaluation on J . Similarly, row 4 is the product of row 2 and the row (that does not appear in the table) that differs from the MIPE x only in the third column. Now, the social position on issue 1 follows from our assumption that SWT , and that on issue 3 follows from the same assumption with the help of monotonicity. The social positions on issues 4, ..., r follow from the Pareto property. As x is infeasible, the social position on issue 2 cannot be x_2 , and by (2) it cannot be *, either. Hence it must be \bar{x}_2 , which shows that SWT' , as required. \square

We are now ready to show that f is D -oligarchic for the set D produced in Claim 3.3. Let j be an issue, let $x_j = (x_j^1, \dots, x_j^n) \in \{0, 1, *\}^n$ be a column of positions on issue j , and let $u \in \{0, 1\}$. Suppose first that $x_j^i = u$ for all $i \in D$. We need to show that $f_j(x_j) = u$. By monotonicity of \mathcal{W} , it suffices to show this under the additional assumption that $x_j^i = \bar{u}$ for all $i \in N \setminus D$. But then the conclusion $f_j(x_j) = u$ follows from the fact, stated in Claim 3.3, that $N \setminus D \notin \mathcal{B}$. Conversely, suppose that $f_j(x_j) = u$ but there exists $i \in D$ such that $x_j^i \neq u$. By monotonicity and robustness to opposition, we may assume that $x_j^k = u$ for all $k \neq i$ and $x_j^i = \bar{u}$. But then $f_j(x_j) = u$ is impossible, because $\{i\} \in \mathcal{B}$ by Claim 3.3.

This completes the proof of $1 \Rightarrow 2$ in Theorem 2.1. As remarked above, the same arguments (without the parts that become irrelevant) yield a proof of $1 \Rightarrow 4$. The implications $2 \Rightarrow 3$ and $4 \Rightarrow 5$ are trivial. The next two claims show together that the negation of clause 1 implies the negations of clauses 3 and 5, thus completing the proof of Theorem 2.1.

Claim 3.6. *If X is not totally blocked, then for every $n \geq 2$ there exists an aggregator $f : (X^*)^n \rightarrow X^*$ so that both f and its restriction to X^n are IIA, Paretian and non-oligarchic.*

Proof. As X is not totally blocked, there exists a partition of the vertices of G_X into two non-empty parts V_1 and V_2 so that there is no arc in G_X from a vertex in V_1 to a vertex in V_2 . Let N be a society, $|N| = n \geq 2$. We define f by specifying the system of binary relations \mathcal{W}_j^u as follows. For any pair of disjoint subsets $S, T \subseteq N$ we have:

$$\text{if } u_j \in V_1 \text{ then } SW_j^u T \Leftrightarrow 1 \in S, \tag{14}$$

$$\text{if } u_j \in V_2 \text{ then } SW_j^u T \Leftrightarrow \{1, 2\} \subseteq S. \tag{15}$$

It is obvious that this definition satisfies (8)–(10), and that the resulting function f and its restriction to X^n have the required properties, provided that f maps $(X^*)^n$ into X^* . Thus, all we need to show is that $\mathbf{x} \in (X^*)^n$ implies $f(\mathbf{x}) \in X^*$.

We consider a fixed $\mathbf{x} \in (X^*)^n$ and let

$$P = \{j \in J \mid f_j(\mathbf{x}) \in \{0, 1\}\},$$

$$Q = \{j \in J \mid x_j^1 \in \{0, 1\}\}.$$

By (14) and (15), we have $P \subseteq Q$ and $f_j(\mathbf{x}) = x_j^1$ for every $j \in P$. We need to check that $f(\mathbf{x})$ satisfies conditions (1) and (2). As x^1 satisfies these two conditions, it follows that $f(\mathbf{x})$ also satisfies (1), and the only part of (2) that remains to be shown is: for every $j \in Q \setminus P$ there exists $y \in X$ that agrees with $f(\mathbf{x})$ and satisfies $y_j \neq x_j^1$. Fix such j and let $x_j^1 = v$. Suppose that there does not exist $y \in X$ as desired. This means that the $P \cup \{j\}$ -evaluation z , where $z_k = f_k(\mathbf{x})$ for all $k \in P$ and $z_j = \bar{v}$, is infeasible. There exists some $K \subseteq P$ so that the restriction of z to $K \cup \{j\}$, which we denote by z' , is a MIPE. Now, consider x^2 . Since $x_j^1 = v$ but $f_j(\mathbf{x}) = *$, we conclude from (14) and (15) that $v_j \in V_2$ and $x_j^2 \neq v$. Furthermore, there must exist $k \in K$ such that, denoting $z'_k = u$, we have $u_k \in V_1$ (otherwise, x^2 would have to coincide with z' on K and satisfy $x_j^2 \in \{\bar{v}, *\}$, which renders it infeasible). For such k , the MIPE z' yields the arc $u_k \rightarrow v_j$ in G_X , contradicting our assumption about V_1 and V_2 . \square

