Discrete Mathematics 117 (1993) 107–112
North-Holland

Packing lines in a hypercube

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Received 6 June 1991

Abstract

Felzenbaum, A., R. Holzman and D.J. Kleitman, Packing lines in a hypercube, Discrete Mathematics 117 (1993) 107–112.

We characterize the n-tuples $(a_1, \ldots, a_n)$ for which one can find a_i lines in the ith direction in the n-cube, $i = 1, \ldots, n$, so that all lines are disjoint.

Let $Q^n = \{0, 1\}^n$ denote the n-dimensional hypercube; we shall refer to it simply as the *n-cube*. An *i-line* (or a line in the ith direction) in Q^n is a pair of vertices of Q^n which differ in the ith coordinate and agree in all other coordinates. We address the following question: Given n nonnegative integers $a_1, \ldots, a_n$, under what conditions can one find a_i i-lines in Q^n, for $i = 1, \ldots, n$, so that all lines are disjoint? When this is possible, we say that the n-tuple $(a_1, \ldots, a_n)$ is *implementable*.

An obvious necessary condition for implementability is $\sum_{i=1}^{n} a_i \leqslant 2^{n-1}$. To see that this condition is not sufficient, try to find two disjoint lines in the 2-cube, one in each direction. We shall treat first the case when $\sum_{i=1}^{n} a_i = 2^{n-1}$. In this case, a system of lines as required would necessarily form a partition of Q^n.

Theorem 1. *Suppose that* $n \geqslant 2$ *and* $\sum_{i=1}^{n} a_i = 2^{n-1}$. *A necessary and sufficient condition for* $(a_1, \ldots, a_n)$ *to be implementable is that all* a_i, $i = 1, \ldots, n$, *be even.*

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Proof. To prove the necessity of the condition, consider a partition of Q^n into lines consisting of a_i i-lines for each i. Fix some i and consider the intersections of the lines in the partition with the half-cube

$$H_0^i=\{x=(x_1,\dots,x_n)\in Q^n\colon\ x_i=0\}.$$

Every line in any direction $j\neq i$ is either contained in H_0^i or disjoint from it, and, thus, has an even intersection with H_0^i. Each of the a_i i-lines intersects H_0^i in one point. Since H_0^i has an even number of points, a_i must be even.

We prove the sufficiency of the condition by induction on n. The few possible even n-tuples for $n=2,3,4$ can be checked directly to be implementable. Assume then that $n\geqslant 5$ and that $(a_1,\dots,a_n)$ is given, with $\sum_{i=1}^n a_i=2^{n-1}$ and all a_i even. Assume also, w.l.o.g., that $a_1\leqslant a_2\leqslant\cdots\leqslant a_n$. We decompose the n-cube into three $(n-2)$-cubes Q_1,Q_2,Q_3 and two $(n-3)$-cubes Q_4,Q_5 as follows:

$$Q_1=\{x\in Q^n\colon\ x=(x_1,0,1,x_4,\dots,x_n)\},$$

$$Q_2=\{x\in Q^n\colon\ x=(1,x_2,0,x_4,\dots,x_n)\},$$

$$Q_3=\{x\in Q^n\colon\ x=(0,1,x_3,x_4,\dots,x_n)\},$$

$$Q_4=\{x\in Q^n\colon\ x=(0,0,0,x_4,\dots,x_n)\},$$

$$Q_5=\{x\in Q^n\colon\ x=(1,1,1,x_4,\dots,x_n)\}.$$

Thus, directions $1,2,3$ appear only in Q_1,Q_2,Q_3, respectively, while directions $4,\dots,n$ appear in all five pieces. The idea is to split our tasks among the five pieces, i.e., to partition each a_i, $i=4,\dots,n$, into even nonnegative integers a_i^k, $k=1,\dots,5$, so that

