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An Extension of Fishburn's Theorem on Extending Orders

RON HOLZMAN

Institute of Mathematics, The Hebrew University, 91904 Jerusalem, Israel

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P. C. Fishburn [*J. Econom. Theory* **31** (1983), 000–000] has shown that one can extend any well ordering of a set to a linear ordering of the set of all non-empty subsets of that set, while satisfying two axioms denoted (GP*) and (M*). By applying the compactness theorem of logic, this note shows that the well ordering assumption can be waived. Definability and well ordering properties of the extension are also discussed and shown to depend on the class of basic orderings considered. *Journal of Economic Literature* Classification Numbers: 025, 026.

Fishburn [1] was interested in the question of whether for every reflexive linear order R on a set Ω there existed a reflexive linear order $\gtrsim$ on the set 2^Ω of non-empty subsets of Ω, so that $\{x\} \gtrsim \{y\}$ whenever xRy and the following two axioms were satisfied:

(GP*): if $A, B \in 2^\Omega$ are such that xRy whenever either $x \in A \setminus B$ and $y \in B$, or $x \in A$ and $y \in B \setminus A$, then $A \gtrsim B$;

(M*): if $A, B, C \in 2^\Omega$ are pairwise disjoint and $B \gtrsim C$, then $A \cup B \gtrsim A \cup C$.

Fishburn proved that such an extension existed whenever R was a well ordering and remarked that the situation in the general case remained an open question. We shall give here an affirmative answer to this question, i.e., we shall prove the following theorem:

THEOREM. *Let R be a reflexive linear order on a set Ω. There exists a reflexive linear order $\gtrsim$ on 2^Ω, so that $\{x\} \gtrsim \{y\}$ whenever xRy and the axioms* (GP*) *and* (M*) *are satisfied.*

The main tool in our proof comes from the field of mathematical logic—it is the compactness theorem for the propositional calculus. An appendix is devoted to a presentation of the necessary notions and to a formulation and intuitive explanation of the theorem.

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Notation. If R is a binary relation on a set Ω and A, B are subsets of Ω, we shall write ARB if xRy whenever $x \in A$ and $y \in B$. Observe that if either of the sets A, B is empty, then ARB holds trivially.

Given a set Ω, we consider the propositional calculus having, for each $(A, B) \in 2^\Omega \times 2^\Omega$, an atomic proposition denoted $A \gtrsim B$. For a binary relation R on Ω, we define $P(R)$ as the union of the following five sets of propositions:

$$P_1 = \{(A \gtrsim B \wedge B \gtrsim C) \to A \gtrsim C : A, B, C \in 2^\Omega\}$$
$$P_2 = \{\sim(A \gtrsim B \wedge B \gtrsim A) : A, B \in 2^\Omega, A \neq B\}$$
$$P_3 = \{A \gtrsim B \vee B \gtrsim A : A, B \in 2^\Omega\}$$
$$P_4(R) = \{A \gtrsim B : A, B \in 2^\Omega, (A \backslash B) RB, AR(B \backslash A)\}$$
$$P_5 = \{B \gtrsim C \to A \cup B \gtrsim A \cup C : A, B, C \in 2^\Omega, A \cap B = A \cap C = B \cap C = \varnothing\}.$$

It is clear that if R is a reflexive linear order on Ω then the existence of an order $\gtrsim$ as required in the Theorem is equivalent to the existence of a model of $P(R)$.

By the compactness theorem, it suffices to show that if R is a reflexive linear order on Ω then every finite $\bar{P} \subset P(R)$ has a model.

CLAIM. Let R be a reflexive linear order on Ω and let $\bar{P}$ be a finite subset of $P(R)$. There exists a reflexive linear order $\bar{R}$ which well orders Ω and satisfies $\bar{P} \subset P(\bar{R})$.

Assuming the truth of the Claim, the Theorem is proved as follows: Given R as above and a finite $\bar{P} \subset P(R)$, for $\bar{R}$ which exists by the Claim, we can apply Fishburn's Theorem; therefore we know that $P(\bar{R})$ has a model, so in particular $\bar{P}$ has one.