Claim 3.7. *If X is an affine subspace, then for every $n \geq 3$ there exists an aggregator $f : (X^*)^n \rightarrow X^*$ so that both f and its restriction to X^n are IIA, Paretian and non-oligarchic.*

Proof. Let N be a society, $|N| = n \geq 3$. We choose some subset R of N of odd cardinality $r \geq 3$. For $\mathbf{x} \in (X^*)^n$ we define $f(\mathbf{x})$ as follows. First, we let

$$P = \{j \in J \mid x_j^i \in \{0, 1\} \text{ for all } i \in R\}.$$

Then we define:

$$f_j(\mathbf{x}) = \begin{cases} \sum_{i \in R} x_j^i & \text{if } j \in P, \\ * & \text{otherwise} \end{cases}$$

where the summation is modulo 2. We will show that f maps $(X^*)^n$ into X^* . The other required properties of f and its restriction to X^n are obvious.

In order to prove that $\mathbf{x} \in (X^*)^n$ implies $f(\mathbf{x}) \in X^*$, we recall the fact (proved in [11]) that an affine subspace of $\{0, 1\}^m$ is closed under summation of an odd number of its elements. This implies immediately that if $P = J$ then $f(\mathbf{x}) \in X$. If $P \subsetneq J$, it suffices to check that $f(\mathbf{x})$ satisfies (2). Let $j \in J \setminus P$ and let $u \in \{0, 1\}$. We need to find $y \in X$ that coincides with $f(\mathbf{x})$ on P and satisfies $y_j = u$. Since each x^i , $i \in R$, belongs to X^* , we can find for each $i \in R$ some $y^i \in X$ that agrees with x^i , and in particular coincides with it on P . Moreover, if $x_j^i = *$ we are free to choose y^i with any specified j -th entry. As $j \notin P$, we have this freedom for at least one $i \in R$, and so we may arrange the choices of y^i , $i \in R$, so that $\sum_{i \in R} y_j^i = u$. Then $y = \sum_{i \in R} y^i$ satisfies our requirements. \square

4. Applications to preference aggregation

In this section we present some corollaries of our main result to preference aggregation. We assume throughout that a finite set of alternatives $A = \{a_1, \dots, a_k\}$ is given, with $k \geq 3$. We will consider various classes of preference relations over A . They will all be subclasses of $\mathcal{A}sym(A)$, the class of all asymmetric binary relations over A (i.e., those binary relations $>$ that satisfy: for

all $a, b \in A$, $a \succ b \Rightarrow \text{not } b \succ a$). The relation $a \succ b$ is interpreted as ‘ a is strictly preferred to b .’ We may define $a \succcurlyeq b$, for $a, b \in A$, to hold if and only if $b \succ a$ does not hold. We will describe our classes of preference relations in terms of the strict relation \succ , although it will sometimes be more intuitive to think of the properties of the derived non-strict relation \succcurlyeq .

First, we introduce a set J of $m = \binom{k}{2}$ issues. Each issue is indexed by a pair rs such that $1 \leq r < s \leq k$. By convention, $x_{rs} = 1$ means that $a_r \succ a_s$, $x_{rs} = 0$ means that $a_s \succ a_r$, and $x_{rs} = *$ means that neither $a_r \succ a_s$ nor $a_s \succ a_r$. In this way, there is a one-to-one correspondence between $\mathcal{A}sym(A)$ and $\{0, 1, *\}^m$. Thus we may (and will) identify asymmetric binary relations over A with pseudo-evaluations on J .