$$\sum_{k=1}^{5} a_i^k=a_i,\quad i=4,\dots,n,$$

and

$$a_k+\sum_{i=4}^{n} a_i^k=2^{n-3},\quad k=1,2,3,$$

$$\sum_{i=4}^{n} a_i^k=2^{n-4},\quad k=4,5.$$

If this splitting can be done, then we can rely on the induction hypothesis to implement the five lists of tasks in the respective five cubes $Q_1,\dots,Q_5$, thus obtaining an implementation of $(a_1,\dots,a_n)$ in Q^n. It remains to verify that such splitting is possible. Indeed, the only obstacle would be if $a_k>2^{n-3}$ for $k=1,2$ or 3. But, since the a_i are even and nondecreasing, this would imply that

$$\sum_{i=3}^{n} a_i\geqslant(n-2)(2^{n-3}+2),$$

which exceeds 2^{n-1} for $n\geqslant 5$. □

We consider next the case when $\sum_{i=1}^{n} a_i < 2^{n-1}$. Since for $\sum_{i=1}^{n} a_i = 2^{n-1}$ implementability requires that all a_i be even, one might expect that implementability would still imply some restrictions on the a_i when their sum is close to 2^{n-1}. Not so, it turns out.

Theorem 2. *If* $\sum_{i=1}^{n} a_i < 2^{n-1}$ *then* $(a_1, \ldots, a_n)$ *is implementable.*

Proof. We proceed by induction on n. The few possible n-tuples for $n=1,2,3$ can be checked directly to be implementable. Assume then that $n \geqslant 4$ and that $(a_1, \ldots, a_n)$ is given. Assume also, w.l.o.g., that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $\sum_{i=1}^{n} a_i = 2^{n-1} - 1$.

We treat first some cases when a_1 is small. The method here will be to decompose the n-cube into two $(n-1)$-cubes H_0^1 and H_1^1, defined by $x_1 = 0$ and $x_1 = 1$, respectively. Then we shall split our tasks in directions $2, \ldots, n$ among the two $(n-1)$-cubes, implement them by the induction hypothesis, and still be able to find a_1 1-lines going across from H_0^1 to H_1^1.

Suppose first that $a_1 = 0$. Then $\sum_{i=2}^{n} a_i = 2^{n-1} - 1$, and we partition each a_i, $i = 2, \ldots, n$, into two nonnegative integers b_i and c_i, so that

$$b_i + c_i = a_i, \quad i = 2, \ldots, n,$$

$$\sum_{i=2}^{n} b_i = 2^{n-2},$$

$$\sum_{i=2}^{n} c_i = 2^{n-2} - 1,$$

and all b_i are even. This can be done as long as there are less than 2^{n-2} odd numbers among $a_2, \ldots, a_n$; for this it suffices that $n - 1 < 2^{n-2}$, which holds true for $n \geqslant 4$. Now, we implement $(b_2, \ldots, b_n)$ in H_0^1 using Theorem 1 and $(c_2, \ldots, c_n)$ in H_1^1 using the induction hypothesis, thus obtaining an implementation of $(a_1, \ldots, a_n)$ in Q^n.

Suppose next that $a_1 = 1$. Then $\sum_{i=2}^{n} a_i = 2^{n-1} - 2$, and we partition each a_i, $i = 2, \ldots, n$, into two nonnegative integers b_i and c_i, so that

$$b_i + c_i = a_i, \quad i = 2, \ldots, n,$$

$$\sum_{i=2}^{n} b_i = \sum_{i=2}^{n} c_i = 2^{n-2} - 1.$$

This can always be done. By the induction hypothesis, we can implement $(b_2, \ldots, b_n)$ in H_0^1 and $(c_2, \ldots, c_n)$ in H_1^1, leaving two free vertices in each half-cube. We can arrange these implementations so that $(k, 0, \ldots, 0)$ remains free in H_k^1, $k = 0, 1$ (indeed, an automorphism of the cube defined by switching the entries in a subset of the coordinates transforms one implementation into another). Then the pair $(0, 0, \ldots, 0), (1, 0, \ldots, 0)$ can serve as the required 1-line, completing the implementation of $(a_1, \ldots, a_n)$.

