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Optimal Fractional Matchings and Covers in Infinite Hypergraphs: Existence and Duality

Ron Aharoni[1] and Ron Holzman[2*]

[1] Department of Mathematics, Technion-Israel Institute of Technology, 32000 Haifa, Israel
[2] Department of Applied Mathematics and Computer Science, The Weizmann Institute of Science, 76100 Rehovot, Israel

Abstract. We study fractional matchings and covers in infinite hypergraphs, paying particular attention to the following questions: Do fractional matchings (resp. covers) of maximal (resp. minimal) size exist? Is there equality between the supremum of the sizes of fractional matchings and the infimum of the sizes of fractional covers? (This is called weak duality.) Are there a fractional matching and a fractional cover that satisfy the complementary slackness conditions of linear programming? (This is called strong duality.) In general, the answers to all these questions are negative, but for certain classes of infinite hypergraphs (classified according to edge cardinalities and vertex degrees) we obtain positive results. We also consider the question of the existence of optimal fractional matchings and covers that assume rational values.

1. Definitions and General Facts

In this paper a *hypergraph* will be a pair $H = (V, \mathscr{E})$, where V is any set (whose elements are the *vertices* of H) and $\mathscr{E}$ is a set of non-empty subsets of V (the members of $\mathscr{E}$ are the *edges* of H). Both V and $\mathscr{E}$ may be infinite; in the special case when both are finite we get the better known concept of a finite hypergraph.

A *matching* in H is a subset $\mathscr{F}$ of $\mathscr{E}$ whose members are pairwise disjoint. A *cover* in H is a subset T of V that intersects every edge of H. We denote:

$$\nu = \nu(H) = \sup\{|\mathscr{F}|: \mathscr{F} \text{ is a matching in } H\},$$

$$\tau = \tau(H) = \min\{|T|: T \text{ is a cover in } H\}.$$

We shall not be interested in the distinction between infinite cardinalities, and so ν and τ will assume either finite values or the value ∞.

A *fractional matching* in H is a function $f: \mathscr{E} \to [0, 1]$ such that, for every vertex $v \in V$, $\sum_{E \ni v} f(E) \leq 1$. The *size* of f is $|f| = \sum_{E \in \mathscr{E}} f(E)$. A *fractional cover* in H is a function $g: V \to [0, 1]$ such that, for every edge $E \in \mathscr{E}$, $\sum_{v \in E} g(v) \geq 1$. The *size* of g is $|g| = \sum_{v \in V} g(v)$. We denote:

* Incumbent of the Robert Edward and Roselyn Rich Manson Career Development Chair.

$$\nu^* = \nu^*(H) = \sup\{|f| : f \text{ is a fractional matching in } H\},$$

$$\tau^* = \tau^*(H) = \inf\{|g| : g \text{ is a fractional cover in } H\}.$$

By identifying subsets with their characteristic functions, matchings (resp. covers) are seen to be special cases of fractional matchings (resp. fractional covers). This implies the inequalities:

$$\nu \leq \nu^*, \tag{1a}$$

$$\tau \geq \tau^*. \tag{1b}$$

If f and g are a fractional matching and a fractional cover respectively, then the following is implied by the definitions:

$$|f| = \sum_{E \in \mathscr{E}} f(E) \leq \sum_{E \in \mathscr{E}} f(E) \sum_{v \in E} g(v) = \sum_{v \in V} g(v) \sum_{E \ni v} f(E) \leq \sum_{v \in V} g(v) = |g|. \tag{2}$$

Hence we have

$$\nu^* \leq \tau^*. \tag{3}$$

The pleasing effect of introducing fractional matchings and covers in the theory of finite hypergraphs is that equality holds in (3). This is a well-known consequence of the duality theorem of linear programming (see e.g. [3]), and we record it here as the benchmark for our investigation in the infinite case.

Proposition 1.1. *If H is a finite hypergraph, then ν^* and τ^* are attained (i.e., the supremum and the infimum are a maximum and a minimum respectively) and equal.*

We give now a simple example that indicates that the situation is not as nice in the infinite case.

Example 1.2. A hypergraph with $\nu^* = 1$, $\tau^* = \infty$.

We take $\mathbb{N}$, the set of natural numbers, as the set of vertices, and let the edges be all subsets of the form $E_i = \{j \in \mathbb{N} : j \geq i\}$, $i \in \mathbb{N}$. Clearly, if $\sum_{i \in \mathbb{N}} f(E_i) > 1$ then a finite partial sum already exceeds 1, but since any finite number of edges intersect this shows that f is not a fractional matching; thus $\nu^* = 1$. On the other hand, if $\sum_{j \in \mathbb{N}} g(j) < \infty$ then a tail of this series that is less than 1 corresponds to an edge that is not covered, showing that g is not a fractional cover; thus $\tau^* = \infty$.

In spite of this counterexample, we may still be able to extend the nice properties of finite hypergraphs to special classes of infinite hypergraphs. We shall study systematically the effects of restrictions on the cardinalities of the edges (namely, that they all be finite, or, moreover, that they be finitely bounded) and of similar restrictions on the degrees of the vertices.

