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Strong Equilibrium in Congestion Games*

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Rosenthal (1973, *Int. J. Game Theory* **2**, 65–67) introduced the class of congestion games and proved that they always possess a Nash equilibrium in pure strategies. Here we obtain conditions for the existence of a strong equilibrium in this class of games, as well as for the equivalence of Nash and strong equilibria. We also give conditions for uniqueness and for Pareto optimality of the Nash equilibrium. Except for a natural monotonicity assumption on the utilities, the conditions are expressed only in terms of the underlying congestion game form. It turns out that avoiding a certain type of bad configuration in the strategy spaces is essential to positive results. *Journal of Economic Literature* Classification Numbers: C71, C72, D62.

1. INTRODUCTION AND OVERVIEW OF RESULTS

A *congestion game* is described as follows. The set of players is $N = \{1, \ldots, n\}$. There is a finite nonempty set M of facilities. Each player i has a nonempty set of strategies, denoted Σ^i. Every strategy $A^i \in \Sigma^i$ is a subset of M. With every facility a and every integer $1 \leq k \leq n$, a real number $u_a(k)$ is associated, having the interpretation: $u_a(k)$ is the utility to each user of a if the total number of users of a is k. Let $\Sigma^N = \Sigma^1 \times \cdots \times \Sigma^n$ and let $A = (A^1, \ldots, A^n) \in \Sigma^N$. The ($|M|$-dimensional) con-

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gestion vector corresponding to A is $\sigma(A) = (\sigma_a(A))_{a \in M}$, where

$$\sigma_a(A) = \left|\{i \in N\colon a \in A^i\}\right|.$$

The payoff function of player i is defined by

$$\pi^i(A) = \sum_{a \in A^i} u_a(\sigma_a(A)).$$

Congestion games have been used to model such situations as rush hour traffic (the players are drivers, the facilities are road segments), demand for factors of production (the players are producers, the facilities are factors of production), foraging animals (the players are animals, the facilities are foraging sites), etc.

Congestion games were introduced by Rosenthal, who proved the following.

THEOREM (Rosenthal, 1973). *Every congestion game possesses a Nash equilibrium in pure strategies.*

Our purpose here is to explore the possibility of replacing a Nash equilibrium with a strong equilibrium. We recall that Aumann (1959) defined a strong equilibrium as a strategy profile with the property that no *coalition* of players can deviate in a manner which is profitable to all its members. More precisely, a strategy profile $A = (A^1, \ldots, A^n) \in \Sigma^N$ is a *strong equilibrium* if for no coalition $\phi \neq S \subseteq N$ is there a choice of $B^i \in \Sigma^i$, $i \in S$, such that $\pi^i(B^S, A^{-S}) > \pi^i(A)$ for all $i \in S$ (where (B^S, A^{-S}) denotes the profile in which each $i \in S$ chooses B^i and each $i \in N \setminus S$ chooses A^i). For a game G, we denote by NE(G) and SE(G) the set of Nash equilibria and strong equilibria, respectively. Clearly, SE(G) $\subseteq$ NE(G), but the converse is not true, and in fact SE(G) is often empty (even if mixed strategies are allowed).

Simple examples show that congestion games in general need not possess a strong equilibrium (in fact, the well-known prisoner's dilemma may be obtained as a congestion game). We will identify classes of congestion games which do possess strong equilibria. One restriction which we shall impose is monotonicity. A congestion game is *monotone* if, for all $a \in M$ and all integers $1 \leq k \leq l \leq n$, we have $u_a(k) \geq u_a(l)$. This is a very natural restriction, which reflects the negative effect of congestion. Rosenthal did not impose this requirement, as it was not needed for his theorem. But without monotonicity one cannot expect to guarantee existence of strong equilibria, even in the simplest congestion setup of two players choosing between two facilities.

Henceforth, we restrict our attention to monotone congestion games. In Section 2 we observe that if strategies are single facilities then strong equilibria exist. In fact, we prove that if G is a monotone congestion game all of whose strategies are singletons, then $\mathrm{SE}(G)$ coincides with $\mathrm{NE}(G)$; the latter is nonempty by virtue of Rosenthal's theorem.

This result serves as the starting point of our investigation, in which we look for structural properties of the strategy spaces which guarantee the existence of strong equilibria. By a *congestion game form* we mean a tuple

$$F = (N, M, \Sigma^1, \dots, \Sigma^n)$$

where $N = \{1, \dots, n\}$ is the set of players, M is the set of facilities, and $\Sigma^1, \dots, \Sigma^n$ are the strategy spaces of the respective players, as in the definition of a congestion game (but without the specification of the utility levels). Given a congestion game form F, one can derive from it a whole family of (monotone) congestion games by assigning (monotone) utility levels $u_a(k)$, for $a \in M$ and $1 \leq k \leq n$. All such congestion games are said to be *derived* from F. We say that F is *strongly consistent* if every monotone congestion game G derived from F possesses a strong equilibrium. We say that F is *strong-Nash equivalent* if every monotone congestion game G derived from F satisfies $\mathrm{SE}(G) = \mathrm{NE}(G)$.