Now, consider the class $\mathcal{L}in(A)$ of all *linear orders* over A , i.e., those relations \succ in $\mathcal{A}sym(A)$ that are transitive and complete. The corresponding set of vectors is

$$X = \{x \in \{0, 1\}^m \mid \forall 1 \leq r < s < t \leq k, (x_{rs}, x_{st}, x_{rt}) \neq (0, 0, 1), (1, 1, 0)\}. \quad (16)$$

Next, consider the class $\mathcal{P}art(A)$ of all *partial orders* over A , i.e., those relations \succ in $\mathcal{A}sym(A)$ that are transitive. To be explicit, we require here that for all $a, b, c \in A$, $a \succ b$ and $b \succ c \Rightarrow a \succ c$, but $a \succcurlyeq b$ and $b \succcurlyeq c \Rightarrow a \succcurlyeq c$ is not required. In the social choice literature, such preferences are usually called quasi-transitive. It is easy to see that the set of vectors that corresponds to $\mathcal{P}art(A)$ is precisely X^* , where X is the set in (16) that corresponds to $\mathcal{L}in(A)$.

It is straightforward to check that the set X in (16) satisfies clause 1 of Theorem 2.1 (this was shown in [11], and for total blockedness earlier in [17]). So we obtain the following two corollaries.

Corollary 4.1. *Let A be a finite set of alternatives, with $|A| \geq 3$. For every n , every IIA and Paretian aggregator $f : (\mathcal{P}art(A))^n \rightarrow \mathcal{P}art(A)$ is oligarchic.*

Corollary 4.2. *Let A be a finite set of alternatives, with $|A| \geq 3$. For every n , every IIA and Paretian aggregator $f : (\mathcal{L}in(A))^n \rightarrow \mathcal{P}art(A)$ is oligarchic.*

We note that the IIA and Pareto properties have their obvious intuitive meaning in this context. Namely, IIA requires that the restriction of the social order to any pair of alternatives depend only on the restrictions of the individual orders to that pair. Pareto requires that whenever every individual has the same linear order, society adopt this order. We observe also that oligarchic aggregators have a transparent set theoretic expression in this context. Indeed, let us view a binary relation as a subset of $A \times A$. Then for a D -oligarchic aggregator, the social order is the intersection of the orders of the individuals in D .

We do not know if these two results have appeared in the literature on preference aggregation. They are reminiscent of a result of Gibbard on aggregation of weak orders. We proceed to recall that result and show that it, too, is a corollary of our main result (though not as immediate as the above two).

The class $\mathcal{W}eak(A)$ of all *weak orders* over A consists of those relations \succ in $\mathcal{A}sym(A)$ that are negatively transitive. To be explicit, we require here that for all $a, b, c \in A$, $c \succ a \Rightarrow b \succ a$ or $c \succ b$, which is equivalent to $a \succcurlyeq b$ and $b \succcurlyeq c \Rightarrow a \succcurlyeq c$. This condition implies, in particular, that $a \succ b$ and $b \succ c \Rightarrow a \succ c$. Thus we have the containments

$$\mathcal{L}in(A) \subseteq \mathcal{W}eak(A) \subseteq \mathcal{P}art(A).$$

The corresponding sets of vectors are

$$X \subseteq X^w \subseteq X^*,$$

Table 5
Construction for Corollary 4.3.

	$a \succ b$	$b \succ c$	$a \succ c$
S	1	0	1
T	0	1	1
$N \setminus (S \cup T)$	*	1	1
			1

where X is the set defined in (16), X^* is derived from it as above, and X^w is given by

$$X^w = \{x \in X^* \mid \forall 1 \leq r < s < t \leq k \text{ the number of } * \text{ entries in } (x_{rs}, x_{st}, x_{rt}) \text{ is not } 2\}. \tag{17}$$

In a vector representation of a weak order, a * entry is interpreted as indifference. The condition that appears in the definition of X^w in (17) amounts to the transitivity of indifference. When this property is added to the transitivity of strict preference already required in X^* , one gets the transitivity of weak preference.