Let us prove the Claim. In order to assure that $\bar{P} \subset P(\bar{R})$, we only have to assure that $\bar{P}_4 = \bar{P} \cap P_4(R)$ be contained in $P_4(\bar{R})$. So we make a finite list $\mathscr{L}_1$ of conditions of the form $C\bar{R}D$ which we have to satisfy. Observe that for every condition $C\bar{R}D$ in $\mathscr{L}_1$, $C \cap D = \varnothing$ and CRD.

Let $C_1, ..., C_n$ be an enumeration of all subsets of Ω that occur in conditions in $\mathscr{L}_1$. For $S \subset \{1, ..., n\}$, let $E_S = \bigcap_{i=1}^n C_i^{(S)}$, where $C_i^{(S)} = C_i$ if $i \in S$ and $C_i^{(S)} = \Omega \backslash C_i$ if $i \notin S$. Denote $\mathscr{S} = \{S : S \subset \{1, ..., n\}, E_S \neq \varnothing\}$. Then the sets E_S for $S \in \mathscr{S}$ form a partition of Ω. Replacing each condition $C_i \bar{R} C_j$ in $\mathscr{L}_1$ by the conditions $E_S \bar{R} E_T$ for all $S, T \in \mathscr{S}$ such that $i \in S$ and $j \in T$, we obtain a list $\mathscr{L}_2$ of conditions which is equivalent to $\mathscr{L}_1$.

We observe two facts:

(i) No condition in $\mathscr{L}_2$ is of the form $E_S \bar{R} E_S$, since this would occur only if $S \in \mathscr{S}$ and there is a condition $C_i \bar{R} C_j$ in $\mathscr{L}_1$ with $i, j \in S$, but then $E_S \subset C_i \cap C_j$, $C_i \cap C_j$ is empty but E_S is not.

(ii) $E_S R E_T$ holds for every condition $E_S \bar{R} E_T$ in $\mathscr{L}_2$, since $\mathscr{L}_1$ and $\mathscr{L}_2$ are equivalent.

By these two facts, there exists an enumeration $S_1, \ldots, S_k$ of $\mathscr{S}$ so that for all $E_S \bar{R} E_T$ in $\mathscr{L}_2$, S appears in the enumeration before T. (To obtain such an enumeration pick $x_S \in E_S$ for each $S \in \mathscr{S}$. Then the x_S are distinct and R induces a linear order on $\{x_S : S \in \mathscr{S}\}$. Let $S = S_m$ if x_S is the mth element in this order.)

Now, for $m = 1, \ldots, k$ let R_m be a reflexive linear order that well orders E_{S_m}. Define $\bar{R}$ by: for $x, y \in \Omega$, $x \bar{R} y$ iff $[x, y \in E_{S_m}$ and $x R_m y$ for some $m \in \{1, \ldots, k\}$, or $x \in E_{S_m}, y \in E_{S_p}$ for $m < p]$. Clearly, $\bar{R}$ is as required in the Claim.

We add two remarks that shed more light on the extension problem.

A. Both Fishburn's theorem and our theorem are only existence theorems. They establish the existence of an order on 2^Ω with certain properties, but do not give us criteria by which we can compare two subsets of Ω if we can compare any two elements of Ω. Formally speaking, it is desirable to have a definable (by a set theoretic formula) function f, so that whenever R is a reflexive linear order on Ω (in Fishburn's case—a well ordering), $f(R)$ is a reflexive linear order on 2^Ω that satisfies the requirements of the theorem with respect to R.

In Fishburn's case, we can define such an f as follows: With every reflexive linear order R that well orders a set Ω and every $A \in 2^\Omega$ we associate a function $\varphi_A^R : \Omega \to \{0, 1, 2\}$ defined by

$$\varphi_A^R(x) = \begin{cases} 1 & x \in A \\ 2 & (\forall y \in \Omega)(xRy \Rightarrow y \notin A) \\ 0 & \text{otherwise.} \end{cases}$$

(The cases in the definition are mutually exclusive as R is reflexive.) We let $Af(R)B$ iff $\varphi_A^R \gtrsim_L^R \varphi_B^R$, where $\gtrsim_L^R$ is the lexicographic order on the functions from Ω into $\{0, 1, 2\}$ that is obtained from the order R on Ω. We leave it to the reader to verify that the above defines a function f as desired. As a by-product, an alternative proof of Fishburn's theorem is obtained.