In this terminology, the result mentioned above states that if $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ is a congestion game form in which the Σ^i consist of singletons only, then F is strong-Nash equivalent (and hence, *a fortiori*, strongly consistent). It turns out that similar positive results can be obtained when larger sets of facilities are allowed in the strategy spaces, as long as we forbid a certain substructure, which we call a bad configuration. Let Σ be a set of strategies on the facility set M. A *bad configuration* in Σ is a tuple

$$(x, y; X, Y, Z)$$

where

$$x, y \in M,$$
$$X, Y, Z \in \Sigma,$$

and the following relations hold:

$$X \cap \{x, y\} = \{x\},$$
$$Y \cap \{x, y\} = \{y\},$$
$$Z \cap \{x, y\} = \{x, y\}.$$

Thus, two facilities x, y give rise to a bad configuration if there are strategies in Σ which use each one of them without the other, and there is also a strategy in Σ which uses both of them. The latter never occurs if Σ consists of singletons. We call Σ *good* if it contains no bad configuration.

In Section 3 we characterize good strategy sets by means of a certain type of tree-representation that they admit. This representation is very useful in proving our results for good strategy sets in the subsequent sections. It also lends more intuition to the condition "no bad configuration." In view of the tree-representation result, this condition may be understood as an acyclicity condition.

In Section 4 we deal with symmetric congestion game forms. A congestion game form $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ is *symmetric* if $\Sigma^1 = \cdots = \Sigma^n$. In this case, we abbreviate the notation and write $F = (N, M, \Sigma)$, where Σ is the common strategy space of the players. We prove that for a symmetric congestion game form $F = (N, M, \Sigma)$ with at least two players, the three conditions

(i) F is strongly consistent,

(ii) F is strong-Nash equivalent,

(iii) Σ is good,

are equivalent. This constitutes a complete solution to our problem in the symmetric case.

In Section 5 we consider the general case, when the players' strategy spaces may be distinct. Let $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ be a congestion game form, and let $\Sigma = \bigcup_{i=1}^{n} \Sigma^i$. We prove that if Σ is good then F is strongly consistent. But, unlike in the symmetric case, Σ being good is neither a sufficient condition nor a necessary one for F to be strong-Nash equivalent.

In Section 6 we demonstrate that the goodness of strategy spaces is instrumental not only in guaranteeing a strong equilibrium but also in establishing some other desirable properties of a Nash equilibrium. The properties that we consider are uniqueness and Pareto optimality.

To conclude the introduction, we mention connections with some recent related works. Monderer and Shapley (1996) introduced the concept of a potential for a game in normal form. A strategy profile which maximizes the potential is a Nash equilibrium. Hence, any finite potential game (i.e., a game with finite strategy spaces which has a potential) possesses a Nash equilibrium. They observed that Rosenthal proved his theorem on congestion games by constructing a potential. Moreover, they proved that every finite potential game is isomorphic to a congestion game, and hence the classes of finite potential games and congestion games coincide. In Section 5 we define the notion of a strong potential for a game in normal form. A strategy profile which maximizes a strong potential is a strong equilibrium.

Hence, any finite strong potential game possesses a strong equilibrium. We prove the main result of Section 5 by showing that, under the assumptions made there, Rosenthal's potential for the congestion game is strong.

Milchtaich (1996) introduced and studied the class of "congestion games with player-specific payoff functions." This class was also investigated, independently and under different names, by Quint and Shubik (1994) and by Konishi *et al*. (1997). They all proved the existence of a Nash equilibrium for games in this class, and some of these authors even showed the existence of a strong equilibrium. In these games, the utility to a user of a facility depends not only on the facility and the number of users of it, but also on the identity of the user in question. In this sense, they are more general than Rosenthal's congestion games, which are studied here. However, in terms of the strategy spaces, they are more restrictive: only singleton strategies are allowed. Thus the main theme of our work—the dependence of equilibrium properties on the structure of the strategy spaces—is not an issue at all in the above-mentioned other works.

Van Megen *et al*. (1996) considered games which lie in the intersection of Rosenthal's class and Milchtaich's class. They proved the equivalence of Nash and strong equilibria for these games. Under the additional assumption that all players have the same strategy space, they showed that the set of Nash (and strong) equilibria coincides with the set of strategy profiles which maximize the potential.

2. THE CASE OF SINGLETON STRATEGIES

As mentioned above, the case of singleton strategies has been studied by several authors. The following result appears, explicitly or implicitly, in some of these works.

THEOREM 2.1. *Let G be a monotone congestion game in which all strategies are singletons*. *Then* $\mathrm{SE}(G) = \mathrm{NE}(G)$.