Gibbard [14] (and others) dealt with aggregators $f : (\text{Weak}(A))^n \rightarrow \text{Part}(A)$, called by Sen [23] quasi-transitive social decision functions. For such aggregators, one cannot expect every IIA and Paretian aggregator to be oligarchic in our sense. For example, one way to aggregate is to let one fixed individual impose his strict preferences, and to adopt another fixed individual’s preferences whenever the former is indifferent. We need a weaker notion of oligarchy. Let us say that f is *weakly D-oligarchic*, for a non-empty $D \subseteq N$, if for every $a, b \in A$ we have:

$$a \succ_i b \text{ for all } i \in D \Rightarrow a \succ b, \tag{18}$$

$$a \succ_i b \text{ for some } i \in D \Rightarrow \text{not } b \succ a, \tag{19}$$

where \succ_i denotes the preference of individual i , and \succ denotes the social preference that results when f is applied to $(\succ_1, \dots, \succ_n)$. This notion is usually called oligarchy, but is weaker than our notion of oligarchy, as it allows the result to be $a \succ b$ when some members of D are indifferent, provided that no member of D actually prefers b to a . We say that f is *weakly oligarchic* if it is weakly D -oligarchic for some non-empty $D \subseteq N$.

Corollary 4.3. (See Gibbard [14].) *Let A be a finite set of alternatives, with $|A| \geq 3$. For every n , every IIA and Paretian aggregator $f : (\text{Weak}(A))^n \rightarrow \text{Part}(A)$ is weakly oligarchic.*

Proof. Consider f^ℓ , the restriction of f to $(\text{Lin}(A))^n$. By Corollary 4.2, there exists a non-empty $D \subseteq N$ such that f^ℓ is D -oligarchic. We will show that f itself is weakly D -oligarchic. Thus, we are given the restrictions of the individual preferences on some two alternatives a and b , and want to show that (18) and (19) hold. Let $S = \{i \in N \mid a \succ_i b\}$ and $T = \{i \in N \mid b \succ_i a\}$. Consider the construction in Table 5 (where c is a third alternative, chosen arbitrarily).

Note that each of the individual rows in the table represents a weak order over $\{a, b, c\}$, and hence can be extended to a vector representation of a weak order over A . The outcome in the $a \succ c$ column follows from the Pareto property. To show (18) holds, suppose indeed that $a \succ_i b$ for all $i \in D$. This implies that $D \subseteq S$. Because the $b \succ c$ column does not contain a *, and f^ℓ is D -oligarchic, it follows that the outcome in this column is 0. As the outcome must represent a partial order, the outcome in the $a \succ b$ column has to be 1, as required in (18). To check (19), suppose that $b \succ a$, i.e., the outcome in the $a \succ b$ column is 0. As the outcome must represent a

Table 6
Construction for Corollary 4.4.

	$a \succ b$	$b \succ c$	$a \succ c$
D_1	1	0	1
D_2	0	1	1
	*	*	1

partial order, it follows that the outcome in the $b \succ c$ column has to be 1. Since the rule there is D -oligarchic, this implies that $D \cap S = \emptyset$, contradicting the premise of (19). \square

Let us say that f is *weakly dictatorial* if there exists an individual $d \in N$ so that for every $a, b \in A$ we have $a \succ_d b \Rightarrow a \succ b$. This is usually called dictatorial, but we reserve that term to the case when $a \succ_d b \Leftrightarrow a \succ b$. Note that f is weakly dictatorial if and only if it is weakly D -oligarchic with $|D| = 1$.

We show next that Arrow’s impossibility theorem for social welfare functions is a corollary of our main result.

Corollary 4.4. (See Arrow [1].) *Let A be a finite set of alternatives, with $|A| \geq 3$. For every n , every IIA and Paretian aggregator $f : (\text{Weak}(A))^n \rightarrow \text{Weak}(A)$ is weakly dictatorial.*

Proof. As $\text{Weak}(A) \subseteq \text{Part}(A)$, we may conclude from Corollary 4.3 that f is weakly D -oligarchic for some non-empty $D \subseteq N$. It remains to show that in our case D must be a singleton. Suppose it is not a singleton, and let D_1, D_2 be a partition of D into two non-empty sets. Consider the construction in Table 6 (where a, b, c are three alternatives, chosen arbitrarily).

The table describes only part of a profile, but this is enough to determine the outcomes in the three columns, which follow from f being weakly D -oligarchic. As the number of * entries among the outcomes in these three columns is 2, the outcome does not represent a weak order. This contradicts our assumption that f maps into $\text{Weak}(A)$. \square

Our approach enables us to handle IIA and Paretian aggregators $f : (\text{Weak}(A))^n \rightarrow \mathcal{R}$, where \mathcal{R} is an arbitrary class of allowed social preferences satisfying $\text{Weak}(A) \subseteq \mathcal{R} \subseteq \text{Part}(A)$. Gibbard’s theorem immediately implies that all such aggregators are weakly oligarchic, and Arrow’s theorem says that when $\mathcal{R} = \text{Weak}(A)$ the oligarchy must be a singleton. For general \mathcal{R} , we will be able to determine the sizes of oligarchies that \mathcal{R} admits, using the parameter $\delta(\mathcal{R})$ that we introduce below.