One question in which we shall be interested is whether or not ν^* and τ^* are attained. Now, if $\tau^* = \infty$ then it is obviously attained, so the question of its attainment arises only when it is finite. We shall show next that the same is true of ν^*.

Proposition 1.3. *If $\nu^* = \infty$ then it is attained.*

Proof. For each $n \in \mathbb{N}$, let f_n be a fractional matching with $|f_n| \geq 2^n$. Then $f = \sum_{n \in \mathbb{N}} 2^{-n} f_n$ is a fractional matching of infinite size. □

It is interesting to note that this is not true of ν, as the following example indicates.

Example 1.4. A hypergraph with $\nu = \infty$ unattained.

Again, we take $\mathbb{N}$ as the set of vertices. For every prime p and integer a, $0 \leq a < p$, we have an edge $E_p^a = \{i \in \mathbb{N}: i \equiv a \pmod p\}$. Then for every prime p we have a matching $\mathscr{F}_p$ of size p consisting of the edges E_p^a, $0 \leq a < p$, but the Chinese Remainder Theorem implies that every matching is contained in some $\mathscr{F}_p$ and hence is finite.

Another question we shall consider is under what conditions does $\nu^* = \tau^*$ hold. If $\nu^*(H) = \tau^*(H)$ we say that H has the *weak duality property*. We note that, by analysing the conditions for equality to hold in (2), the weak duality property follows from the existence of a fractional matching f and a fractional cover g, satisfying:

$$\sum_{v \in E} g(v) = 1 \qquad \text{whenever } f(E) > 0, \tag{4a}$$

$$\sum_{E \ni v} f(E) = 1 \qquad \text{whenever } g(v) > 0. \tag{4b}$$

These are the so-called *complementary slackness conditions* of linear programming duality. A pair consisting of a fractional matching f and a fractional cover g that satisfy (4) is called *orthogonal*. A hypergraph H has the *strong duality property* if it has an orthogonal pair. The following fact follows easily from (2).

Proposition 1.5. *Suppose $\nu^*(H) < \infty$. Then H has the strong duality property if and only if it has the weak duality property and ν^* and τ^* are attained.*

When $\nu^* = \infty$, however, weak duality and the attainment of ν^* and τ^* follow automatically from (3) and Proposition 1.3, whereas strong duality is more demanding and seems like the meaningful property to look for. Indeed, it has been established that all *graphs* have the strong duality property (in [1] for the countable case, in [2] for general graphs).

In the case of graphs, an orthogonal pair whose members assume values in $\{0, \frac{1}{2}, 1\}$ always exists. In the case of finite hypergraphs this is not true, but an orthogonal pair whose members assume rational values always exists. This leads to the following question about infinite hypergraphs: to the extent that optimal fractional matchings and covers exist, may we restrict their values to be rational? We shall have some things to say, both positive and negative, on this question.

We end this section with a few additional facts that will be useful in our investigation.

Lemma 1.6. *$\tau^* < \infty$ if and only if $\tau < \infty$.*

Proof. One direction follows from (1b). Conversely, suppose $\tau^* < \infty$, and let g be a fractional cover with $|g| < \infty$. Let U be a finite subset of V with $\sum_{v \in U} g(v) > |g| - 1$. Then, clearly, U is a cover. □

It is interesting to note that ν and ν^* are not tied in the same way, as the following example illustrates.

Example 1.7. A hypergraph with $\nu = 1$, $\nu^* = \infty$.

We take the plane $\mathbb{R}^2$ as the set of vertices, and let the edges be infinitely many lines in general position (i.e., no two parallel, no three through a point). Then clearly $\nu = 1$, but the fractional matching assigning $\frac{1}{2}$ to every edge shows that $\nu^* = \infty$.

We shall consider two ways of obtaining a hypergraph on a subset of V from a given hypergraph on V.

Notation 1.8. Let $H = (V, \mathscr{E})$ be a hypergraph and let $U \subseteq V$.
(a) $H^U = (U, \mathscr{E}^U)$, where $\mathscr{E}^U = \{E \in \mathscr{E}: E \subseteq U\}$. Given $f: \mathscr{E}^U \to [0,1]$, the extension of f to $\mathscr{E}$ that assumes the value 0 on $\mathscr{E} \backslash \mathscr{E}^U$ is denoted by $\tilde{f}$. Given $g: V \to [0,1]$, the restriction of g to U is denoted $g \lceil U$.
(b) Suppose that U is a cover in H. $H_U = (U, \mathscr{E}_U)$, where $\mathscr{E}_U = \{E \cap U: E \in \mathscr{E}\}$. Given $E' \in \mathscr{E}_U$, $\mathscr{E}(E') = \{E \in \mathscr{E}: E \cap U = E'\}$. Given $f: \mathscr{E} \to [0,1]$ and $E' \in \mathscr{E}_U$, $f_U(E') = \sum_{E \in \mathscr{E}(E')} f(E)$. Given $g: U \to [0,1]$, the extension of g to V that assumes the value 0 on $V \backslash U$ is denoted $\tilde{g}$.