Now, suppose that $a_1=2$. Then $\sum_{i=2}^{n} a_i=2^{n-1}-3$, and we partition each $a_i, i=2,\ldots,n$, into two nonnegative integers b_i and c_i, so that

$$b_i+c_i=a_i, \quad i=2,\ldots,n,$$

$$\sum_{i=2}^{n} b_i=2^{n-2}-1,$$

$$\sum_{i=2}^{n} c_i=2^{n-2}-2,$$

and all b_i but one are even. This can be done, by an argument similar to the one given above for $a_1=0$. Say b_2 is odd. We implement $(b_2+1,b_3,\ldots,b_n)$ in H_0^1 using Theorem 1 and $(c_2+1,c_3,\ldots,c_n)$ in H_1^1 using the induction hypothesis, arranging it so that $(k,0,\ldots,0)$ is on a 2-line in H_k^1, $k=0,1$. We then replace these two 2-lines by two 1-lines in Q^n using the same vertices: $(0,0,\ldots,0),(1,0,\ldots,0)$ and $(0,1,0,\ldots,0),(1,1,0,\ldots,0)$. Thus, we obtain an implementation of $(a_1,\ldots,a_n)$.

From now on, we assume that $a_1\geqslant 3$; therefore, either $n=5$ and the 5-tuple is $(3,3,3,3,3)$ or $n\geqslant 6$. An implementation of the former is depicted in Fig. 1. We assume henceforth that $n\geqslant 6$, and come to our main argument. For this argument, we use the decomposition of the n-cube into three $(n-2)$-cubes Q_1,Q_2,Q_3 and two $(n-3)$-cubes Q_4,Q_5 as in the proof of Theorem 1. We distinguish two possible cases according to the parity of a_1,a_2,a_3.

Suppose first that a_1,a_2 and a_3 are even. We partition each a_i, $i=4,\ldots,n$, into nonnegative integers a_i^k, $k=1,\ldots,5$, so that

$$\sum_{k=1}^{5} a_i^k=a_i, \quad i=4,\ldots,n,$$

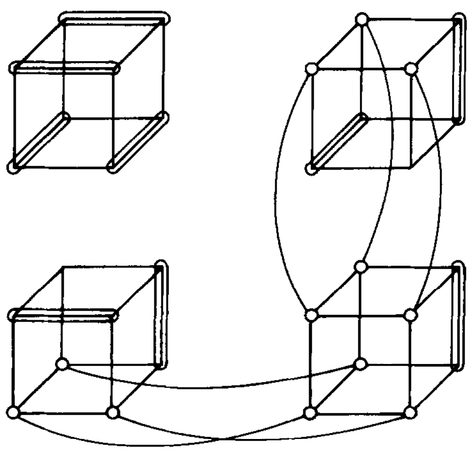

Fig. 1. An implementation of $(3,3,3,3,3)$ in Q^5.

and

$$a_k + \sum_{i=4}^{n} a_i^k = 2^{n-3}, \quad k = 1, 2, 3,$$

$$\sum_{i=4}^{n} a_i^4 = 2^{n-4},$$

$$\sum_{i=4}^{n} a_i^5 = 2^{n-4} - 1,$$

and a_i^k are even for $i = 4, \ldots, n$, $k = 1, \ldots, 4$. For this to be feasible, we need that (i) $a_3 \leqslant 2^{n-3}$, and (ii) there are less than 2^{n-4} odd numbers among $a_4, \ldots, a_n$. Indeed, (i) follows as in the proof of Theorem 1, and (ii) follows from $n - 3 < 2^{n-4}$, which holds for $n \geqslant 6$. Now, we can implement the respective lists of tasks in Q_1, Q_2, Q_3, Q_4 (by Theorem 1) and Q_5 (by the induction hypothesis), thus obtaining an implementation of $(a_1, \ldots, a_n)$ in Q^n.

Suppose next that at least one of a_1, a_2, a_3 is odd. Say a_1 is odd (the argument for a_2 or a_3 odd is similar). We partition each a_i, $i = 4, \ldots, n$, into nonnegative integers a_i^k, $k = 1, \ldots, 5$, so that