Proof. It suffices to prove that $\mathrm{NE}(G) \subseteq \mathrm{SE}(G)$. Let $A = (A^1, \ldots, A^n)$ be a Nash equilibrium. Suppose, for contradiction, that there is a coalition $\phi \neq S \subseteq N$ that has a profitable deviation to (B^S, A^{-S}). We claim that the congestion vectors corresponding to A and to (B^S, A^{-S}) are identical. Indeed, if this is not the case, then since

$$\sum_{a \in M} \sigma_a(A) = n = \sum_{a \in M} \sigma_a(B^S, A^{-S}),$$

there must exist some $a \in M$ for which $\sigma_a(B^S, A^{-S}) \geq \sigma_a(A) + 1$. Let $i \in S$ be such that $B^i = \{a\}$. (Such a player i exists, since the deviation increased the number of users of a.) Then

$$\pi^i(B^S, A^{-S}) = u_a\big(\sigma_a(B^S, A^{-S})\big) \leq u_a\big(\sigma_a(A) + 1\big) \leq \pi^i(A)$$

(here we used monotonicity and the fact that A is a Nash equilibrium). This, however, shows that the deviation is not worthwhile for i. Thus, we have proved that $\sigma(A) = \sigma(B^S, A^{-S})$. It follows that the deviation consists in permuting the choices of the members of S in some way. But then their payoffs undergo the same permutation, which makes it impossible for all of them to gain. ■

3. BAD CONFIGURATIONS AND TREE-REPRESENTATIONS

In this section we consider a set Σ of strategies on the facility set M (that is, Σ is a set of subsets of M). We recall that Σ is said to be good if it contains no bad configuration (see the Introduction).

LEMMA 3.1. *Let Σ be a good set of strategies on M, and let $Z \in \Sigma$. Then there exists $z \in M$ such that for all $X \in \Sigma$,*

$$Z \not\subseteq X \Rightarrow z \in Z \setminus X.$$

Proof. Let $\Sigma_1 = \{X \in \Sigma: Z \not\subseteq X\}$. We may assume that $\Sigma_1 \neq \phi$ (otherwise there is nothing to prove). We claim that for any $X, Y \in \Sigma_1$, either $X \cap Z \subseteq Y \cap Z$ or $Y \cap Z \subseteq X \cap Z$. Otherwise, we could choose $x \in (X \cap Z) \setminus (Y \cap Z)$ and $y \in (Y \cap Z) \setminus (X \cap Z)$ and obtain that $(x, y; X, Y, Z)$ is a bad configuration. Thus, the family of intersections $X \cap Z$, for $X \in \Sigma_1$, forms a (finite) chain with respect to inclusion. Hence there exists a set $\overline{X} \in \Sigma_1$ such that

$$\bigcup_{X \in \Sigma_1} (X \cap Z) = \overline{X} \cap Z.$$

We may choose $z \in Z \setminus \overline{X}$, and it will satisfy the requirement of the lemma. ■

By an *M-tree*, we shall mean the following:

a tree with a root r,

a labeling of the nodes of the tree (except r) by elements of M; not all elements of M must appear, but each can appear at most once;

a designated subset D of the nodes, which contains all terminal nodes (and possibly other nodes as well).

An example of an M-tree appears in Fig. 1.

Given an M-tree T, we associate with it a set Σ of strategies on M, as follows: to each node in D there corresponds a strategy in Σ consisting of the labels which appear on the path from r to that node. For instance, if T is the M-tree depicted in Fig. 1, then

$$\Sigma = \{\{a,b\},\{a,b,c\},\{a,d\},\{a,e,f\},\{a,e,g\},\{h\},\{h,i\},\{h,j\}\}.$$

If Σ is obtained from T in this way, we say that T is a *tree-representation* of Σ.

PROPOSITION 3.2. *Let Σ be a nonempty set of strategies on M. Then Σ is good if and only if it has a tree-representation.*

Proof. Suppose first that Σ has a tree-representation T. If $(x, y; X, Y, Z)$ is a bad configuration in Σ, then x and y appear on the path in T corresponding to Z. Without loss of generality, we may assume that x precedes y on that path. But then it is impossible for Y to contain y without containing x. This proves that Σ is good.

Suppose next that Σ is good. We prove by induction on $|\cup \Sigma|$ that Σ has a tree-representation. If $|\cup \Sigma| = 0$, that is, $\Sigma = \{\phi\}$, then the tree which consists of the root alone represents Σ. Assume then that $|\cup \Sigma| > 0$.

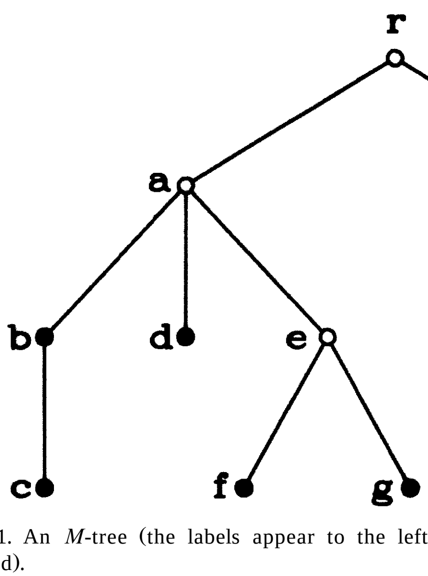

FIG. 1. An M-tree (the labels appear to the left of the nodes; the nodes in D are blackened).