First, recall that the *dimension* of a partial order \succ over A , denoted by $\dim(\succ)$, is the smallest number p so that there exist p linear orders over A whose intersection is precisely \succ . Equivalently, $\dim(\succ)$ is the smallest p so that (A, \succ) may be embedded in $(\mathbb{R}^p, >)$, i.e., the p -dimensional Euclidean space with the standard partial order on it (see [24, p. 39]). This means that we can think of $\dim(\succ)$ as the smallest number of criteria, so that \succ may result from comparing the alternatives according to each of the criteria, with strict overall preference requiring weak preference by all criteria and strict preference by at least one of them.

For a given \mathcal{R} , $\text{Weak}(A) \subseteq \mathcal{R} \subseteq \text{Part}(A)$, we define

$$\delta(\mathcal{R}) = \min\{\dim(\succ) \mid \succ \in \text{Part}(A) \setminus \mathcal{R}\},$$

with the convention that $\delta(\mathcal{R}) = \infty$ if $\mathcal{R} = \text{Part}(A)$. Thus, $\delta(\mathcal{R})$ is the minimum, over all partial orders \succ excluded by \mathcal{R} , of $\dim(\succ)$.

Corollary 4.5. *Let A be a finite set of alternatives, with $|A| \geq 3$. Let $Weak(A) \subseteq \mathcal{R} \subseteq Part(A)$. Let N be a society of n individuals. Then:*

- (i) *All D -oligarchic rules with $D \subseteq N$, $0 < |D| < \delta(\mathcal{R})$, map $(Weak(A))^n$ into \mathcal{R} , and thus define IIA and Paretian aggregators $f : (Weak(A))^n \rightarrow \mathcal{R}$.*
- (ii) *Conversely, every IIA and Paretian aggregator $f : (Weak(A))^n \rightarrow \mathcal{R}$ is weakly D -oligarchic for some $D \subseteq N$ satisfying $0 < |D| < \delta(\mathcal{R})$.*

Proof. (i) If $|D| = 1$ the conclusion follows from $Weak(A) \subseteq \mathcal{R}$. So we assume that $|D| = p \geq 2$. Let \succ_1, \dots, \succ_p be the weak orders of the individuals in D , and let the partial order \succ be their intersection, which is the corresponding social order by the D -oligarchic rule. We have to prove that $\succ \in \mathcal{R}$. We will do so by showing that there exist linear orders $\succ \succ_1, \dots, \succ \succ_p$ that extend \succ_1, \dots, \succ_p respectively (in the sense that for all $a, b \in A$, $a \succ_i b \Rightarrow a \succ \succ_i b$), and whose intersection is still \succ . This will imply that $\dim(\succ) \leq p < \delta(\mathcal{R})$ and hence, by the definition of $\delta(\mathcal{R})$, the partial order \succ must belong to \mathcal{R} .

We will pick the linear extensions $\succ \succ_1, \dots, \succ \succ_p$ one-by-one, making sure that in each step, as we pass from

$$\succ \succ_1, \dots, \succ \succ_{i-1}, \succ_i, \succ_{i+1}, \dots, \succ_p$$

to

$$\succ \succ_1, \dots, \succ \succ_{i-1}, \succ \succ_i, \succ_{i+1}, \dots, \succ_p,$$

the intersection of the latter list remains the same as that of the former list, namely \succ . To extend the weak order \succ_i to a linear order, we need to break the ties in each indifference class of \succ_i . For every such class I , we define an auxiliary binary relation \triangleleft over I as follows: $a \triangleleft b$ if we have $a \succ \succ_1 b, \dots, a \succ \succ_{i-1} b, a \succ_{i+1} b, \dots, a \succ_p b$. Note that the comparisons $a \triangleleft b$ are precisely those that must be avoided in $\succ \succ_i$ in order to keep the intersection from growing when passing from \succ_i to $\succ \succ_i$. Note also that \triangleleft is a partial order, because $\succ \succ_1, \dots, \succ \succ_{i-1}, \succ_{i+1}, \dots, \succ_p$ are, and $p \geq 2$. Hence the inverse \triangleright of \triangleleft , defined by $a \triangleright b$ if $b \triangleleft a$, is also a partial order, and therefore can be extended to a linear order over I . The linear orders over the indifference classes of \succ_i obtained in this way yield the desired linear extension $\succ \succ_i$ of \succ_i .