Lemma 1.9. *Let $H = (V, \mathscr{E})$ be a hypergraph and let $U \subseteq V$.*
(a) *If f is a fractional matching in H^U then $\tilde{f}$ is a fractional matching in H. If g is a fractional cover in H then $g \lceil U$ is a fractional cover in H^U. Consequently, $\nu^*(H^U) \le \nu^*(H)$ and $\tau^*(H^U) \le \tau^*(H)$.*
(b) *Suppose that U is a cover in H. If f is a fractional matching in H, then f_U is a fractional matching in H_U. If g is a fractional cover in H_U then $\tilde{g}$ is a fractional cover in H. Consequently, $\nu^*(H) \le \nu^*(H_U)$ and $\tau^*(H) \le \tau^*(H_U)$.* □

2. Hypergraphs with Finite Edges

$H = (V, \mathscr{E})$ is said to be a *hypergraph with finite edges* if $|E| < \infty$ for all $E \in \mathscr{E}$.

Theorem 2.1. *If H is a hypergraph with finite edges, then τ^* is attained and $\nu^* = \tau^*$.*

Proof. We shall work in the topological space $Y = [0,1]^V$, whose elements are all functions $g: V \to [0,1]$ and whose topology is the product topology. By Tychonoff's theorem, Y is compact. For each finite subset U of V, we let

$$\Gamma^U = \{g \in Y: |g| \le \nu^*(H) \text{ and } g \lceil U \text{ is a fractional cover in } H^U\}$$

(see Notation 1.8(a)). It is easy to verify that Γ^U is closed. We claim also that $\Gamma^U \ne \varnothing$. Indeed, let g_0 be a fractional cover in H^U of size $\nu^*(H^U)$; the existence of such g_0 follows from Proposition 1.1. Then extending g_0 to be 0 on $V \backslash U$ yields $g \in \Gamma^U$ (since $|g| = |g_0| = \nu^*(H^U) \le \nu^*(H)$ by Lemma 1.9(a)). The family of sets Γ^U has the finite intersection property, because $\bigcap_{i=1}^n \Gamma^{U_i}$ contains $\Gamma^{\bigcup_{i=1}^n U_i}$ which, by the above, is non-empty. We conclude, therefore, from compactness, that $\Gamma = \bigcap \Gamma^U$ (where the intersection ranges over all finite subsets U of V) is non-empty. Let $g \in \Gamma$. Since every edge of H, being finite, is an edge of some H^U, it follows that g is a fractional cover in H. But $|g| \le \nu^*(H) \le \tau^*(H)$, so both of these inequalities must hold with equality. □

We have already seen (Example 1.2) that $\nu^* = \tau^*$ cannot be asserted for general hypergraphs. We shall show now that the attainment of τ^* is not guaranteed either (without the assumption that the edges are finite).

Example 2.2. A hypergraph with τ^* unattained.

As is well-known, for every rational number $t \geq 1$ there exists a finite hypergraph H with $\tau^*(H) = t$ (indeed, if $t = m/n$ let V be an m-element set and let the edges be all its n-element subsets). Hence we can take a sequence $(H_i = (V_i, \mathscr{E}_i))_{i \in \mathbb{N}}$ of finite hypergraphs with V_i pairwise disjoint and $\tau^*(H_i) = t_i$ forming a strictly decreasing sequence of rational numbers ≥ 1. We write $t = \lim_{i \to \infty} t_i$, and let $H = (V, \mathscr{E})$, where $V = \bigcup_{i \in \mathbb{N}} V_i$ and $E \in \mathscr{E}$ if and only if $E \cap V_i \in \mathscr{E}_i$ for each $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, there is a fractional cover in H of size t_i. Indeed, we may take a fractional cover in H_i of size t_i and extend it to be 0 on $V \backslash V_i$. Now let g be an arbitrary fractional cover of finite size in H. We write $g_i = g \lceil V_i$, $s_i = |g_i|$, $c_i = \min\{\sum_{v \in E} g_i(v) \colon E \in \mathscr{E}_i\}$ $(i \in \mathbb{N})$. For any $i \in \mathbb{N}$ with $c_i > 0$, the function $\bar{g}_i(v) = \min\{g_i(v)/c_i, 1\}$ $(v \in V_i)$ is a fractional cover in H_i of size $\leq s_i/c_i$; hence $t_i \leq s_i/c_i$, or $c_i \leq s_i/t_i$. The latter also holds if $c_i = 0$, so it holds for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $E_i \in \mathscr{E}_i$ satisfy $\sum_{v \in E_i} g_i(v) = c_i$. Let $E = \bigcup_{i \in \mathbb{N}} E_i$. Then $E \in \mathscr{E}$ and we get:

$$1 \leq \sum_{v \in E} g(v) = \sum_{i \in \mathbb{N}} \sum_{v \in E_i} g_i(v) = \sum_{i \in \mathbb{N}} c_i \leq \sum_{i \in \mathbb{N}} \frac{s_i}{t_i} < \frac{1}{t} \sum_{i \in \mathbb{N}} s_i = \frac{|g|}{t}.$$

Thus $|g| > t$ for any fractional cover g in H, and, as we have seen, there exist fractional covers whose sizes come arbitrarily close to t. Hence $\tau^*(H) = t$ and is unattained.