Let $Z \in \Sigma$ be maximal with respect to inclusion. Then, according to Lemma 3.1, we can find an element $z \in M$ which belongs to Z and to no other set in Σ. Let $\Sigma^- = (\Sigma \setminus \{Z\}) \cup \{Z \setminus \{z\}\}$. Then it is easy to check that Σ^- is good. Let T^- be a tree-representation of Σ^- (which exists by the induction hypothesis). Let ν be the end-node of the path in T^- corresponding to $Z \setminus \{z\}$. Then we obtain a tree-representation T of Σ by adding a new node, pending from ν, labeling it z, and adding it to the designated subset D (ν itself has to be removed from D, unless $Z \setminus \{z\} \in \Sigma$). ■

In the remainder of this section we develop a technique which is based on the tree-representation and will serve us in the following sections. Let $F = (N, M, \Sigma^1, \ldots, \Sigma^n)$ be a congestion game form, let $A = (A^1, \ldots, A^n)$ be a strategy profile, and let (B^S, A^{-S}) be another profile where each $i \in S$ chooses B^i instead of A^i. For every $x \in M$ we define $\delta(x)$ as the change in the congestion at x, that is,

$$\delta(x) = \sigma_x(B^S, A^{-S}) - \sigma_x(A).$$

Assume that T is a tree-representation of $\Sigma = \bigcup_{i=1}^n \Sigma^i$. Then each strategy A^i can be written uniquely in the form

$$A^i = (a_1, \ldots, a_k, b_1, \ldots, b_l), \tag{3.1}$$

where the elements are listed in the order of their appearance on the corresponding path in T, and b_1 is the first element at which the congestion decreases. That is,

$$\delta(a_1) \geq 0, \ldots, \delta(a_k) \geq 0, \qquad \delta(b_1) < 0.$$

(Possibly, $k = 0$ or $l = 0$.) If $i \in S$ and A^i is as in (3.1), we say that player $j \in S$ *replaces* i, if

$$B^j = (a_1, \ldots, a_k, c_1, \ldots, c_m), \tag{3.2}$$

where

$$\delta(c_1) > 0, \ldots, \delta(c_m) > 0.$$

(Possibly, $m = 0$.) The term "replaces" is suggested by the case $l = m = 0$, in which $A^i = B^j$; in this case, when the players move from A to (B^S, A^{-S}), player j takes the place of player i. Note that a player may replace himself.

CLAIM 3.3. *For every player $i \in S$ there exists a player $j \in S$ who replaces i.*

Proof. Let $i \in S$, and let $A^i = (a_1, \ldots, a_k, b_1, \ldots, b_l)$ as in (3.1). Let $X = (a_1, \ldots, a_k, c_1, \ldots, c_m)$ be a sequence of labels appearing on a path Q from the root r of T, such that $\delta(c_1) > 0, \ldots, \delta(c_m) > 0$, and m is as large as possible (possibly $m = 0$). Let Δ be the change in the number of players who choose X as their strategy, that is,

$$\Delta = |\{p \in S\colon B^p = X\}| - |\{p \in S\colon A^p = X\}|.$$

At this point, we do not know yet that X is a strategy, but Δ is defined anyway (it is zero if $X \notin \Sigma$). Let v be the end-node of Q, and let δ be the change in the congestion at the element labeling v; more precisely,

$$\delta = \begin{cases} \delta(c_m) & \text{if } m \geq 1, \\ \delta(a_k) & \text{if } k \geq 1, m = 0, \\ 0 & \text{if } k = m = 0. \end{cases}$$

Let C be the set of labels of the nodes which immediately follow v in T ($C = \phi$ if v is a terminal node). Then simple accounting shows that

$$\delta = \Delta + \sum_{c \in C} \delta(c).$$

Now $\delta(c) \leq 0$ for all $c \in C$, because otherwise we could extend X by letting $c_{m+1} = c$ for some $c \in C$ with $\delta(c) > 0$. On the other hand, $\delta \geq 0$. It follows that $\Delta \geq 0$, with equality only if

(i) $\delta(c) = 0$ for all $c \in C$,

and

(ii) $\delta = 0$.

However, (ii) implies that $m = 0$ (otherwise, $\delta = \delta(c_m) > 0$), and then (i) implies that $l = 0$ (otherwise, $b_1 \in C$ and $\delta(b_1) < 0$). Thus, either $\Delta > 0$ or else $\Delta = 0$ and $A^i = X$. In either case, there must exist $j \in S$ such that $B^j = X$. This j replaces i. ■

4. THE SYMMETRIC CASE

In this section we deal with symmetric congestion game forms; this is the case when all players have the same strategy space Σ. We give a full characterization of strong consistency and of strong–Nash equivalence (see the Introduction for definitions) in this case.