$$\sum_{k=1}^{5} a_i^k = a_i, \quad i = 4, \ldots, n,$$

and

$$a_1 - 1 + \sum_{i=4}^{n} a_i^1 = 2^{n-3},$$

$$a_k + \sum_{i=4}^{n} a_i^k = 2^{n-3} - 1, \quad k = 2, 3,$$

$$\sum_{i=4}^{n} a_i^k = 2^{n-4}, \quad k = 4, 5,$$

and a_i^k are even for $i = 4, \ldots, n$, $k = 1, 4, 5$. For this to be feasible, we need that (i) $a_3 \leqslant 2^{n-3} - 1$, and (ii) there are at most $2(2^{n-3} - 1) - (a_2 + a_3)$ odd numbers among $a_4, \ldots, a_n$. Indeed, (i) follows as above; for (ii) it suffices to show that

$$n - 3 \leqslant 2^{n-2} - 2 - (a_2 + a_3).$$

If this is false, then $a_2 + a_3 \geqslant 2^{n-2} - n + 2$, implying that

$$\sum_{i=1}^{n} a_i \geqslant a_1 + \frac{n-1}{2}(a_2 + a_3) \geqslant 3 + \frac{n-1}{2}(2^{n-2} - n + 2),$$

which exceeds 2^{n-1} for $n \geqslant 6$. Thus, a partition as above can be obtained. Now, we implement $(a_1 - 1, a_4^1, \ldots, a_n^1)$ in Q_1 and $(a_4^k, \ldots, a_n^k)$ in Q_k, $k = 4, 5$, using Theorem 1. Furthermore, we implement $(a_k, a_4^k, \ldots, a_n^k)$ in Q_k, $k = 2, 3$, using the induction hypothesis, and arranging it so that the points $(1, 1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$ remain free in

Q_2 and Q_3, respectively. These two points form the extra 1-line required to complete an implementation of $(a_1, \ldots, a_n)$ in Q^n. $\square$

Theorem 2 is a packing result for lines in the n-cube. It is natural to formulate a counterpart covering result. The latter is an easy consequence of the former.

Corollary. *Let* $a_1, \ldots, a_n$ *be integers satisfying* $0 \leqslant a_i \leqslant 2^{n-1}$, $i = 1, \ldots, n$, *and* $\sum_{i=1}^{n} a_i > 2^{n-1}$. *Then one can cover the vertices of* Q^n *using* a_i *i-lines, for* $i = 1, \ldots, n$.

Proof. We may assume that $\sum_{i=1}^{n} a_i = 2^{n-1} + 1$. Then we may reduce two of the a_i by 1, rendering the sum $2^{n-1} - 1$, implement the reduced n-tuple by Theorem 2, then cover the remaining two vertices with two lines in the reduced directions. $\square$

We conclude with a discussion of possible generalizations of our results. One way to go would be to consider, instead of lines, planes and/or higher-dimensional objects. To be concrete, an *ij-plane* is a set of 4 vertices of Q^n which agree in all coordinates except i, j. Given $\binom{n}{2}$ nonnegative integers a_{ij} summing to at most 2^{n-2}, under what conditions can one find a_{ij} *ij*-planes in Q^n, for each pair ij, so that all planes are disjoint? It is not clear what the analogue of Theorem 1 should be (evenness of the a_{ij} is still necessary when $n \geqslant 3$ and $\sum a_{ij} = 2^{n-2}$, but simple examples show that it is not sufficient). The analogue of Theorem 2 fails: it can be shown that if $n = 6$ and $a_{ij} = 1$ for each of the 15 pairs ij, no implementation exists.

Another type of generalization involves replacing the hypercube with 2 vertices on a line by one with k vertices on a line, for some integer $k \geqslant 2$. Thus, we consider $Q_k^n = \{0, 1, \ldots, k-1\}^n$, and by an *i-line* we mean a set of k vertices of Q_k^n which agree in all coordinates except i. Implementability of $(a_1, \ldots, a_n)$ is defined as in the case $k = 2$. If $n \geqslant 2$ and $\sum a_i = k^{n-1}$, a necessary condition for implementability is that all a_i be divisible by k. For $k = 3$, this condition can be shown to be sufficient too, with one exception: $n = 3$ and $a_1 = a_2 = a_3 = 3$. For higher k, there seem to be more exceptions, but it is plausible that the condition becomes sufficient if $n \geqslant N = N(k)$. If $\sum a_i < k^{n-1}$, there are low-dimensional examples where there is no implementation, but we do not know what happens in high dimensions.