Our next example shows that one cannot add the attainment of ν^* to the assertions of Theorem 2.1, and therefore that not all hypergraphs with finite edges have the strong duality property.

Example 2.3. A hypergraph with finite edges and ν^* unattained.

We take the set of vertices to consist of two copies of the set of nonnegative integers: formally,

$$V = \{v_j^i \colon i = 0, 1 \text{ and } j = 0, 1, 2, \ldots\}.$$

The edges are all sets of the form:

$$E_j^i = \{v_0^i, v_1^i, \ldots, v_j^i, v_j^{1-i}\}, \qquad i = 0, 1 \text{ and } j = 1, 2, \ldots$$

For each $n \in \mathbb{N}$, we define a fractional matching f_n by:

$$f_n(E_j^i) = \begin{cases} \dfrac{1}{2^{n-j}} & \text{if } j < n, \\ 0 & \text{if } j \geq n. \end{cases}$$

Since $|f_n| = 2 - 2^{-(n-2)}$, we have $\nu^* \geq 2$. Now, let f be an arbitrary fractional matching of positive size. Let $j_0 = \min\{j \colon f(E_j^0) + f(E_j^1) > 0\}$, and assume w.l.o.g. that $f(E_{j_0}^0) = a > 0$. Then $\sum_{j=1}^{\infty} f(E_j^1) = \sum_{j=j_0}^{\infty} f(E_j^1) \leq 1 - a$ since all E_j^1, $j \geq j_0$, share the vertex $v_{j_0}^1$ with $E_{j_0}^0$. Also, $\sum_{j=1}^{\infty} f(E_j^0) \leq 1$ since all these edges share v_0^0. Hence $|f| \leq 2 - a$, and so every fractional matching has size <2. Thus $\nu^* = 2$ and is unattained.

In the last example, the edges are finite but their cardinalities are not finitely bounded. A hypergraph $H = (V, \mathscr{E})$ is said to have *finite rank* if there exists $r < \infty$ such that $|E| \leq r$ for all $E \in \mathscr{E}$; the smallest such r is the *rank* of H. We shall see that the problem illustrated by the example disappears when finite rank is assumed.

Theorem 2.4. *If H has finite rank then ν^* is attained.*

Proof. Let $H = (V, \mathscr{E})$ have rank r. By Proposition 1.3 we may assume that $\nu^* < \infty$. This implies, by Theorem 2.1 and Lemma 1.6, that $\tau < \infty$. Let U_1 be a finite cover in H. We proceed to define inductively a chain $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r$ of finite subsets of V. Suppose U_i is already defined, and $i < r$. For each $A \subseteq U_i$ with $|A| = i$, we consider the hypergraph $H_i\langle A\rangle = (V \backslash U_i, \mathscr{E}_i\langle A\rangle)$, where

$$\mathscr{E}_i\langle A\rangle = \{B \subseteq V\backslash U_i \colon B \neq \varnothing \text{ and } A \cup B \in \mathscr{E}\}.$$

We let

$$\mathscr{F}_i = \{A \subseteq U_i \colon |A| = i \text{ and } \tau(H_i\langle A\rangle) < \infty\}.$$

For each $A \in \mathscr{F}_i$, we choose a finite cover $T_i(A)$ in $H_i\langle A\rangle$. We define

$$U_{i+1} = U_i \cup \left(\bigcup_{A \in \mathscr{F}_i} T_i(A) \right).$$

Having completed the inductive definition, we write $U = U_r$, and consider the hypergraph H_U (see Notation 1.8(b)). Let f' be a fractional matching in H_U of size $\nu^*(H_U)$. Let $E'_1, E'_2, \ldots, E'_s$ be an enumeration of the support of f'.

Assertion. It is possible to choose distinct edges $E_1, E_2, \ldots, E_s$ of H, so that the function f defined on $\mathscr{E}$ by $f(E_j) = f'(E'_j)$ for $j = 1, \ldots, s$ and $f(E) = 0$ otherwise is a fractional matching in H.

The theorem follows from the assertion, because $|f| = |f'| = \nu^*(H_U) \geq \nu^*(H)$ by Lemma 1.9(b). To prove the assertion, we assume that E_j have been chosen for all $j < k$, and show how to choose E_k. If $E'_k \in \mathscr{E}$, let $E_k = E'_k$. Assume now that $E'_k \notin \mathscr{E}$. By the definition of H_U, we have $E'_k = E \cap U$ for some $E \in \mathscr{E}$. For $i = 1, \ldots, r$, we denote $E^i_k = E'_k \cap U_i = E \cap U_i$. Since U_1 is a cover in H and $E \in \mathscr{E}$, we have $|E^1_k| \geq 1$. Since $|E| \leq r$ and $E^r_k \subset E$ (the containment is strict because otherwise $E'_k = E \in \mathscr{E}$), we have $|E^r_k| < r$. Hence there is some i, $1 \leq i < r$, such that $|E^i_k| \geq i$ but $|E^{i+1}_k| < i + 1$. In fact, for this i we have $|E^i_k| = |E^{i+1}_k| = i$. If E^i_k belonged to $\mathscr{F}_i$, then U_{i+1}, by construction, would intersect $E \backslash E^i_k$, resulting in $|E^{i+1}_k| > i$. Thus $E^i_k \notin \mathscr{F}_i$, so it must be the case that $\tau(H_i\langle E^i_k\rangle) = \infty$. Hence there exists $B \in \mathscr{E}_i\langle E^i_k\rangle$ that is disjoint from $U \cup (\bigcup_{j<k} E_j)$. Let $E_k = E^i_k \cup B$. It is straightforward to verify that this construction satisfies the requirements of the assertion. □