THEOREM 4.1. *Let $F = (N, M, \Sigma)$ be a symmetric congestion game form. If Σ is good then F is strong–Nash equivalent.*

Proof. Let F be as in the assumptions, and let G be a monotone congestion game derived from F. We have to prove that $\mathrm{SE}(G) = \mathrm{NE}(G)$, and it suffices to prove that $\mathrm{NE}(G) \subseteq \mathrm{SE}(G)$. Let $A = (A^1, \dots, A^n)$ be a Nash equilibrium. Suppose, for contradiction, that there is a coalition $\phi \neq S \subseteq N$ that has a profitable deviation to (B^S, A^{-S}). Let $i \in S$ be a player who gets the lowest payoff among the members of S after the deviation, that is,

$$\pi^i(B^S, A^{-S}) \leq \pi^j(B^S, A^{-S}) \qquad \text{for all } j \in S. \tag{4.1}$$

By the results of Section 3, we can find a tree-representation T of Σ, write $A^i = (a_1, \dots, a_k, b_1, \dots, b_l)$ as in (3.1), and find a $j \in S$ who replaces i, i.e., chooses $B^j = (a_1, \dots, a_k, c_1, \dots, c_m)$ as in (3.2). Let A^* be the profile obtained from A when i alone deviates from A^i to B^j (here we use the symmetry assumption). Then for $p = 1, \dots, k$ we have

$$\sigma_{a_p}(A^*) = \sigma_{a_p}(A) \leq \sigma_{a_p}(B^S, A^{-S}),$$

where the equality follows from the fact that a_p appears in both A^i and B^j, and the inequality holds because $\delta(a_p) \geq 0$. For $q = 1, \dots, m$ we have

$$\sigma_{c_q}(A^*) = \sigma_{c_q}(A) + 1 \leq \sigma_{c_q}(B^S, A^{-S}),$$

where the equality follows from the fact that c_q appears in B^j but not in A^i, and the inequality holds because $\delta(c_q) > 0$. It follows from these comparisons that

$$\pi^i(A^*) \geq \pi^j(B^S, A^{-S}). \tag{4.2}$$

The fact that A is a Nash equilibrium implies that

$$\pi^i(A) \geq \pi^i(A^*). \tag{4.3}$$

The profitability of the deviation to (B^S, A^{-S}) implies that

$$\pi^i(B^S, A^{-S}) > \pi^i(A). \tag{4.4}$$

From (4.2)–(4.4) we obtain that $\pi^i(B^S, A^{-S}) > \pi^j(B^S, A^{-S})$, which contradicts (4.1). ■

THEOREM 4.2. *Let $F = (N, M, \Sigma)$ be a symmetric congestion game form. If $|N| \geq 2$ and Σ contains a bad configuration then F is not strongly consistent.*

Proof. Let F be as in the assumptions. We have to exhibit a monotone congestion game G derived from F which has no strong equilibrium. Let

$(x, y; X, Y, Z)$ be a bad configuration in Σ. Let G be derived from F by assigning the utility levels $u_a(k)$, for $a \in M$ and $1 \leq k \leq n$, as follows:

$$u_a(k) = \begin{cases} 3 & \text{if } a \in \{x, y\} \text{ and } 1 \leq k \leq n-1, \\ 1 & \text{if } a \in \{x, y\} \text{ and } k = n, \\ 0 & \text{if } a \in M \setminus \{x, y\}. \end{cases}$$

Let $A = (A^1, \dots, A^n)$ be a Nash equilibrium of G. Then $x, y \in A^i$ for each i, since otherwise i could gain by switching from A^i to Z. Hence $\pi^i(A) = 2$ for all i. Now, if players 1 and 2 deviate to X and Y, respectively, then their payoffs increase to 3 each. Thus, A is not a strong equilibrium. As A was an arbitrary Nash equilibrium, we have proved that G has no strong equilibrium. ■

Since, by Rosenthal's theorem, strong−Nash equivalence implies strong consistency, our two theorems yield the following.

COROLLARY 4.3. *Let $F = (N, M, \Sigma)$ be a symmetric congestion game form with $|N| \geq 2$. Then the following three conditions are equivalent*:

(i) *F is strongly consistent*,

(ii) *F is strong−Nash equivalent*,

(iii) *Σ is good*.

The results of this section can be slightly generalized as follows. Let $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ be a congestion game form. We say that F is *quasi-symmetric* if for any two players i and j and any two strategies $X \in \Sigma^i$ and $Y \in \Sigma^j$ such that $X \cap Y \neq \phi$ we have $Y \in \Sigma^i$. Clearly, every symmetric congestion game form is quasi-symmetric, and so is every congestion game form in which all strategies are singletons (or the empty set). Theorems 2.1 and 4.1 have the following common generalization.

THEOREM 4.4. *Let $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ be a quasi-symmetric congestion game form, and let $\Sigma = \bigcup_{i=1}^{n} \Sigma^i$. If Σ is good then F is strong−Nash equivalent*.

Theorem 4.2 generalizes as follows.

THEOREM 4.5. *Let $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ be a quasi-symmetric congestion game form, and let $\Sigma = \bigcup_{i=1}^{n} \Sigma^i$. If Σ contains a bad configuration $(x, y; X, Y, Z)$, and $\Sigma^i \cap \{X, Y, Z\} \neq \phi$ for at least two players $i \in N$, then F is not strongly consistent*.