(ii) By Corollary 4.3, f is weakly D -oligarchic for some non-empty $D \subseteq N$. It remains to show that $|D| < \delta(\mathcal{R})$. Let $\delta(\mathcal{R}) = p < \infty$, and suppose, for the sake of contradiction, that $|D| \geq p$. Let D_1, \dots, D_p be a partition of D into p non-empty sets. Using the definition of $\delta(\mathcal{R})$, let \succ_1, \dots, \succ_p be p linear orders whose intersection \succ is not in \mathcal{R} . Now, consider a profile in which the individuals in D_i hold the linear order \succ_i , $i = 1, \dots, p$. As f is weakly D -oligarchic, it maps this profile to \succ , which is a contradiction because \succ is not in \mathcal{R} . \square

Let us call a class \mathcal{R} of allowed social preferences, $Weak(A) \subseteq \mathcal{R} \subseteq Part(A)$, an *impossibility range* if for every n , every IIA and Paretian aggregator $f : (Weak(A))^n \rightarrow \mathcal{R}$ is weakly dictatorial. Otherwise we say that \mathcal{R} is a *possibility range*. We get a full characterization of impossibility ranges.

Corollary 4.6. *Let A be a finite set of alternatives, with $|A| \geq 3$. Let $Weak(A) \subseteq \mathcal{R} \subseteq Part(A)$. Then \mathcal{R} is an impossibility range if and only if \mathcal{R} excludes at least one partial order of dimension 2.*

Proof. If \mathcal{R} excludes a partial order of dimension 2, then $\delta(\mathcal{R}) = 2$, and hence \mathcal{R} is an impossibility range by Corollary 4.5(ii). Conversely, if \mathcal{R} does not exclude any partial order of dimension 2, then $\delta(\mathcal{R}) > 2$, and Corollary 4.5(i) shows that \mathcal{R} is a possibility range. \square

Following Arrow's theorem for $Weak(A)$, there were several results in the literature showing that certain natural intermediate classes \mathcal{R} between $Weak(A)$ and $Part(A)$ are impossibility ranges, when $|A| \geq 4$. For the class of semiorders, this was shown by Wilson [25], Brown [4], Blair and Pollak [2], and Blau [3]. Blau also handled the class of interval orders, and the class of semitransitive orders.⁵ Each of these results can be deduced from Corollary 4.6, simply by producing two linear orders over a 4-element set whose intersection is not in the class \mathcal{R} in question. Indeed, the intersection of the two linear orders $a \succ_1 b \succ_1 c \succ_1 d$ and $c \succ_2 d \succ_2 a \succ_2 b$ is the partial order in which the only comparisons are $a \succ b, c \succ d$. The latter is not a semiorder, and not even an interval order. The intersection of the two linear orders $a \succ_1 b \succ_1 c \succ_1 d$ and $d \succ_3 a \succ_3 b \succ_3 c$ is the partial order in which $a \succ b \succ c$ and d is not comparable to any of them. The latter is not semitransitive.

We are not aware of any treatment in the social choice literature of the class of all partial orders over A of dimension at most 2. Denoting it by $Dim_{\leq 2}(A)$, we have the containments

$$Lin(A) \subseteq Weak(A) \subseteq Dim_{\leq 2}(A) \subseteq Part(A).$$

A reformulation of Corollary 4.6 says that \mathcal{R} is a possibility range if and only if it contains $Dim_{\leq 2}(A)$. This reveals the importance of the class $Dim_{\leq 2}(A)$ in preference aggregation: it is the smallest possibility range.

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⁵ A partial order \succ over A is an *interval order* if for all $a, b, c, d \in A$, $a \succ b \succ c \succ d \Rightarrow a \succ d$. It is *semitransitive* if for all $a, b, c, d \in A$, $a \succ b \succ c \succ d \Rightarrow a \succ d$. It is a *semiorder* if it satisfies both of the above.

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