Corollary 2.5. *If H has finite rank and $\nu^* < \infty$ then H has the strong duality property.*

Proof. By Theorems 2.1 and 2.4 and Proposition 1.5. □

The question of whether or not all hypergraphs of finite rank have the strong duality property is left open.

We mention briefly another result that can be proved for hypergraphs of finite rank by a method similar to that employed to prove Theorem 2.4. If $H = (V, \mathscr{E})$ is a hypergraph and $T \subseteq V$, we say that T is *fractionally matchable* if there exists a fractional matching f in H that covers T, in the sense that $\sum_{E \ni v} f(E) = 1$ for all $v \in T$.

Theorem 2.6. *Let $H = (V, \mathscr{E})$ have finite rank and let T be a finite subset of V. Then T is fractionally matchable if and only if there does not exist a function h: $V \to \mathbb{R}$ such that:*
(i) $h(v) \geq 0$ *for all* $v \in V \setminus T$,
(ii) $\sum_{v \in E} h(v) \geq 0$ *for all* $E \in \mathscr{E}$,
(iii) $\sum_{v \in V} h(v) < 0$.

Proof. (sketch) The necessity of the condition is easily verified. The sufficiency follows, in the case of finite hypergraphs, from Farkas' Lemma. To extend this to infinite hypergraphs of finite rank, we use the same construction of a chain $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_r$ of finite subsets of V as in the proof of Theorem 2.4, starting here from $U_1 = T$. We write $U = U_r$, we let H' be the hypergraph obtained from H by keeping only those edges that intersect T, and consider H'_U. A function h with properties (i)–(iii) does not exist for H'_U (or else it could be extended to one for H), and so, by the finite case, T is fractionally matchable in H'_U. Given a fractional matching f' in H'_U that covers T, it can be transformed into a fractional matching f in H' (hence in H) that covers T, by the argument given to prove the assertion in the proof of Theorem 2.4. □

3. Hypergraphs with Finite Degrees

Given a hypergraph $H = (V, \mathscr{E})$ and $v \in V$, we write $\mathscr{E}_v = \{E \in \mathscr{E}: v \in E\}$. The *degree* of a vertex v is $d(v) = d_H(v) = |\mathscr{E}_v|$. We say that H is a *hypergraph with finite degrees* if $d(v) < \infty$ for all $v \in V$.

Proposition 3.1. *If H is a hypergraph with finite degrees and infinitely many edges then $\tau^* = \infty$.*

Proof. By Lemma 1.6 it suffices to show that $\tau = \infty$. Assume, to the contrary, that T is a finite cover in H. Through every vertex in T there are finitely many edges, so only a finite number of edges can intersect T. This contradicts the assumption that there are infinitely many edges and all of them intersect T. □

Clearly, infinite hypergraphs with finitely many edges are just finite hypergraphs in disguise, so they are not interesting from our point of view. Thus, in the case of finite degrees we have $\tau^* = \infty$ and the question of its attainment does not arise. The other questions are answered in the negative by the following construction.

Example 3.2. A hypergraph with finite degrees and ν^* unattained.

We indicate a general method to transform an arbitrary hypergraph into one with finite degrees and the same fractional matchings (under a one-to-one correspondence between their edge sets). Given an arbitrary hypergraph $H = (V, \mathscr{E})$, we let

$$\tilde{V} = \{(v, \mathscr{F}) : v \in V, \mathscr{F} \subseteq \mathscr{E}_v \text{ and } |\mathscr{F}| < \infty\}.$$

Given an edge $E \in \mathscr{E}$, we let

$$\tilde{E} = \{(v, \mathscr{F}) \in \tilde{V} : v \in E \in \mathscr{F}\}.$$

We write $\tilde{\mathscr{E}} = \{\tilde{E} : E \in \mathscr{E}\}$ and $\tilde{H} = (\tilde{V}, \tilde{\mathscr{E}})$. It is easy to verify that the transformation $H \to \tilde{H}$ has the required properties. Now, we can apply the transformation to Example 2.3 and obtain a hypergraph with finite degrees and $\nu^* = 2$ unattained. As $\tau^* = \infty$ by Proposition 3.1, the weak duality property is violated too.

In the above example, the degrees are finite but not finitely bounded. Also, the edges are infinite. So, we ask now what happens if we rule out one or the other of these features. First, a hypergraph $H = (V, \mathscr{E})$ is said to have *finitely bounded degrees* if there exists $D < \infty$ such that $d(v) \leq D$ for all $v \in V$.

