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Fractional Kernels in Digraphs

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We define a fractional version of the notion of "kernels" in digraphs and prove that every clique–acyclic digraph (i.e., one in which no clique contains a cycle) has a fractional kernel. Using this we give a short proof of a recent result of Boros and Gurvich (proving a conjecture of Berge and Duchet) that every clique–acyclic orientation of a perfect graph has a kernel. © 1998 Academic Press

1. FRACTIONAL KERNELS

The in-neighborhood, $I(v)$, of a vertex v in a digraph $D=(V, A)$ is v together with the set of all vertices sending an arc to v, i.e., vertices u such that $(u, v) \in A$. A subset of V is called *dominating* if it meets $I(v)$ for every $v \in V$. (To avoid confusion, it must be noted that some authors require in the definition meeting every *out*-neighborhood.) A set of vertices is called *independent* if no two distinct elements in it are connected by an arc. We shall allow in a digraph pairs of oppositely directed arcs. An arc (u, v) is called *irreversible* if (v, u) is not an arc of the graph.

Like many other combinatorial concepts, these two have fractional counterparts. A non-negative function f on V is called *fractionally dominating* if $\sum_{u \in I(v)} f(u) \geqslant 1$ for every vertex v. This requirement can be strengthened, to demand that $\sum_{u \in K} f(u) \geqslant 1$ for some clique K contained in $I(v)$. (A set K of vertices is a *clique* if every two vertices in K are connected by at least one arc.) If this holds for every $v \in V$ then f is called *strongly dominating*. A non-negative function f on V is called *fractionally independent* if $\sum_{u \in K} f(u) \leqslant 1$ for every clique K.

A *kernel* in D is an independent and dominating set of vertices. A *fractional kernel* is a function on V which is both fractionally independent and fractionally dominating. In case that it is also strongly dominating, it is

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called a *strong fractional kernel.* The characteristic function of a kernel is obviously a strong fractional kernel.

A cycle in D is called *proper* if all of its arcs are irreversible. It is easily seen that a complete digraph (i.e., a clique) containing a proper Hamiltonian cycle does not possess a fractional kernel. The main aim of this paper is to show that this is the only possible obstacle for fractional kernels. A graph in which no clique contains a proper cycle is called *clique-acyclic.* We shall prove:

THEOREM 1.1. *Every clique-acyclic digraph has a strong fractional kernel.*

The main tool in our proof is the following theorem of Scarf (its proof is given, for the convenience of the reader, in the last section):

THEOREM 1.2 [7]. *Let $m<n$ and let B be an $m\times n$ real matrix such that $b_{ij}=\delta_{ij}$ for $1\leqslant i, j\leqslant m$. Let b be a non-negative vector in $\mathbb{R}^m$, such that the set $\{\alpha\in\mathbb{R}^n_+ : B\alpha=b\}$ is bounded. Let C be an $m\times n$ matrix such that $c_{ii}\leqslant c_{ik}\leqslant c_{ij}$ whenever $i, j\leqslant m$, $i\neq j$ and $k>m$. Then there exists a subset J of size m of $[n]$ such that*

(a) *$B\alpha=b$ for some $\alpha\in\mathbb{R}^n_+$ such that $\alpha_j=0$ whenever $j\notin J$, and*

(b) *For every $k\in[n]$ there exists $i\in[m]$ such that $c_{ik}\leqslant c_{ij}$ for all $j\in J$.*

(As usual, $[n]$ denotes the set $\{1, ..., n\}$.)

Proof of Theorem 1.1. Let $D=(V, A)$ be a clique-acyclic digraph. Let $K_1, ..., K_m$ be an enumeration of all maximal cliques in D. Form a new digraph D' by adding to D vertices $z_1, ..., z_m$, and all arcs of the form (u, z_i), $u\in K_i$. Let $K'_i=K_i\cup\{z_i\}$ $(i\leqslant m)$.