Theorems 4.4 and 4.5 can be proved by adapting the proofs of Theorems 4.1 and 4.2, respectively. We omit the details.

5. THE GENERAL CASE

Let $G = (N, \Sigma^1, \dots, \Sigma^n, \pi^1, \dots, \pi^n)$ be a game in normal form; here N is the set of players, Σ^i is the strategy space of player i, and π^i is the payoff function of player i. Let $\Sigma^N = \Sigma^1 \times \cdots \times \Sigma^n$. Monderer and Shapley (1996) defined several notions of potential for G. We recall here two of them. A function $P: \Sigma^N \to \mathbb{R}$ is an *exact potential* for G if for every $A = (A^1, \dots, A^n) \in \Sigma^N$, every $i \in N$, and every $B^i \in \Sigma^i$,

$$\pi^i(B^i, A^{-i}) - \pi^i(A) = P(B^i, A^{-i}) - P(A);$$

it is a *generalized ordinal potential* for G if, for every $A = (A^1, \dots, A^n) \in \Sigma^N$, every $i \in N$, and every $B^i \in \Sigma^i$,

$$\pi^i(B^i, A^{-i}) > \pi^i(A) \Rightarrow P(B^i, A^{-i}) > P(A).$$

Monderer and Shapley observed that if P is a generalized ordinal potential (in particular, if it is an exact potential) for G, then a strategy profile that maximizes P is a Nash equilibrium of G.

We introduce here variants of the notion of potential which are suitable for establishing the existence of a strong equilibrium. If $A = (A^1, \dots, A^n)$ is a strategy profile, $\phi \neq S \subseteq N$ is a coalition, and $B^i \in \Sigma^i$ for $i \in S$, we say that (B^S, A^{-S}) is an *improvement* of S (relative to A) if $\pi^i(B^S, A^{-S}) > \pi^i(A)$ for all $i \in S$. We say that it is a *minimal improvement* if, in addition, no proper subcoalition of S has an improvement relative to A. A function $P: \Sigma^N \to \mathbb{R}$ is a *strong potential* for G if for every $A = (A^1, \dots, A^n) \in \Sigma^N$ and every minimal improvement (B^S, A^{-S}),

$$\sum_{i \in S} \pi^i(B^S, A^{-S}) - \sum_{i \in S} \pi^i(A) \leq P(B^S, A^{-S}) - P(A);$$

it is a *generalized strong potential* for G if for every $A = (A^1, \dots, A^n) \in \Sigma^N$ and every minimal improvement (B^S, A^{-S}),

$$P(B^S, A^{-S}) > P(A).$$

We note that a strong potential is a generalized strong potential, which in turn is a generalized ordinal potential. The following is obvious.

Observation 5.1. If P is a generalized strong potential for G, then a strategy profile that maximizes P is a strong equilibrium of G.

Let us return to congestion games. For a congestion game G, Rosenthal defined the function

$$P(A) = \sum_{a \in M} \sum_{k=1}^{\sigma_a(A)} u_a(k). \tag{5.1}$$

He observed that P is an exact potential for G (though he did not use this terminology), thereby proving his existence theorem for the Nash equilibrium. Here we shall prove that, under certain conditions, the same function is a strong potential; existence of a strong equilibrium will follow.

THEOREM 5.2. *Let* $F = (N, M, \Sigma^1, \dots, \Sigma^n)$ *be a congestion game form, and let* $\Sigma = \bigcup_{i=1}^n \Sigma^i$. *If* Σ *is good and* G *is a monotone congestion game derived from* F, *then* (5.1) *is a strong potential for* G.

Proof. Let F and G be as in the assumptions. Let $A = (A^1, \dots, A^n) \in \Sigma^N$, and let (B^S, A^{-S}) be a minimal improvement of S relative to A.

Let T be a tree-representation of Σ (see Proposition 3.2). Referring to the replacement relation defined in Section 3, it follows from Claim 3.3 and the finiteness of S that there exists a replacement cycle, that is, a sequence $i_1, i_2, \dots, i_t$ of distinct players in S so that i_2 replaces i_1; i_3 replaces i_2; ...; i_t replaces i_{t-1}; and i_1 replaces i_t. (Note that $t = 1$ is allowed: this is the case when i_1 replaces himself.) Let us choose a replacement cycle which is as short as possible. To simplify notation, let this cycle be $1, 2, \dots, t$. The strategies chosen by these players can be written, in accordance with (3.1) and (3.2), as follows:

$$A^i = \left(a_1^i, \dots, a_{k_i}^i, b_1^i, \dots, b_{l_i}^i\right),$$

$$B^{i+1} = \left(a_1^i, \dots, a_{k_i}^i, c_1^i, \dots, c_{m_i}^i\right).$$

(Here and in the following, $t + 1$ is to be understood as 1.) Let us denote

$$\mathsf{A} = \left\{a_p^i : i = 1, \dots, t,\ p = 1, \dots, k_i\right\},$$

$$\mathsf{B} = \left\{b_q^i : i = 1, \dots, t, q = 1, \dots, l_i\right\},$$

$$\mathsf{C} = \left\{c_s^i : i = 1, \dots, t,\ s = 1, \dots, m_i\right\}.$$

CLAIM. (1) $\mathsf{B} \cap (\mathsf{A} \cup \mathsf{C}) = \phi$.