Proposition 3.3. *If H has finitely bounded degrees and infinitely many edges then $\nu^* = \infty$.*

Proof. We can define a fractional matching of infinite size by $f(E) = 1/D$ for all $E \in \mathscr{E}$, where $D = \max\{d(v) : v \in V\}$. □

Thus, in the case of finitely bounded degrees we automatically get the attainment of ν^* and τ^* and the weak duality property. We do not know, however, if the strong duality property must hold in this case.

Coming back to the other feature of Example 3.2 mentioned above (the edges being infinite), we shall show now that if we rule that out and keep the assumption of finite degrees we get strong duality.

Theorem 3.4. *If H is a hypergraph with finite edges and finite degrees then H has the strong duality property.*

Proof. We shall work in the topological space $X \times Y$, where $X = [0,1]^{\mathscr{E}}$, $Y = [0,1]^V$ and the topology is the product topology. By Tychonoff's theorem, $X \times Y$ is compact. For each finite subset U of V, we let

$$\Delta^U = \left\{ (f, g) \in X \times Y : \begin{array}{l} f \text{ is a fractional matching in } H, \\ g \lceil U \text{ is a fractional cover in } H^U, \\ f \text{ and } g \text{ satisfy (4)} \end{array} \right\}$$

(see Notation 1.8(a)). It is easy to verify, using the finiteness of the edges and the degrees, that Δ^U is closed. We can take an orthogonal pair in H^U and extend its members to assume the value 0 elsewhere, thus obtaining $(f, g) \in \Delta^U$ and showing that $\Delta^U \neq \varnothing$. The family of sets Δ^U has the finite intersection property, because $\bigcap_{i=1}^n \Delta^{U_i} \supseteq \Delta^{\bigcup_{i=1}^n U_i} \neq \varnothing$. Thus, it follows from compactness that $\Delta = \bigcap \Delta^U$ (where the intersection ranges over all finite subsets U of V) is non-empty. Let $(f, g) \in \Delta$.

Since every edge of H is an edge of some H^U, we conclude that g is a fractional cover in H and therefore (f, g) is an orthogonal pair in H. □

We summarize our results, examples an open questions for the various classes of hypergraphs in Table 1.

Table 1. A summary

<table>
<tr><th>Degrees
Edges</th><th>finitely bounded degrees</th><th>finite degrees</th><th>arbitrary degrees</th></tr>
<tr><td>finite rank</td><td rowspan="2" colspan="2">strong duality
$\nu^* = \tau^* = \infty$
(unless $|\mathscr{E}| < \infty$)</td><td>strong duality if $\nu^* < \infty$, otherwise open</td></tr>
<tr><td>finite edges</td><td>weak duality
τ^* attained
ν^* need not be attained</td></tr>
<tr><td>arbitrary edges</td><td>$\nu^* = \tau^* = \infty$
(unless $|\mathscr{E}| < \infty$)
strong duality open</td><td>$\tau^* = \infty$
(unless $|\mathscr{E}| < \infty$)
ν^* may be finite and unattained</td><td>ν^*, τ^* need not be attained or equal</td></tr>
</table>

4. Rationality

We say that a fractional matching or a fractional cover is *rational* if it assumes only rational values. Even in the case of finite hypergraphs, not all optimal fractional matchings and covers are rational. But, in that case, irrationality arises only in an inessential way: there always exist rational optimal fractional matchings and covers, and, in fact, any optimal fractional matching (resp. cover) is a convex combination of rational optimal fractional matchings (resp. covers). In the infinite case, as we have seen, optimal fractional matchings and covers need not exist at all. But when they do exist, it is interesting to know if rational ones exist too.

We start by observing that in the case of finite rank there always exist rational fractional matchings and covers of size $\nu^* = \tau^*$. Indeed, if $\nu^* = \infty$ this is trivial, and if $\nu^* < \infty$ it follows easily from the proof of Theorem 2.4.

If, instead of finite rank, we require only finite edges, then τ^* still has to be attained, but not necessarily at a rational fractional cover.

Example 4.1. A hypergraph with finite edges and a unique minimal fractional cover that is not rational.

We take the set of vertices to consist of two copies of $\mathbb{N}$ and two additional vertices: formally,

$$V = \{v_j^i \colon i = 0, 1 \text{ and } j \in \mathbb{N}\} \cup \{u, \bar{u}\}.$$

We choose an arbitrary irrational number $\alpha \in (0, 1)$, and define a weight function w

on V by:

$$w(v_j^i) = \frac{1}{2^j}(i = 0, 1, j \in \mathbb{N}),\ w(u) = \alpha \quad \text{and} \quad w(\bar{u}) = 1 - \alpha.$$