Since D is clique-acyclic, so is D'. Hence, on each clique K'_i there exists a linear order $>_i$ compatible with the irreversible arcs in K'_i, i.e., if $u, v\in K'_i$ and (u, v) is an irreversible arc in D' then $u>_i v$.

Let $w_1=z_1, ..., w_m=z_m, w_{m+1}, ..., w_n$ be an enumeration of the vertices of D'. Define an $m\times n$ matrix C in the following way: for each i, j such that $w_j\notin K'_i$ define $c_{ij}=M$, where M is some number larger than $|V|$. If $w_j\in K'_i$ let c_{ij} be the height of w_j in the order $>_i$ (that is, $c_{ij}=0$ if w_j is minimal in K'_i, namely if $w_j=z_i$; $c_{ij}=1$ if w_j is second from bottom in K'_i, etc.) Also let B be the incidence matrix of the cliques K'_i, i.e., b_{ij} is defined as 1 if $w_j\in K'_i$, and as 0 otherwise.

Apply now Theorem 1.2 to the matrices B and C, with b being the vector of all 1-s. Clearly, the conditions of the theorem are satisfied. Let J be a subset of $[n]$ as in the conclusion of the theorem, and let $\alpha\in\mathbb{R}^n_+$ be as in part (a) of the conclusion. Define a function f on V by $f(w_j)=\alpha_j$ for $j=m+1, ..., n$. Clearly, f is fractionally independent in D. We shall conclude the proof by showing that it is also strongly dominating.

Let $w_k \in V$, $k>m$. By (b) there exists $i \in [m]$ such that $c_{ik} \leqslant c_{ij}$ for all $j \in J$. Let $K_i^J = \{w_j \in K'_i \colon j \in J\}$. We claim that $K_i^J \subseteq I(w_k)$. (Note that $I(w_k)$ is contained in V and is the same set whether defined with respect to D or to D'.) Indeed, assume that $w_j \in K_i^J \backslash I(w_k)$. Then, since $w_j \in K'_i$, it follows that $c_{ij} < M$. As $c_{ik} \leqslant c_{ij}$ we have that $c_{ik} < M$, and hence $w_k \in K'_i$. Since w_j and w_k are both in K'_i, they are joined by at least one arc. But $w_j \notin I(w_k)$, and therefore (w_k, w_j) is an irreversible arc in D'. This implies that $w_k >_i w_j$, and hence $c_{ik} > c_{ij}$, which is a contradiction. Hence $K_i^J \subseteq I(w_k) \subseteq V$ and by (a) we have $\sum_{w_j \in K_i^J} \alpha_j = 1$. We have thus shown that f is strongly dominating. ▮

2. KERNELS IN PERFECT GRAPHS

By an *orientation* (or, in fact, a *super orientation*) of an undirected graph we shall understand the replacement of every edge by either an irreversible arc or a pair of oppositely directed arcs. A graph is called *kernel solvable* if every clique-acyclic orientation of it has a kernel. Berge and Duchet conjectured [1] that a graph is perfect if and only if it is kernel solvable.

Recently, Boros and Gurvich proved one direction of the conjecture, namely:

THEOREM 2.1 [2]. *A perfect graph is kernel solvable.*

(Many special cases of this conjecture had been proved before. Notably, the case of *i*-triangulated graphs [5] and that of the line graphs of bipartite graphs [3], and perfect line graphs in general [6].)