The situation is different in the general case. Here we cite the following impossibility result (whose proof applies advanced set theoretic methods): If set theory is consistent then there exists no definable linear order on $2^{\mathbb{R}}$ (where $\mathbb{R}$ is the set of real numbers). Since there exists a definable linear order on $\mathbb{R}$ itself (e.g., the ordering by magnitude), the existence of a definable function f as desired would imply, in particular, the existence of a definable linear order on $2^{\mathbb{R}}$—to obtain one, apply f to a definable reflexive linear order on $\mathbb{R}$. Hence, assuming the consistency of set theory, there exists no definable f as desired.

B. It is natural to ask whether in Fishburn's theorem one can demand that $\gtrsim$ be a well ordering too. The answer is affirmative if and only if Ω is finite. Indeed, if Ω is finite then so is 2^{Ω} and every linear order on it is a well ordering. If, on the other hand, Ω is infinite then we contend that no well ordering of 2^{Ω} satisfies (M*), which implies, in particular, a negative answer to the question.

To prove our contention, assume that $\gtrsim$ is a well ordering of 2^{Ω} that satisfies (M*). Let $x_1, x_2, \ldots, x_n, \ldots$ be the first, the second,..., the nth,... elements of Ω in the order induced by $\gtrsim$ on Ω. For any distinct $x, y \in \Omega$ it follows by (M*) that $\{x\} \gtrsim \{y\}$ iff $\Omega\backslash\{y\} \gtrsim \Omega\backslash\{x\}$ (take $A = \Omega\backslash\{x, y\}$). Hence, the sequence $\Omega\backslash\{x_1\}$, $\Omega\backslash\{x_2\}, \ldots$, $\Omega\backslash\{x_n\}, \ldots$ contradicts the fact that $\gtrsim$ is a well ordering.

APPENDIX: THE COMPACTNESS THEOREM

The propositional calculus is the most elementary part of logic—it is concerned with the relations between propositions from the point of view of truth or falsehood of propositions.

The starting point is a set of propositions, called atomic, that have no inner structure (as far as the calculus is concerned). The propositions we deal with are the atomic ones as well as compound ones, which are obtained from the atomic propositions by applications of the logical connectives: $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication). For example, if p, q, r are atomic propositions, then the following are (compound) propositions: $p \rightarrow q$, $\sim r$, $\sim(q \vee \sim p)$, $(p \rightarrow p) \wedge q$, $(p \vee q) \rightarrow (r \rightarrow \sim p)$.

As we are not concerned with the content of atomic propositions, no truth values are, a priori, associated with them (a truth value is one of the two: "true," "false"). However, one truth values are (arbitrarily) assigned to the atomic propositions, the truth values of all the propositions are determined by means of the truth tables of the connectives, which correspond to the usual interpretations of the connectives in mathematics. Returning to the examples in the previous paragraph, let us assume that p is true while q and r are false. Then $p \rightarrow q$ is false, $\sim r$ is true, $\sim(q \vee \sim p)$ is true, $(p \rightarrow p) \wedge q$ is false, $(p \vee q) \rightarrow (r \rightarrow \sim p)$ is true.

A model of a set of propositions P is an assignment of truth values to the atomic propositions, such that the truth value of every proposition in P

(determined as explained above) is "true." Intuitively, the fact that a set of propositions P has a model means that the propositions in P are mutually compatible.

The compactness theorem asserts that a set of propositions P has a model if every finite subset of P has a model. Intuitively, this can be explained as follows: If P does not have a model there must be an incompatibility between its elements. This incompatibility can be described, and a description mentions only finitely many propositions in P. Hence there is an incompatibility between finitely many propositions in P. But then they form a finite subset of P which does not have a model, contradicting the assumption.

Acknowledgments

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Reference

1. P. C. Fishburn, Comment on the Kannai–Peleg impossibility theorem for extending orders, *J. Econom. Theory* **32** (1984), 176–179.