For any subset S of V, we denote by $w(S)$ the total weight of the elements of S. We let

$$\mathscr{E} = \{E \subset V: |E| < \infty \text{ and } w(E) \geq 1\}.$$

We let $H = (V, \mathscr{E})$. By the definition of $\mathscr{E}$, the function w: $V \to [0, 1]$ is a fractional cover in H, and therefore $\tau^*(H) \leq |w| = 3$. For $k \in \mathbb{N}$ we denote

$$V_k = \{v_j^i: i = 0, 1 \text{ and } j < k\},$$

and consider the hypergraph H^{V_k} (see Notation 1.8(a)). We introduce names for the following edges of H^{V_k}:

$$E_1 = \{v_1^0, v_1^1\}, \qquad E_j^i = \{v_1^i, v_2^i, \ldots, v_j^i, v_j^{1-i}\}, \quad i = 0, 1 \text{ and } 1 < j < k.$$

The following function f_k: $\mathscr{E}^{V_k} \to [0, 1]$ is a fractional matching in H^{V_k}:

$$f_k(E) = \begin{cases} \dfrac{1}{2^{k-j}} & \text{if } E = E_j^i, \\[2ex] \dfrac{1}{2^{k-2}} & \text{if } E = E_1, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Hence $\nu^*(H^{V_k}) \geq |f_k| = 2 - 2^{-(k-2)}$. Each f_k can be extended to a fractional matching in H by assigning the value 1 to the edge $\{u, \bar{u}\}$ and the value 0 elsewhere, and therefore $\nu^*(H) \geq 3$. This, together with $\tau^*(H) \leq |w| = 3$, implies that $\tau^*(H) = 3$ and w is a minimal fractional cover in H. It remains to show that w is unique.

Suppose that g is a fractional cover in H of size 3. We denote:

$$g(v_j^i) = x_j^i \ (i = 0, 1, j \in \mathbb{N}), \quad g(u) = x \quad \text{and} \quad g(\bar{u}) = \bar{x}.$$

Since $\sum_{j \in \mathbb{N}} x_j^0 \leq 3$, it follows that $x_j^0 \to 0$. For each $j \in \mathbb{N}$, we must have $\sum_{l=1}^{j} x_l^1 + x_j^0 \geq 1$ in order to cover the edge E_j^1. Letting $j \to \infty$ we get $\sum_{l=1}^{\infty} x_l^1 \geq 1$. By an analogous argument $\sum_{l=1}^{\infty} x_l^0 \geq 1$. Moreover, $x + \bar{x} \geq 1$ in order to cover $\{u, \bar{u}\}$. As $|g| = 3$, these three inequalities must hold with equality. It follows that $\bar{x} = 1 - x$ and $\sum_{l=1}^{\infty} x_l^i = 1$ for $i = 0, 1$. But, in fact, the above argument can be adapted to show that every series of the form $\sum_{l=1}^{\infty} x_l$, where $x_l \in \{x_l^0, x_l^1\}$ for each l, sums to 1. This implies that $x_l^0 = x_l^1$ for each l. From now on we drop the superscripts and write $x_j = g(v_j^0) = g(v_j^1)$, and remember that $\sum_{j \in \mathbb{N}} x_j = 1$.

Next, we define $A \subset \mathbb{N}$ by $\alpha = \sum_{j \in A} 2^{-j}$, and note that, since α is irrational, both A and $\bar{A} = \mathbb{N} \backslash A$ are infinite. For any $l \in \bar{A}$, we denote:

$$A_l = \{j \in A: j < l\} \quad \text{and} \quad B_l = A_l \cup \{l\}.$$

We have $\sum_{j \in B_l} 2^{-j} > \alpha$ and therefore the following is an edge in H for every $l \in \bar{A}$:

$$C_l = \{v_j^0: j \in B_l\} \cup \{\bar{u}\}.$$

It follows that $\sum_{j\in A_l} x_j + x_l + 1 - x \geq 1$ for every $l \in \bar{A}$. Letting $l \to \infty$ through $\bar{A}$ we get $\sum_{j\in A} x_j \geq x$. By analogous argument $\sum_{j\in\bar{A}} x_j \geq 1 - x$, and since $\sum_{j\in\mathbb{N}} x_j = 1$ we must have equality in both:

$$\sum_{j\in A} x_j = x \quad \text{and} \quad \sum_{j\in\bar{A}} x_j = 1 - x.$$

Now, let k, l, $m \in \mathbb{N}$ with $k < l < m$, $k \in A$ and $l \in \bar{A}$. Then $(C_l \backslash \{v_k^0\}) \cup \{v_{k+1}^1, v_{k+2}^1, \ldots, v_m^1, v_m^0\}$ is an edge in H, because C_l is and the total weight of the additional vertices equals that of the removed v_k^0. It follows that

$$\sum_{j\in A_l} x_j + x_l + 1 - x - x_k + \sum_{j=k+1}^{m} x_j + x_m \geq 1.$$

Letting first $m \to \infty$, then $l \to \infty$ through $\bar{A}$, we get that $x_k \leq \sum_{j=k+1}^{\infty} x_j$ for every $k \in A$. An analogous argument shows this to be true for every $k \in \bar{A}$, hence for every $k \in \mathbb{N}$. Since $g \lceil V_{k+1}$ is a fractional cover in $H^{V_{k+1}}$ (see the definition of V_k above and Lemma 1.9(a)), we have

$$2 \sum_{j=1}^{k} x_j \geq v^*(H^{V_{k+1}}) \geq 2 - 2^{-(k-1)},$$

implying that $\sum_{j=1}^{k} x_j \geq 1 - 2^{-k}$ and therefore $\sum_{j=k+1}^{\infty} x_j \leq 2^{-k}$. Combining this with a previous inequality we get $x_k \leq 2^{-k}$ for every $k \in \mathbb{N}$. But since $\sum_{k\in\mathbb{N}} x_k = 1$, we must have $x_k = 2^{-k}$ for each $k \in \mathbb{N}$. Finally,

$$x = \sum_{j\in A} x_j = \sum_{j\in A} 2^{-j} = \alpha.$$

We have shown that g coincides with w as required.