Theorem 2.1 will clearly follow from Theorem 1.1 and the following:

THEOREM 2.2. *An orientation of a perfect graph has a kernel if and only if it has a strong fractional kernel.*

For the proof we recall a well known fact (implicitly proved in [4]):

LEMMA 2.3. *Given any perfect graph with a non-negative weight function (not identically zero) on its vertices, one can find an independent set of the graph which meets all cliques of maximal weight.*

Proof of Theorem 2.2. As remarked above, a kernel is also a strong fractional kernel. It remains to be shown that the existence of a strong fractional kernel implies the existence of a kernel. Let D be an orientation of a perfect graph G, and let f be a strong fractional kernel in it. Since f is fractionally independent, the maximal f-weight of cliques in it is 1. Let S be an independent set meeting all cliques of weight 1, as guaranteed by Lemma 2.3. Since f is strongly dominating, for every vertex v there is a clique K of weight 1 contained in $I(v)$. Since S meets K, it follows that S dominates v. Thus S is a kernel. ∎

Remark. The original proof of Theorem 2.1 also used Theorem 1.2, but indirectly. Theorem 1.2 was used by Scarf in the proof of another (better known) theorem. Variations of the latter, formulated in terms of "effectivity functions," were used by Boros and Gurvich in their proof. The proof given here is much shorter, and does not require familiarity with game theoretic concepts.

3. A PROOF OF SCARF'S THEOREM

For the sake of completeness we include here a proof of Theorem 1.2, essentially the one given by Scarf himself.

Let $J \subseteq [n]$. A column c^k of C is said to be *J-subordinated at the index i* if $c_{ik} \leqslant c_{ij}$ for every $j \in J$. It is said to be *J-subordinated* if it is J-subordinated at some i. We say that J is *subordinating* (for C) if every column of C is J-subordinated. Clearly, if $J' \subseteq J$ and J is subordinating for C then so is J'. A subset J of size m of $[n]$ is called a *feasible basis* (*for the pair* (B, b)) if the columns b^j, $j \in J$, are linearly independent, and there exist non-negative numbers α_j, $j \in J$, such that $\sum_{j \in J} \alpha_j b^j = b$ (that is, b belongs to the cone spanned by the columns b^j, $j \in J$). Our aim is to prove that there exists a set J of size m which is both subordinating and a feasible basis.

We say that the pair (B, b) is *non-degenerate* if b is not in the cone spanned by fewer than m columns of B. We call C *ordinal-generic* if all the elements in each row of C are distinct.

From the definition of a feasible basis it follows that there exists a small perturbation b' of b such that the pair (B, b') is non-degenerate and every feasible basis for (B, b') is also a feasible basis for (B, b). Similarly, by slightly perturbing C we can obtain an ordinal-generic matrix C' satisfying the assumptions of the theorem, and if the perturbation is small enough then any subordinating set for C' is also subordinating for C. Thus we may (and henceforth will) assume that (B, b) is non-degenerate, and that C is ordinal-generic.

We shall need two lemmas, the first a basic and well known tool in linear programming, the other a fact about subordinating sets which forms the core of the proof of the theorem.

LEMMA 3.1. *Let J be a feasible basis for (B, b), and $k \in [n] \setminus J$. Then there exists a unique $j \in J$ such that $J + k - j$ (which is a common abuse of notation for $(J \cup \{k\}) \setminus \{j\}$) is a feasible basis.*

(This is the fact which is at the base of the simplex algorithm. The validity of the lemma requires the assumptions that the set $\{\alpha \in \mathbb{R}^n_+ : B\alpha = b\}$ is bounded and that (B, b) is non-degenerate.)

LEMMA 3.2. *Let K be a subordinating set for C of size $m-1$. Then there are precisely two elements $j \in [n] \setminus K$ such that $K + j$ is subordinating for C, unless $K \subseteq [m]$, in which case there exists precisely one such j.*

Proof. For each $i \in [m]$ let $k = t(i)$ be the element of K for which c_{ik} is minimal. (Remember that we are assuming distinctness of the elements in each row of C, hence k is unique, and the function t is well defined.) Since every column c^k, $k \in K$, is K-subordinated, it contains an element c_{ik} which is minimal among all c_{ij}, $j \in K$. That is, the map t is onto. Thus there exists precisely one element $h \in K$ such that $h = t(i)$ for two values of i, say i_1 and i_2, while every other $k \in K$ is equal to $t(i)$ for precisely one i.