(2) *For each* $a \in \mathsf{A}$, *the number of its appearances as* a_p^i *in the strategies* $A^1, \dots, A^t$ *equals the number of its appearances as* a_p^i *in the strategies* $B^1, \dots, B^t$.

(3) *Each* $b \in \mathsf{B}$ *is equal to* b_q^i *for exactly one pair* i, q.

(4) *Each* $c \in \mathsf{C}$ *is equal to* c_s^i *for exactly one pair* i, s.

Proof of the Claim. (1) For every b_q^i, there is an element x, which appears on the path from the root of T to the node labeled b_q^i and which satisfies $\delta(x) < 0$ (take $x = b_1^i$). This is not the case for any a_p^i or c_s^i.

(2) Any such appearance in A^i is offset by an appearance in B^{i+1}.

(3) Suppose that $b = b_q^i = b_{q'}^{i'}$ for two distinct pairs i, q and i', q'. Then clearly $i \neq i'$, and the portions of A^i and $A^{i'}$ up to b are identical. Hence, any $j \in S$ replaces i if and only if he replaces i'. Say $i < i'$; then $i + 1, \ldots, i'$ is a replacement cycle, which contradicts the choice of $1, \ldots, t$ as a shortest cycle.

(4) If $c = c_s^i = c_{s'}^{i'}$ for $i < i'$, then again $i + 1, \ldots, i'$ is a shorter replacement cycle.

Let $S' = \{1, \ldots, t\}$, and consider what would happen if only the players in S' were to deviate, each $i \in S'$ switching from A^i to B^i. For every $x \in M$, let us denote by $\delta'(x)$ the change that would occur in the congestion at x, that is,

$$\delta'(x) = \sigma_x(B^{S'}, A^{-S'}) - \sigma_x(A).$$

It follows from the claim that

$$\delta'(x) = \begin{cases} -1 & \text{if } x \in \mathbf{B}, \\ 1 & \text{if } x \in \mathbf{C}, \\ 0 & \text{otherwise}. \end{cases} \tag{5.2}$$

Thus, for $x \in \mathbf{A} \cup \mathbf{C}$ we have $\delta'(x) \leq \delta(x)$. Hence, it follows by monotonicity that for all $i \in S'$,

$$\pi^i(B^{S'}, A^{-S'}) \geq \pi^i(B^S, A^{-S}) > \pi^i(A).$$

This implies that $(B^{S'}, A^{-S'})$ is an improvement of S' relative to A. Since (B^S, A^{-S}) was assumed to be a minimal improvement, we conclude that $S' = S$. Thus, the values of $\delta(x)$ are given by (5.2), and can be used in the following computation. For each $i \in S$,

$$\begin{aligned} &\pi^{i+1}(B^S, A^{-S}) - \pi^i(A) \\ &\quad= \sum_{p=1}^{k_i} u_{a_p^i}\Big(\sigma_{a_p^i}(B^S, A^{-S})\Big) + \sum_{s=1}^{m_i} u_{c_s^i}\Big(\sigma_{c_s^i}(B^S, A^{-S})\Big) \\ &\qquad - \left[\sum_{p=1}^{k_i} u_{a_p^i}\Big(\sigma_{a_p^i}(A)\Big) + \sum_{q=1}^{l_i} u_{b_q^i}\Big(\sigma_{b_q^i}(A)\Big)\right] \\ &\quad\leq \sum_{s=1}^{m_i} u_{c_s^i}\Big(\sigma_{c_s^i}(B^S, A^{-S})\Big) - \sum_{q=1}^{l_i} u_{b_q^i}\Big(\sigma_{b_q^i}(A)\Big). \end{aligned}$$

(The inequality follows from monotonicity and the fact that $\delta(a_p^i) \geq 0$.

Note that $\delta(a_p^i)$ may equal one (this is the case if $a_p^i \in \mathbf{A} \cap \mathbf{C}$) and hence the inequality may be strict.) Adding up the above inequalities over all $i \in S$, we obtain

$$\sum_{i \in S} \pi^i(B^S, A^{-S}) - \sum_{i \in S} \pi^i(A) \leq \sum_{c \in \mathbf{C}} u_c(\sigma_c(B^S, A^{-S})) - \sum_{b \in \mathbf{B}} u_b(\sigma_b(A)).$$

From (5.1) and (5.2) it can be seen that the right-hand side is exactly $P(B^S, A^{-S}) - P(A)$. This is what was needed in order to show that P is a strong potential for G. ■

In view of Observation 5.1, we obtain from Theorem 5.2 the following.

COROLLARY 5.3. *Let $F = (N, M, \Sigma^1, \ldots, \Sigma^n)$ be a congestion game form, and let $\Sigma = \bigcup_{i=1}^n \Sigma^i$. If Σ is good then F is strongly consistent.*

The following example shows that the conclusion of strong consistency cannot be sharpened to strong–Nash equivalence (compare with Theorem 4.1 in the symmetric case).