In this example v^* is not attained (in fact, the example is a development of Example 2.3). If we want to study the rationality of maximal fractional matchings in hypergraphs with finite edges, we must make the assumption that v^* is attained. Then we have the following result.

Theorem 4.2. *Let H be a hypergraph with finite edges and $v^* < \infty$, and suppose that v^* is attained. Then v^* is rational and there exists an orthogonal pair whose members are rational.*

Proof. Let (f, g) be an orthogonal pair in $H = (V, \mathscr{E})$; the existence of such a pair follows from Proposition 1.5, Theorem 2.1 and our assumptions. Let

$$U = \left\{ v \in V : \sum_{E\ni v} f(E) > \frac{1}{2} \right\}.$$

Assertion. U is finite.

Suppose not. Take $v_1 \in U$ and a finite subset $\mathscr{E}_1$ of $\mathscr{E}_{v_1}$ (the set of edges containing v_1) such that $\sum_{E\in\mathscr{E}_1} f(E) > \frac{1}{2}$. Now $U \backslash \bigcup \mathscr{E}_1 \neq \varnothing$, since U is infinite and $\bigcup \mathscr{E}_1$ is a finite union of finite sets. Take $v_2 \in U \backslash \bigcup \mathscr{E}_1$ and a finite subset $\mathscr{E}_2$ of $\mathscr{E}_{v_2}$ such that $\sum_{E\in\mathscr{E}_2} f(E) > \frac{1}{2}$. Again, $U \backslash [(\bigcup \mathscr{E}_1) \cup (\bigcup \mathscr{E}_2)] \neq \varnothing$. Continuing this way, one obtains infinite sequences $v_1, v_2, v_3, \ldots$ and $\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \ldots$ with the properties: (a) v_i

belongs to every edge in $\mathscr{E}_i$ but to no edge in $\mathscr{E}_j$, $j < i$, and (b) $\sum_{E \in \mathscr{E}_i} f(E) > \frac{1}{2}$. Now (a) implies that the $\mathscr{E}_i$ are pairwise disjoint, so we can add up the inequalities (b) to get $|f| = \infty$, contradicting $\nu^* < \infty$.

By (4b) the support of g is contained in U, and therefore $g \upharpoonright U$ is a fractional cover in H_U (see Notation 1.8(b)). Hence $\tau^*(H_U) \leq \tau^*(H)$. Since the inverse inequality is true in general (see Lemma 1.9(b)), we have $\nu^*(H_U) = \tau^*(H_U) = \tau^*(H) = \nu^*(H)$. This implies immediately that $\nu^*(H)$ is rational and there exists a rational minimal fractional cover in H (take one in H_U and extend it to be 0 on $V \setminus U$). It remains to show the existence of a rational maximal fractional matching in H.

By Lemma 1.9(b), f_U is a fractional matching in H_U. Since $|f_U| = |f| = \nu^*(H) = \nu^*(H_U)$, it is a maximal one. Now, f_U can be approximated to an arbitrary degree by a rational maximal fractional matching in H_U (just approximate by rational numbers the coefficients in a representation of f_U as a convex combination of rational maximal fractional matchings). Let $\varepsilon = 1/(2|\mathscr{E}_U|)$ and let f_0 be a rational maximal fractional matching in H_U such that $|f_0(E') - f_U(E')| < \varepsilon$ for all $E' \in \mathscr{E}_U$. For each $E' \in \mathscr{E}_U$ we choose an arbitrary $E_0 \in \mathscr{E}(E')$ and assign nonnegative rational numbers $\tilde{f}(E)$ to all $E \in \mathscr{E}(E')$ so that:

(a) $\tilde{f}(E_0) < f(E_0) + \varepsilon$,

(b) $\tilde{f}(E) \leq f(E)$ for all $E \in \mathscr{E}(E') \setminus \{E_0\}$,

(c) $\Sigma_{E \in \mathscr{E}(E')} \tilde{f}(E) = f_0(E')$.

Such an assignment is possible, because $\sum_{E \in \mathscr{E}(E')} f(E) = f_U(E') > f_0(E') - \varepsilon$. When this is done for all $E' \in \mathscr{E}_U$, the resulting $\tilde{f}: \mathscr{E} \to [0, 1]$ is easily seen to be a rational fractional matching in H of size ν^*, as required. □

We remark that the foregoing proof shows, in fact, that any orthogonal pair can be approximated to an arbitrary degree by a rational one.

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