If $K + j$ is subordinating for C then, by a similar argument to one made above, every column c^p, $p \in K + j$, contains a single element c_{ip} which is minimal among all c_{iq}, $q \in K + j$. Hence there is $a \in \{1, 2\}$ such that $c_{i_a j} < c_{i_a h}$, while c^j is not K-subordinated at i for any $i \neq i_a$. Denoting by $S_a (a = 1, 2)$ the set of those $j \in [n] \setminus K$ such that j is not K-subordinated at any $i \in [m] \setminus \{i_a\}$, it is easy to see that $K + j$ is subordinating for C if and only if j belongs to some S_a, $a = 1, 2$, and $c_{i_a j}$ is maximal among all $c_{i_a q}$, $q \in S_a$. Thus the lemma will be proved if we show that if $K \nsubseteq [m]$ then both S_a's are non-empty, while if $K \subseteq [m]$ then precisely one of them is non-empty.

Consider first the case that $K \nsubseteq [m]$, and let $j \in K \setminus [m]$. Then $i_1, i_2 \notin K$; indeed, if for example $i_1 \in K$ then $t(i_1) = i_1$ and hence also $t(i_2) = i_1$, but the latter is impossible because $c_{i_2 j} < c_{i_2 i_1}$. Moreover, $i_a \in S_a$ for $a = 1, 2$, since the fact that $c_{i i_a} > c_{ij}$ for all $i \in [m] \setminus \{i_a\}$ implies that c^{i_a} is not K-subordinated at any $i \in [m] \setminus \{i_a\}$. Thus $S_a \neq \emptyset$, $a = 1, 2$.

Consider next the case that $K \subseteq [m]$. Then $t(i) = i$ for $i \in K$, and therefore one of the i_a's, say i_1, is in K, and the other, i_2, is the member of $[m]$ not in K. Then $S_1 = \emptyset$ since for every $j \in [n] \setminus K$ the column c^j is K-subordinated at i_2. On the other hand, $S_2 = [n] \setminus K \neq \emptyset$. This, as remarked above, proves the lemma.

Let us now return to the proof of the theorem. Form a bipartite graph Γ with respective sides $\mathscr{F}$ and $\mathscr{S}$, where $\mathscr{F}$ is the set of all feasible bases containing 1, and $\mathscr{S}$ is the set of all subordinating sets of size m not containing 1. An element F of $\mathscr{F}$ and an element S of $\mathscr{S}$ are joined by an edge in Γ if $F \setminus S = \{1\}$.

Consider a set $F \in \mathscr{F}$ which is not subordinating, and assume that F has positive degree in Γ. Then the set $K = F \backslash \{1\}$ is subordinating. Applying Lemma 3.2 to K, we see that F has degree 2 in Γ, unless $F = [m]$, in which case it has degree 1. Note that $[m]$ is in $\mathscr{F}$, and, indeed, is not subordinating.

Next, consider a set $S \in \mathscr{S}$ which is not a feasible basis, and assume that S has positive degree in Γ. Let F be a neighbor of S, and let s be the single element of $S \backslash F$. By Lemma 3.1 there exists a unique element f of F such that $F' = F + s - f$ is a feasible basis. If $f = 1$ then $F' = S$, which contradicts our assumption about S. Thus $f \neq 1$, and F' is the unique element of $\mathscr{F}$, different from F, which is connected to S.

By the above, every vertex of Γ which is not both subordinating and a feasible basis has degree 0 or 2, apart from $[m]$, which has degree 1. Similar arguments show that a vertex which is both subordinating and a feasible basis, if it exists, has degree 1. Thus the connected component of Γ containing $[m]$ is a path, which must end at another vertex of degree 1, namely at a vertex which is both subordinating and a feasible basis. This proves the theorem. ∎

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