EXAMPLE 5.4. Let $N = \{1, 2\}$, and let $M = \{a, b, c\}$. Let $\Sigma^1 = \{\{a, b\}, \{c\}\}$ and $\Sigma^2 = \{\{a\}, \{c\}\}$. Then $\Sigma^1 \cup \Sigma^2$ is good. Suppose

$$u_a(1) = 3, \quad u_b(1) = -2, \quad u_c(1) = 2,$$
$$u_a(2) = 0, \quad u_c(2) = 0.$$

Then the normal form is:

	a	c
ab	$-2, 0$	$1, 2$
c	$2, 3$	$0, 0$

The boxes with payoffs $(1, 2)$ and $(2, 3)$ are both Nash equilibria, but only the latter is a strong equilibrium.

We note also that the goodness of the union of the strategy spaces is not a necessary condition for strong consistency, in fact it is not even necessary for strong–Nash equivalence (compare with Theorem 4.2 in the symmetric case). A trivial example is when the union of the strategy spaces contains a bad configuration, but each player has just one strategy. Less trivial examples can also be given.

6. UNIQUENESS AND PARETO OPTIMALITY OF NASH EQUILIBRIUM

First, we consider uniqueness properties of the Nash equilibrium in symmetric congestion games (i.e., games in which the players have identical strategy spaces). If A is a Nash equilibrium of such a game, then any strategy profile obtained from A by permuting the players' strategies is also a Nash equilibrium. In view of this fact, it makes sense to speak of the uniqueness of the Nash equilibrium up to permutation.

Furthermore, the uniqueness of the Nash equilibrium may fail due to indifference in the choice between strategies. Hence we shall confine attention to generic games. More precisely, let F be a congestion game form and let P be a property of congestion games. We shall say that a *generic* (monotone) congestion game derived from F has property P, if every assignment of (monotone) utility levels, with the possible exception of assignments which satisfy one of a finite number of linear equations, results in a congestion game that has property P. In other words, we tolerate exceptions that lie in the union of finitely many hyperplanes in the space of utility levels.

We are now ready to state our result.

THEOREM 6.1. *Let $F = (N, M, \Sigma)$ be a symmetric congestion game form. If Σ is good, then a generic monotone congestion game derived from F has a unique Nash equilibrium up to permutation.*

We omit the proof, which resembles that of Theorem 4.1. We remark that the assumptions of symmetry and of goodness in the theorem are essential, as can be shown by suitable (generic) examples.

Next, we consider Pareto optimality of Nash equilibria. Let G be a congestion game, and let A be a strategy profile in G. We say that A is *strictly Pareto optimal* if there is no profile B such that

$$\pi^i(B) \geq \pi^i(A) \qquad \text{for } i = 1, \dots, n \tag{6.1}$$

and at least one of these inequalities is strict. We say that A is *weakly Pareto optimal* if there is no profile B for which all the inequalities (6.1) are strict. We denote by SPO(G) and WPO(G), respectively, the sets of strictly and weakly Pareto optimal profiles in G. Of course, $\phi \neq \text{SPO}(G) \subseteq \text{WPO}(G)$.

As $\text{SE}(G) \subseteq \text{WPO}(G)$, wherever we proved above that all (or some) Nash equilibria are strong, we know that under the same conditions all (respectively some) Nash equilibria are weakly Pareto optimal. In particular, consider our results in Section 4. Observe that for the game constructed in the proof of Theorem 4.2, our argument in fact showed that no

Nash equilibrium is weakly Pareto optimal. Therefore, Corollary 4.3 has the following Pareto optimality analog. Let $F = (N, M, \Sigma)$ be a symmetric congestion game form with $|N| \geq 2$. Then for either of the properties,

(i) $\mathrm{NE}(G) \cap \mathrm{WPO}(G) \neq \phi$,

(ii) $\mathrm{NE}(G) \subseteq \mathrm{WPO}(G)$,

in order for every monotone congestion game G derived from F to have that property, it is necessary and sufficient that Σ be good.

For strict Pareto optimality, all of the above still holds true generically; indeed, it can be checked that for any congestion game form F, a generic congestion game G derived from F satisfies $\mathrm{SE}(G) \subseteq \mathrm{SPO}(G)$. But if we want to look beyond the generic case, things become less obvious. For quasi-symmetric congestion game forms (see Section 4) we have the following.

THEOREM 6.2. *Let $F = (N, M, \Sigma^1, \ldots, \Sigma^n)$ be a quasi-symmetric congestion game form, and let $\Sigma = \bigcup_{i=1}^{n} \Sigma^i$. If Σ is good, then every monotone congestion game derived from F possesses a strong equilibrium which is strictly Pareto optimal.*

In the interest of brevity, we omit the proof, which involves a new idea in addition to the tree-representation technique. We remark that a (non-generic) example can be given to show that under the conditions of Theorem 6.2 (in fact, even under the stronger conditions of singleton strategies and symmetry), not every strong equilibrium has to be strictly Pareto optimal.

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